## Chapter –7

## A REPRESENTATON OF AFFINE VARIETIES

The aim of this chapter is to find a representation of affine sub-varieties of an affine algebraic set in an affine n-space. Classical algebraic geometry arose from the study of solutions of system of polynomial equations over the field k of real or complex numbers. But for the general case we take k an arbitrary field.

Let $k[x_1, x_2, \ldots, x_n]$ denote the polynomial ring over k in n variables and $f_1, f_2, \ldots, f_m \in k[x_1, x_2, \ldots, x_n]$. Then the algebraic set V(S) defined by $S = \{ f_1, f_2, \ldots, f_m\} \subset k[x_1, x_2, \ldots, x_n]$ is the set of solutions in $k^n$ of the system of equations,
$f_1(x_1, x_2, \ldots, x_n) = 0,\ f_2(x_1, x_2, \ldots, x_n) = 0, \ldots, f_m(x_1, x_2, \ldots, x_n) = 0.$
i.e., $V(S) = \{(x) \in k^n : f(x) = 0, \forall\ f \in S\}$ where
$(x) = (x_1, x_2, \ldots, x_n) \in k^n$.

If I is the ideal of $k[x_1, x_2, \ldots, x_n]$ generated by S i.e., $I = < S >$, then V(I) = V(S) [in **Proposition 7.1.2**].

As k is a field, $k[x] = k[x_1, x_2, \ldots, x_n]$ is a Noetherian ring and every ideal I of k[x] is finitely generated. Hence there is no need to consider infinite set S of polynomial equations to determine the set of solutions in $k^n$.

### 7.1 Algebraic sets, Affine variety and Ideal of a set of points in $k^n$.

In this section we need the following some elementary properties of Algebraic sets, Affine variety and Ideals of a set of points in $k^n$.

**Definition 7.1 .1**

Let k be an algebraically closed field. An affine n-space over k, denoted by $k^n$ is the set of all n-tuples $(x) = (x_1, x_2, \ldots, x_n)$ of elements of k. A subset A in $k^n$ is called an affine algebraic set if A = V(S), where $V(S) = \{(x) \in k^n : f(x) = 0, \forall\ f \in S\}$ for some $S \subset k[x]$. Then any algebraic set A is defined by a finite set of polynomial in k[x].

**Definition 7.1.2**

Given a subset A in $k^n$, the ideal of A denoted by I(A) is defined by $I(A) = \{f \in k[x] : f(x) = 0; \forall (x) \in A\}$ is an ideal in k[x].

**Definition 7.1.3**

An algebraic set A in $k^n$ is called irreducible or an affine variety if $A \neq B \cup C$, where B and C are algebraic set in $k^n$ and $A \neq B$, $A \neq C$, otherwise A is called reducible.

**Proposition 7.1. 1.**

i) $P \subseteq Q$ in $k[x_1, x_2, \ldots, x_n] \Rightarrow V(P) \supseteq V(Q)$ in $k^n$.

ii) $V(0) = k^n$, $V(a) = \varnothing$ where $a \neq 0$ and $a \in k$;

iii) $\{P_\alpha\}$ is a family of ideals in $k[x_1, x_2, \ldots, x_n]$
$\Rightarrow V(\bigcup_\alpha P_\alpha) = \bigcap_\alpha V(P_\alpha)$.

iv) P and Q are ideals in $k[x_1, x_2, \ldots, x_n]$ and $A = V(P)$ and $B = V(Q) \Rightarrow A \cup B = V(P \cap Q) = V(PQ)$.

v) Any finite subset of $k^n$ is an algebraic set.

Proof :

i) Let $(x) \in V(Q) \Leftrightarrow (x)$ is a common zero of polynomials of $Q \Rightarrow (x)$ is also a common zero of polynomials of $P \Rightarrow (x) \in V(P) \Rightarrow V(Q) \subseteq V(P)$.

ii) As every point of $k^n$ satisfies the zero polynomial, $V(0) = k^n$ .As the constant non-null polynomial 'a' has no zero, $V(a) = \varnothing$.

iii) Let $(x) \in V(\bigcup_\alpha P_\alpha) \Leftrightarrow f(x) = 0 \; \forall f \in \bigcup_\alpha P_\alpha \Leftrightarrow f(x) = 0 \; \forall f \in P_\alpha$ where $P_\alpha$ is any member of the collection $\{P_\alpha\}$
$\Leftrightarrow (x) \in V(P_\alpha), \forall \alpha \Rightarrow (x) \in \cap V(P_\alpha)$.

iv) Let $(x) \in A \cup B = V(P) \cup V(Q) \Rightarrow (x)$ is a zero of all polynomials in P or (x) is zero of all polynomials in Q
$\Rightarrow (x)$ is a zero of all polynomials in $P \cap Q$

$\Rightarrow (x)$ is a zero of all polynomials in $P \cdot Q$

(since $P \cdot Q \subseteq P \cap Q) \Rightarrow A \cup B \subseteq V(P \cap Q) \subseteq V(PQ)$.

Again $(x) \notin A \cup B \Rightarrow \exists\, f \in P$ and $g \in Q$ such that $f(x) \neq 0$ and $g(x) \neq 0 \Rightarrow f \cdot g \in P \cap Q$(also to $P \cdot Q$) does not vanish at $(x) \Rightarrow (x) \notin V(P \cap Q)$ and also $(x) \notin V(P \cdot Q) \Rightarrow$ the zeros of $P \cap Q$(as well as $P \cdot Q$) are the points $A \cup B$ and only these.

vi) Any point $(x_1, x_2, \ldots\ldots, x_n) \in k^n$ is an algebraic set ,because $V(X_1 - x_1, X_2 - x_2, \ldots\ldots\ldots, X_n - x_n) = \{(x_1, x_2, \ldots\ldots\ldots, x_n)\}$.
Hence, any finite subset A of $k^n$ is an algebraic set by iv).

**Proposition 7.1.2**

i) $A \subseteq B$ in $k^n \Rightarrow I(A) \supseteq I(B)$ in $k[x_1, x_2, \ldots\ldots\ldots, x_n]$.

ii) $S \subseteq I(V(S))$ for any set of polynomials S in $k[x_1, x_2, \ldots\ldots\ldots, x_n]$.

iii) $A \subseteq V(I(A))$ for any set of points A in $k^n$.

iv) $V(S) = V(I(V(S)))$ for any set of polynomials S in $k[x_1, x_2, \ldots\ldots\ldots, x_n]$.

v) $I(A) = I(V(I(A)))$ for any set of points A in $k^n$.

vi) $I(\varnothing) = k[x_1, x_2, \ldots\ldots\ldots, x_n]$.

vii) $I(k^n) = 0$, if k is an infinite field.

Proof:

i) Let $f \in I(B) \Leftrightarrow$ f vanishes on $B \Rightarrow$ f vanishes on A, (since $A \subseteq B$) $\Leftrightarrow f \in I(A) \Rightarrow I(B) \subseteq I(A)$.

ii) This follows from the fact that S is a set of some polynomials which vanish on V(S), while I(V(S)) is the set of all polynomials which vanish on V(S).

iii) This follows from the fact that A is a set of some common zeros of polynomials in I(A) while V(I(A)) is the set of all common zeros of polynomials in I(A).

iv) $V(S) \subseteq V(I(V(S)))$ by ii) ( takeV(S) from A in i)). Conversely, $V(I(V(S))) \subseteq V(S)$. Because
$(x) \in V(I(V(S))) \Leftrightarrow f(x) = 0 \;\forall f \in I(V(S)) \Rightarrow f(x) = 0 \;\forall f \in S$ by ii) $\Leftrightarrow (x) \in V(S)$.

v) $I(A) \subseteq I(V(I(A)))$ by iii) (take I(A) for S in ii)). Conversely, $I(V(I(A))) \subseteq I(A)$ by iii). Because , $f \in I(V(I(A))) \Leftrightarrow f(x) = 0$ $\forall (x) \in V(I(A)) \Rightarrow f(x) = 0 \;\forall (x) \in A$ by iii) $\Rightarrow f \in I(A)$.

**Remark**

i) If A is an algebraic set and A = V(S) then A = V(I(A)); and if P is an ideal of an algebraic set A i.e., if P = I(A), then P = I(V(P).

ii) $V(I) = \varnothing \Rightarrow I(V(I)) = I(\varnothing)$. Now by ii), $I \in I(V(I)) \Rightarrow I(\varnothing) = k[x_1, x_2, \ldots\ldots, x_n]$.

iii) This part says that if k is an infinite field and
$f \in k[x_1, x_2, \ldots\ldots, x_n]$, then f vanishes on $k^n$ implies $f = 0$ .We shall prove this by induction on n .If n =1, a nonzero $f \in k[x_1]$ has a finite number of zeros $\leq$ deg f, by the fundamental theorem of algebra, and hence f cannot vanish on the infinite set k, unless f = 0. Next , suppose that the result is true for (n-1) variables.
Let $f \in k[x_1, x_2, \ldots\ldots, x_n]$ vanish on $k^n$. We write
$f(x_1, x_2, \ldots\ldots, x_n) = \sum f_i(x_1, x_2, \ldots\ldots, x_n)\; x_n^i$ as a polynomial in $x_n$ with coefficients in $k[x_1, x_2, \ldots\ldots, x_{n-1}]$ . If $\exists$ a point $(a_1, a_2, \ldots\ldots, a_{n-1}) \in k^{n-1}$ such that for some j, $f_j(a_1, a_2, \ldots\ldots, a_{n-1}) \neq 0$, then the nonzero polynomial $f(a_1, a_2, \ldots\ldots, a_{n-1}, x_n) \in k[x_n]$ vanishes on k, since $f(x_1, x_2, \ldots\ldots, x_n)$ vanishes on $k^n$ .

This implies , by the fundamental theorem of algebra that $f(a_1,a_2 \ldots\ldots a_{n-1},x_n) = 0$ , which is impossible , since $f_j(a_1,a_2,\ldots\ldots a_{n-1}) \neq 0$. Therefore $f_j$ must vanish on $k^{n-1}$ $\forall$ j, and hence $f_j = 0$ $\forall$ j, by inductive hypothesis. Consequently, f = 0.

**Proposition 7.1.3**

An algebraic set A is an affine variety $\Leftrightarrow$ I(A) is a prime ideal.

Proof :-

Suppose A is an algebraic set and P is its ideal, i.e., P = I(A). Then A = V(P) by **Proposition 7.1.2 iv**). If P is not prime, we can find f, g $\in k[x_1, x_2,\ldots\ldots x_n]$ such that
$f \notin P$, $g \notin P$ but $fg \in P$. Now by **Proposition 7.1.1**(i) and iii), we get $f \notin P$ $\Rightarrow P \cup (f) \underset{\neq}{\supset} P \Rightarrow V(P) \cap V(f) = V(P \cap (f)) \underset{\neq}{\subset} V(P)$; $g \notin P$
$\Rightarrow P \cup (g) \underset{\neq}{\supset} P \Rightarrow V(P) \cap V(g) = V(P \cup (g)) \underset{\neq}{\subset} V(P)$.
Also $[V(P) \cap V(f)] \cup [V(P) \cap V(g)] = [V(P) \cap [V(f) \cup (g)]$ (by distributive laws of sets) = $V(P) \cap V(fg)$ (by **Proposition 7.1.1 iv**))
$= V(P \cap (fg)) = V(P)$, since $fg \in P$.

Thus the algebraic set A = V(P) is the union of two algebraic sets $V(P) \cap V(f)$ and $V(P) \cap V(g)$, which are distinct from A, and hence A is reducible.
Conversely, if A is reducible, then $A = B \cup C$ where B and C are algebraic sets different from A. Then
$A \underset{\neq}{\supset} B \Rightarrow I(A) \underset{\neq}{\subset} I(B)$ and $A \underset{\neq}{\supset} C \Rightarrow I(A) \underset{\neq}{\subset} I(C)$.

Therefore , if $f \in I(B)$, $g \in I(C)$ and f, $g \notin I(A)$, then $fg \in I(A)$. Because , $A = B \cup C = V(I(B)) \cup V(I(C))$ (by **Proposition 7.1.2 ii**)) $= V(I(B) \cdot I(C))$ (by **Proposition 7.1.1(iv)**) and hence
$I(A) = I(V(I(B) \cdot I(C)) \supseteq I(B) \cdot I(C)$
(by **Proposition ( 7.1.2 ii**) )$\Rightarrow$I(A) is not prime .

We have proved that an algebraic set A is reducible $\Leftrightarrow$ I(A) is not a prime ideal. Equivalently, A is an affine variety $\Leftrightarrow$ I(A) is a prime ideal.

We can define a topology in $k^n$ by taking a subset A of $k^n$ to be closed $\Leftrightarrow$ A is an affine variety in $k^n$. This topology is called **Zarisky topology**. Thus a subset A is closed $\Leftrightarrow$ its ideal I(A) is a prime ideal by **Proposition 7.1.3.**

In this way we can introduce a topology in the prime spectrum Spec( R ).

In **section 7.2** by applying **Yoneda's Lemma**, we obtain a representation of affine subvarieties of an affine algebraic set in $k^n$.

## 7.2 Representations :

Let k be an algebraically closed field. A basic problem in algebraic geometry is the representation of subvarieties of an algebraic set in $k^n$ .

Any polynomial $f \in k[x_1, x_2, \ldots, x_n]$ defines a polynomial function $\psi_f : k^n \longrightarrow k$ by
$\psi_f(x) = \psi_f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n), \forall (x) \in k^n$.
Let $A \subset k^n$ be an algebraic set . Then $\psi_f | A$ is a polynomial functions defined on A.

All polynomial functions defined on A form a ring $\mathcal{R}[A]$ under usual compositions of functions.

Let R be a ring . Define prime spectrum Spec( R ) of R by
Spec( R ) = {p : p is a prime ideal of R}.
Let $\mathcal{R}$ denote the category of rings and ring homomorphisms and $\mathcal{S}$ denote the category of sets and functions.

We define a function
Spec : $\mathcal{R} \rightarrow \mathcal{S}$ by
$R \mapsto$ Spec( R) , for every object $R \in \Re$ and for a ring homomorphism $f : R \rightarrow T$ in $\mathcal{R}$,
$f^* =$ Spec( f ) : Spec(T)$\rightarrow$ Spec( R ) by
$f^*(Q) = f^{-1}(Q), \forall Q \in$ Spec(T).
Since $f : R \rightarrow T$ in a ring homomorphism and Q is an ideal of T and hence $\pi_1 : T \rightarrow T/Q$ is a natural epimorphism.

Now $\psi(=\pi_1 \circ f) : R\to T/Q$ is a ring homomorphism. Then by **first isomorphism theorem** for rings, $\ker \psi = f^{-1}(Q)$ is an ideal of R and hence $\pi : R\to R/f^{-1}(Q)$ is a natural epimorphism .Therefore $\psi : R/f^{-1}(Q) \to T/Q$ is a monomorphism.

Since Q is a prime ideal of T and hence T/Q is an integral domain and hence $R/f^{-1}(Q)$ is an integral domain.
Therefore $f^{1}(Q)$ is a prime ideal of R and hence
$f^{1}(Q) \in$ Spec( R).

Thus $f^{*}$ : Spec(T)$\to$Spec( R) is well defined .

Let $f : R\to T$ and $g : T\to M$ be ring homomorphisms in $\mathscr{R}$. Then $gf : R\to M$ is also ring homomorphism from R to M .

Therefore $\mathrm{Spec}(gf) = (gf)^{*} : \mathrm{Spec}(M)\to\mathrm{Spec}(R)$ by

$$
\begin{aligned}
(gf)^{*}(Q) &= (gf)^{-1}(Q)\forall Q \in \mathrm{Spec}(M)\\
&=( f^{1}\cdot g^{-1})(Q)\\
&= f^{1}(g^{-1}(Q))\\
&= \mathrm{Spec}(f)\,(\mathrm{Spec}(g)(Q))
\end{aligned}
$$

$$\Rightarrow \mathrm{Spec}(gf)(Q) = (\mathrm{Spec}(f)\cdot\mathrm{Sec}(g))(Q),\forall\ Q\in\mathrm{Spec}(M)$$

$$\Rightarrow \mathrm{Spec}(gf) = \mathrm{Spec}(f)\cdot\mathrm{Spec}(g).$$

Also , for the identity homomorphism $I_R : R\to R$ in $\mathscr{R}$,

$$
\begin{aligned}
\mathrm{Spec}(I_R)(R) &= I^{-1}{}_R(R) ,\forall\ R\in \mathrm{Spec}(R)\\
&= I_R(R) = I_{\mathrm{Spec}(R)}(R)
\end{aligned}
$$

$$\Rightarrow \mathrm{Spec}(I_R) = I_{\mathrm{Spec}(R)},\forall\ R\in\mathscr{R}.$$

Hence we have the following theorem.

**Theorem 7.2.1**

Spec: $\mathscr{R}\to\mathscr{S}$ is a contravariant functor .

It is well known that an algebraic set A in $k^n$ is a affine variety iff its ideal I(A) is a prime ideal (**Proposition 7.1.3)** in $k[x_1, x_{2,\dots\dots}, x_n]$.

This shows that the set of all subvarieties of A has a bijective correspondence with the set of all prime ideals of the ring $\mathscr{R}[A]$.

Let R be a ring in the category $\mathscr{R}$ of rings and ring homomorphisms. Let $h^{\mathscr{R}[A]}(R)$ be the set of all ring homomorphisms from the ring R to the ring $\mathscr{R}[A]$. i.e., $h^{\mathscr{R}[A]}(R) = \text{Hom}(R, \mathscr{R}[A])$.

We define for each $f : R \to T$ in $\mathscr{R}$ .

$$h^{\mathscr{R}[A]}(f) : \text{Hom}(T, \mathscr{R}[A]) \to \text{Hom}(R, \mathscr{R}[A]) \text{ by}$$

$h^{\mathscr{R}[A]}(f)(\alpha) = \alpha \circ f \quad \forall\ \alpha \in \text{Hom}(T, \mathfrak{R}[A])$. Then it follows that

**Theorem 7.2.2**

$h^{\mathscr{R}[A]} : \mathscr{R} \to \mathscr{S}$ is a contravariant functor.

Thus we have two contravariant functors 'Spec' and $h^{\mathscr{R}[A]}$ from the category $\mathscr{R}$ to $\mathscr{S}$.

**Theorem 7.2.3**

For each algebraic set A is a affine n-space $k^n$, there is an equivalence $\theta : (\text{Spec}, h^{\mathscr{R}[A]}) \to \text{Spec}(\mathscr{R}[A])$.

Proof :

Using **Yoneda's Lemma** for the contravariant functors Spec and $h^{\mathscr{R}[A]}$ from the category $\mathscr{R}$ to category $\mathscr{S}$, the theorem follows.

The **Theorem 7.2.3** shows that there is a bijective correspondence between the set of all prime ideals of the ring $\mathscr{R}[A]$ of polynomial functions of the affine algebraic set A and the set of all natural transformations from Spec to $h^{\mathscr{R}[A]}$. Again as there is a bijective correspondence between the set of all prime ideals of the ring $\mathscr{R}[A]$ and the set of all subvarieties of A in $k^n$, it follows the following **Theorem**

**Theorem 7.2.4**

For each algebraic set A in $k^n$, the set of all subvarieties in $k^n$ is equipotent to the set (Spec, $h^{\mathcal{K}[A]}$) of all natural transformations from the contravariant functor Spec to the contravariant functor $h^{\mathcal{K}[A]}$.

**Remark :**

The **Theorem 7.2.4** yields a representation of subvarieties of an affine algebraic set A in $k^n$