A Generalization of M. H. Stone's Theorem

Our study shall start with the definition of the different function rings. A large majority of the results obtained here are adaptations of similar kind of results that are obtained in Gillman & Jerison [22], although some extrinsic characterization of the real line $\mathbb{R}$ have emerged in the process.

In what follows $X$ shall stand for a Hausdorff topological space and $F$ shall stand for a (linearly) ordered field equipped with its order topology. Also for sets $S$ and $T$ the symbol $T^S$ shall stand for the set of all functions $f : S \to T$. Moreover, unless otherwise specified, the words "topological space" shall always be used to mean a Hausdorff topological space and the word "order" shall always be used to mean a linear order.

1. Function Rings & Their Basic Properties

We start with the definition of the different function rings.
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II. M. H. STONE'S THEOREM : GENERALIZATION

**DEFINITION 1.1.**

\[ \mathcal{C}(X, F) = \{ f \in F^X : f \text{ is continuous on } X \}; \]

\[ (12) \quad \mathfrak{B}(X, F) = \{ f \in \mathcal{C}(X, F) : (\exists t \in F)(\forall x \in X)(-t \leq f(x) \leq t) \} \]

\[ \mathcal{C}^*(X, F) = \{ f \in \mathcal{C}(X, F) : \text{cl}_F(f(X)) \text{ is compact} \}. \]

We shall now define the binary operations of addition (+), multiplication (·), join (\lor) and meet (\land), the unary operations of negation (-) and modulus (|·|), the nullaries of the constant functions and finally the binary relation of a partial order (\leq) on the above mentioned sets.

**DEFINITION 1.2.** Let \( f, g \in \mathcal{C}(X, F) \).

i.

\[ f + g : X \rightarrow F \quad \left\{ \begin{array}{c}
  x \mapsto f(x) + g(x) \\
\end{array} \right\}; \]

ii.

\[ fg : X \rightarrow F \quad \left\{ \begin{array}{c}
  x \mapsto f(x)g(x) \\
\end{array} \right\}; \]

iii.

\[ f \lor g : X \rightarrow F \quad \left\{ \begin{array}{c}
  x \mapsto f(x) \lor g(x) \\
\end{array} \right\}; \]
iv. 
\[ f \land g : X \rightarrow F \]
\[ x \mapsto f(x) \land g(x) \]

v. 
\[ -f : X \rightarrow F \]
\[ x \mapsto -f(x) \]

vi. 
\[ |f| : X \rightarrow F \]
\[ x \mapsto |f(x)| \]

vii. for any \( t \in F \) the symbol \( t \) is used to denote the function that takes the constant value \( t \);

viii. \( f \leq g \), if and only if, for each \( x \) in \( X \) we have \( f(x) \leq g(x) \).

\[ \Box \]

Since \( F \) is a topological field it is clear from these definitions that:

**Theorem 1.1.** Each of the sets \( \mathcal{C}(X, F) \), \( \mathfrak{B}(X, F) \) and \( \mathcal{C}^*(X, F) \) with the operations defined in Definition 1.2 make lattice ordered commutative rings with 1 as the unity.

Furthermore

\[ \mathcal{C}(X, F) \supseteq \mathfrak{B}(X, F) \supseteq \mathcal{C}^*(X, F) \supseteq F. \]
provided we agree to identify the member \( t \in F \) with the constant function \( t \in \mathcal{C}^*(X, F) \).

Finally, if \( X \) is compact then \( \mathcal{C}^*(X, F) = \mathcal{C}(X, F) \).

**Proof.** Trivial. \( \square \)

We would now solve a series of simple-minded questions: when do the inequalities in the second part of Theorem 1.1 turn to an equality? To this end let us first see when should \( \langle t^* (X, F) \rangle f \in F \)? To answer this question we would require to introduce the following notations.

**Definition 1.3.** Let \( f, g \in \mathcal{C}(X, F) \).

i. The set \( Z_{X,F}(f) \) defined by

\[
Z_{X,F}(f) = \{ x \in X : f(x) = 0 \}
\]

is called the zero set of \( f \).

ii. The set \( [f \neq 0] \) defined by

\[
[f \neq 0] = X \setminus Z_{X,F}(f)
\]

is called the cozero set of \( f \).

iii.

\[
[f = g] = \{ x \in X : f(x) = g(x) \}.
\]
iv. 
\[ [f \leq g] = \{ x \in X : f(x) \leq g(x) \} \]

v. 
\[ [f \geq g] = \{ x \in X : f(x) \geq g(x) \} \]

vi. 
\[ [f < g] = \{ x \in X : f(x) < g(x) \} \]

vii. 
\[ [f > g] = \{ x \in X : f(x) > g(x) \} \]

viii. \( S \subseteq X \) is said to be a \textit{zero set of} \( X \) \textit{with respect to} \( F \), if and only if, there exists one \( f \in \mathcal{C}(X, F) \) such that \( S = Z_{X,F}(f) \) and \( T \subseteq X \) is said to be a \textit{cozero set of} \( X \) \textit{with respect to} \( F \), if and only if, there exists one \( g \in \mathcal{C}(X, F) \) such that \( T = X \setminus Z_{X,F}(g) \).

ix. \( \mathcal{Z}(X, F) \) is the set of all \textit{zero sets of} \( X \) \textit{with respect to} \( F \).
For the time being, let us assume that $X$ has at least two points and that $f \in \mathcal{C}^*(X, F)$ is a non-constant function. Then there exists $a, b \in F$ such that $a < b$ and that both the sets $[f = a]$ and $[f = b]$ are non-empty. Consequently each of the sets $[f \leq \frac{a+b}{2}]$ and $[f \geq \frac{a+b}{2}]$ are non-empty, so that both the functions $g$ and $h$ defined by

$$g = (f \bigwedge \frac{a+b}{2}) - \frac{a+b}{2}, \quad \text{and} \quad h = (f \bigvee \frac{a+b}{2}) - \frac{a+b}{2},$$

are members of $\mathcal{C}^*(X, F)$, with $g \neq 0$ and $h \neq 0$ and $gh = 0$.

So the moment we assume $X$ has at least two points and $\mathcal{C}^*(X, F)$ has a non-constant function, $\mathcal{C}^*(X, F)$ fails to be a field and possesses divisors of zero. For the case of $X$ a singleton, it is very clear that all the function rings of Definition 1.1 are same as the ordered field $F$ itself. Thus the previous question boils down to the investigation of the question: when does $\mathcal{C}^*(X, F)$ possess non-constant functions?

For the case of $F = \mathbb{R}$, $\mathcal{C}^*(X, \mathbb{R})$ is a field, if and only if, $\mathcal{C}(X, \mathbb{R})$ is a field; and a partial answer to this is known from Gillman & Jerison [22], i.e., if $X$ be a Tychonoff space with at least two points then $\mathcal{C}(X, \mathbb{R})$ is not a field.

If $F \neq \mathbb{R}$ then from Theorem I.2.6 we find that $F$ is zero dimensional, so that if $X$ be a connected topological space then $\mathcal{C}(X, F)$ is a
field — indeed \( \mathcal{C}(X, F) = F \). On the other hand, if \( X \) be disconnected and \( X = A \cup B \) be a disconnection of \( X \) then the function \( f : X \to F' \) defined by

\[
f(x) = \begin{cases} 
0 & \text{if } x \in A; \\
1 & \text{if } x \in B;
\end{cases}
\]

is indeed a member of \( \mathcal{C}^*(X, F) \) so that in this case \( \mathcal{C}^*(X, F) \) possesses a non-constant member, and thus by what was previously done, follows that \( \mathcal{C}^*(X, F) \) is not a field. All these are summarized by the next theorem.

**Theorem 1.2.**

i. If \( X \) has at least two points, then barring \( F \), no lattice ordered subring \( \mathcal{A} \) of \( \mathcal{C}^*(X, F) \) which contains \( F \) and has at least one non-constant function can afford to be a field.

ii. For \( F \neq \mathbb{R} \), \( \mathcal{C}^*(X, F) \) possesses a non-constant function, if and only if, \( X \) is disconnected.

iii. For \( F = \mathbb{R} \), \( \mathcal{C}^*(X, F) \) is a field, if and only if, \( \mathcal{C}(X, F) \) is a field, and that if \( \mathcal{C}(X, F) \) be a field then \( X \) cannot be a Tychonoff topological space.

iv. For any ordered field \( F \), \( \mathcal{C}(X, F) \) is a field, if and only if, \( \mathcal{B}(X, F) \) is a field.
Proof. The proofs of (i), (ii) and (iii) have already been done. The proof of (iv) requires some other facts and is provided after the proof of Theorem 1.8.

Indeed one can look upon Theorem 1.2(ii) as a ring-theoretic analogue of the topological property of connectedness: to check whether a Hausdorff topological space $X$ is connected it is enough to see whether $\mathcal{C}^*(X, F)$ is a field or not, where $F$ could be any ordered field other than $\mathbb{R}$, say the field $\mathbb{Q}$ of all the rational numbers. Apart from this Theorem 1.2(ii) also suggests that to do an interesting study of the rings $\mathcal{C}(X, F)$ for $F \neq \mathbb{R}$ one has to forego the nicety of connected topological spaces.

The next question that we shall ask is — can $\mathcal{B}(X, F) = \mathcal{C}^*(X, F)$?

This is settled immediately now using Theorem 1.2.7.

Theorem 1.3. For all topological spaces $X$, $\mathcal{C}^*(X, F) = \mathcal{B}(X, F)$, if and only if, $F = \mathbb{R}$.

Proof. The "$\leftarrow$" part follows immediately from the Heine-Borel Theorem for $\mathbb{R}$. So we prove the "$\Rightarrow$" part only.
Let \( F \neq \mathbb{R} \) and we do consider the function \( f : F \to F \) defined by

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
x & \text{if } 0 \leq x \leq 1 \\
1 & \text{if } x \geq 1 
\end{cases}
\]

It is then clear that \( f \in \mathcal{C}(F, F) \) and that \( \text{cl}_F(f(F)) = [0, 1] \), and that from Theorem 2.7 we have \([0, 1]\) to be a closed, bounded and a non-compact subset of \( F \), so that \( f \in \mathcal{B}(X, F) \setminus \mathcal{C}^*(X, F) \).

The final question in this direction would be on the equality of the two rings \( \mathcal{C}(X, F) \) and \( \mathcal{B}(X, F) \). The answer to this remains unanswered and we propose it as a question separately.

**Problem 1.** Given a topological space \( X \) is it possible to provide a construction of an ordered field \( F \) so that \( \mathcal{B}(X, F) = \mathcal{C}(X, F) \)?

**Problem 2.** Given an ordered field \( F \), is it possible to provide a construction of a topological space \( X \) so that \( \mathcal{B}(X, F) = \mathcal{C}(X, F) \)?

Since \( F \) is an ordered field we get the following immediate consequences of Definition 1.3.
**Theorem 1.4.**

i. The zero sets of $X$ with respect to $F$ are closed subsets of $X$ and the cozero sets of $X$ with respect to $F$ are open subsets of $X$.

ii. The clopen subsets of $X$ are both zero subsets and cozero sets of $X$ with respect to $F$.

iii. For any $s$ and $t$ from $F$ and $f$ from $\mathcal{C}(X, F)$ the sets $[f \geq t]$, $[f \leq s]$ and $[t \leq f \leq s]$ are each zero sets of $X$ with respect to $F$ while the sets $[f > t]$, $[f < s]$ and $[t < f < s]$ are each cozero sets of $X$ with respect to $F$.

Conversely, any $Z \in \mathcal{Z}(X, F)$ is expressible in the form $[f \leq 0]$ or $[f \geq 0]$ and any cozero set in $X$ with respect to $F$ is expressible in the form $[f < 0]$ or $[f > 0]$.

iv. For every $f \in \mathcal{C}(X, F)$ and $n \in \mathbb{N}$,

$$Z_{X,F}(f) = Z_{X,F}(f^n)$$

$$= Z_{X,F}(|f|)$$

$$= Z_{X,F}(|f|^n).$$

v. For every $f, g \in \mathcal{C}(X, F)$,

$$Z_{X,F}(f^2 + g^2) = Z_{X,F}(|f| + |g|)$$

$$= Z_{X,F}(f) \cap Z_{X,F}(g).$$
vi. For every $f, g \in \mathcal{C}(X, F)$,

$$\mathcal{Z}_{X,F}(fg) = \mathcal{Z}_{X,F}(f) \cup \mathcal{Z}_{X,F}(g).$$

vii. $\mathcal{J}(X, F)$ is closed under finite intersections and unions and possesses both a largest member, namely $X$, and a least member, namely $\emptyset$.

viii. For any $f \in \mathcal{C}(X, F)$,

$$\mathcal{Z}_{X,F}(f) = \bigcap_{\alpha \in F_+} |f| < \alpha|,$$

where $F_+$ stands for the set of all positive members of $F$.

PROOF. It is enough to prove the “converse” part of (iii). For if $Z = \mathcal{Z}_{X,F}(f) \in \mathcal{J}(X, F)$, for some $f \in \mathcal{C}(X, F)$ then $Z = \mathcal{Z}_{X,F}(|f|) = |f| \leq 0 = [-|f| \geq 0].$ Since the cozero sets in $X$ with respect to $F$ are complements of zero sets in $X$ with respect to $F$ their representation follows from the representation for the zero sets in $X$ with respect to $F$.

Thus the entire family of zero sets in $X$ with respect to $F$ is given by the collection of sets $[f \geq s]$ and $[f \leq t]$, as $f$ varies over $\mathcal{C}(X, F)$ and $s, t$ varies over the field $F$, and the family of cozero sets in $X$ with respect to $F$ is given by the collection of sets $[f < s]$ and $[f > t]$ as $f$
varies over \( C(X, F) \) and \( s, t \) vary over the field \( F \) — a consequence of Theorem 1.4(iii) that shall be used heavily in this work.

We shall now characterize the situation when the zero sets are closed \( G_\delta \)-subsets of \( X \).

**Theorem 1.5.** Let \( \text{cf}(F) = \aleph_\alpha \). Then every \( Z \in \mathcal{Z}(X, F) \) is an intersection of an \( \omega_\alpha \)-sequence of cozero sets of \( X \) with respect to \( F \).

In particular, for any Hausdorff topological space \( X \) the zero sets of \( X \) with respect to \( F \) are \( G_\delta \) subsets of \( X \), if and only if, \( \text{cf}(F) = \aleph_0 \).

**Proof.** By Theorem 1.1(i) as \( \text{cf}(F) = \aleph_\alpha \), a copy of \( \aleph_\alpha \) is cofinally embedded in \( F \) and we may as well identify \( \aleph_\alpha \) with this copy. Then \( \{ \beta^{-1} : \beta \in \aleph_\alpha \} \) is coinitial in \( F_+ = \{ x \in F : x > 0 \} \) and thus by Theorem 1.4(viii) we have

\[
Z_{X, F}(f) = \bigcap_{t > 0} [\{ f \} < t]
\]

\[
= \bigcap_{\beta \in \aleph_\alpha} [\{ f \} < \beta^{-1}] .
\]

For the second part the "\( \Leftarrow \)" follows from above, and we prove the "\( \Rightarrow \)" part.
1. BASIC PROPERTIES

Let \( \text{cf}(F) > \aleph_0 \). Then the identity map \( F : F \to F \) defined by \( F : x \mapsto x \), is surely a member of \( \mathcal{C}(F, F) \) with \( Z_{F,F}(F) = \{0\} \) which is not a \( G_\delta \) set as \( \text{ci}(F_+) > \aleph_0 \).

Thus the function spaces \( \mathcal{C}(X, F) \) also describe certain properties pertaining to the order of the field \( F \) in a nice way. Such capabilities were actually hidden in the theory of \( \mathcal{C}(X, \mathbb{R}) \) on fixing the range, whereas here \( F \) is as much unknown as the base space \( X \) itself. The explanation of the fact that the zero sets of \( X \) with respect to \( \mathbb{R} \) are closed \( G_\delta \) subsets of \( X \) is provided more vividly here; in fact the above and the next example suggests that there could be as well many more sites, quite distinct from that of \( \mathbb{R} \) or its subfields where the zero sets with respect to the ordered field in consideration could be all closed \( G_\delta \) sets.

Example 1.1. Here we do provide a non-Archimedean field with a countable cofinal subset.

Let \( \theta \) be an indeterminate over the real field \( \mathbb{R} \) and consider the polynomial ring \( \mathbb{R}[	heta] \). Let \( \mathcal{P} \) denote the set of all members from \( \mathbb{R}[	heta] \) which has their leading term non-negative.
It is clear that:

\[ \mathbb{P} + \mathbb{P} \subseteq \mathbb{P} \]

\[ \mathbb{P}\mathbb{P} \subseteq \mathbb{P} \]

\[ \mathbb{P} \cup (-\mathbb{P}) = \mathbb{R}[\theta] \]

\[ \mathbb{P} \cap (-\mathbb{P}) = \{0\} \].

Thus, if for \( f \) and \( g \) from \( \mathbb{R}[\theta] \) we define \( f \leq g \), if and only if, \( g - f \in \mathbb{P} \) then it is clear that \( \leq \) defines an order on \( \mathbb{R}[\theta] \).

Let \( \mathbb{R}(\theta) \) be the field of fractions of the ordered ring \( \mathbb{R}[\theta] \). Without any loss of generality we may agree to denote the elements of \( \mathbb{R}(\theta) \) as ratios \( \frac{f}{g} \) where \( g > 0 \). On \( \mathbb{R}(\theta) \) we now define \( \frac{f}{g} \geq 0 \), if and only if, \( f \geq 0 \). This extends the order of the original ring \( \mathbb{R}[\theta] \) and thus now \( \mathbb{R}(\theta) \) is an ordered field. Furthermore, this field also contains an order isomorphic copy of \( \mathbb{R} \) and that

\[ 1 \ll \theta \ll \theta^2 \ll \cdots \ll \theta^n \ll \theta^{n+1} \ll \cdots, \]

where in an ordered ring \( \mathbb{R} \) we write \( x \ll y \), to denote the fact that

\((\forall n \in \mathbb{N})(y > nx)\), i.e., \( y \) is infinitely larger than \( x \) or that \( x \) is infinitely smaller than \( y \).
Clearly the set \{1, \theta, \theta^2, \theta^3, \ldots, \theta^n, \theta^{n+1}, \ldots \} is cofinal in \mathbb{R}(\theta) so that \text{cf}(\mathbb{R}(\theta)) = \mathbb{R}_0. But then from the above construction we find that \mathbb{R}(\theta) is a non-Archimedean ordered field and hence distinct from \mathbb{R} and all its subfields.

We shall now proceed towards the description of the units of the rings \mathcal{C}(X, F), \mathcal{B}(X, F) and \mathcal{C}^*(X, F).

From the continuity of the map \( x \mapsto x^{-1} \) from \( F \setminus \{0\} \) onto \( F \setminus \{0\} \) it quickly follows that the units of \( \mathcal{C}(X, F) \) are precisely those continuous functions \( f \) for which \( Z_{X,F}(f) = 0 \). Further as \( \mathcal{B}(X, F) \) is a lattice ordered subring of \( \mathcal{C}(X, F) \), the units of \( \mathcal{B}(X, F) \) must also be the units of \( \mathcal{C}(X, F) \), so that all the units of \( \mathcal{B}(X, F) \) must of necessity possess the property that \( Z_{X,F}(f) = 0 \). Since, it is quite natural a choice that for \( f \in \mathcal{C}(X, F), Z_{X,F}(f) = 0 \Rightarrow \text{its inverse is } f^{-1} : x \mapsto f(x)^{-1}, \)

we find that for \( f \in \mathcal{B}(X, F) \) with \( Z_{X,F}(f) = 0, f^{-1} \in \mathcal{B}(X, F) \), if and only if, \( f^{-1} \) is also bounded, i.e., \( f \) is itself bounded and bounded away from zero. Thus the units of \( \mathcal{B}(X, F) \) are precisely those continuous functions from the said ring which are bounded and bounded away from zero.
Finally, as before, the units of $C^*(X, F)$ must also be units of $B(X, F)$, so that they must be non-vanishing bounded functions bounded away from zero with pre-compact range. We assert that any member $f$ from $C^*(X, F)$ which is bounded and bounded away from zero has the function $f^{-1}$ also to possess pre-compact range.

To prove the assertion let us note that the map $x \mapsto x^{-1}$ is a homeomorphism on $F \setminus \{0\}$ onto itself so that for any $A \subseteq F \setminus \{0\}$ the sets $A$ and $A^{-1}$ are homeomorphic under the same map, where $A^{-1} = \{x^{-1} : x \in A\}$. Consequently, it is easy to see that $cl_F(A)$ and $cl_F(A^{-1})$ are also homeomorphic under the same map, provided $0 \not\in cl_F(A)$ and $0 \not\in cl_F(A^{-1})$. Now, if $f \in C^*(X, F)$ be a unit of $B(X, F)$ then $f$ is bounded and bounded away from $0$ so that $0 \not\in cl_F(f(X))$ and that $0 \not\in cl_F((f(X))^{-1})$. Accordingly, $cl_F(f(X))$ and $cl_F((f(X))^{-1})$ are homeomorphic. Since $cl_F(f(X))$ is compact, it follows that the set $cl_F((f(X))^{-1})$, which is equal to the set $cl_F(f^{-1}(X))$, is compact, where $f^{-1} : x \mapsto f(x)^{-1}$ is the inverse of $f$. Thus $f^{-1} \in C^*(X, F)$ so that $f$ is a unit of $C^*(X, F)$ too.

All these are summarized in the next theorem.

**Theorem 1.6.** i. $f \in C(X, F)$ is a unit of $C(X, F)$, if and only if, $Z_{X,F}(f) = \emptyset$, i.e., $f$ is non-vanishing.
ii. \( f \in \mathcal{B}(X, F) \) is a unit of \( \mathcal{B}(X, F) \), if and only if, \( f \) is bounded and bounded away from zero, i.e., there exist \( s, t \in F_+ \) such that \( X = [t < |f| \leq s] \).

iii. \( f \in \mathcal{C}^*(X, F) \) is a unit of \( \mathcal{C}^*(X, F) \), if and only if, \( f \) is bounded and bounded away from zero, i.e., there exist \( s, t \in F_+ \) such that \( X = [t < |f| \leq s] \).

**Proof.** Done.

We shall now put forward some divisibility properties of these function rings which shall be of use later on.

**Theorem 1.7.** Let \( f \) and \( g \) be two members from \( \mathcal{C}(X, F) \).

i. If \( Z_{X,F}(f) \) be a neighborhood of \( Z_{X,F}(g) \) then \( g|f \).

If further, \( [f \neq 0] \subseteq A \), where \( A \) is a compact subset of \( X \), then \( h \in \mathcal{C}^*(X, F) \).

ii. If \( |f| \leq |g|^n \), for some \( n \in \mathbb{N} \) and \( n \geq 2 \) then \( g|f \).

**Proof.** i. On taking

\[
h(x) = \begin{cases} 
    f(x) & \text{if } x \notin \text{int}_X(Z_{X,F}(f)) \\
    \frac{f(x)}{g(x)} & \text{if } x \in \text{int}_X(Z_{X,F}(f)) \\
    0 & \text{if } x \in Z_{X,F}(f) 
\end{cases}
\]

we find that \( h \in \mathcal{C}(X, F) \) and that \( f = gh \).
Finally, since \([f \neq 0] \subseteq A\), it follows that \(X = A \cup Z_{X,F}(f)\) \(\Rightarrow\)
\[h(x) = h(A) \cup h(Z_{X,F}(f)) = h(A) \cup \{0\},\]
so that the condition of compactness of \(A\) and the continuity of \(h\) ensures that \(h(X)\) is also compact, so that \(h \in \mathcal{C}^*(X,F)\).

ii. On taking
\[
h(x) = \begin{cases} 
\frac{f(x)}{g(x)} & \text{if } g(x) \neq 0 \\
0 & \text{if } g(x) = 0
\end{cases}
\]
we find that \(h \in \mathcal{C}(X,F)\) and that \(f = gh\).

The next proposition states that \(\mathcal{Z}(X,F)\) is completely determined by \(\mathcal{B}(X,F)\).

**Theorem 1.8.** For any \(f \in \mathcal{C}(X,F)\) there exists one positive unit \(u \in \mathcal{C}(X,F)\) such that \(uf = (-1 \lor f) \land 1\).

Further, if \(f \in \mathcal{B}(X,F)\) then the unit may be so chosen that it belongs to \(\mathcal{B}(X,F)\).

In particular, both \(\mathcal{C}(X,F)\) and \(\mathcal{B}(X,F)\) determine the same zero sets.
PROOF. Taking

\[ u(x) = \begin{cases} \frac{1}{f(x)} & \text{if } |f(x)| \geq 1 \\ 1 & \text{if } |f(x)| \leq 1 \end{cases} \]

the first part easily follows.

For the second part, if \( f \in \mathcal{B}(X, F) \) then for all \( x \in X \), \( |f(x)| \leq M \) for some constant \( M \geq 1 \), so that from above, it follows that \( M^{-1} \leq |u| \leq 1 \), so that the assertion follows.

The last part follows from the fact that as \( u \) is a unit, \( Z_{X,F}(u) \) is an empty set so that \( Z_{X,F}(f) = Z_{X,F}(uf) \) and the second function is a member of \( \mathcal{B}(X, F) \) whenever the first is a member of \( \mathcal{C}(X, F) \).

We are now prepared for the proof of Theorem 1.2(iv).

PROOF. (Proof of Theorem 1.2(iv)) : If \( X \) has exactly one point then there is nothing to prove. So we assume that \( X \) has at least two points.

If \( \mathcal{C}(X, F) \) is a field then there are no non-constant functions and thus \( \mathcal{B}(X, F) \) is also a field.

Conversely, if \( \mathcal{B}(X, F) \) is a field then \( \mathcal{C}^*(X, F) \) possesses no non-constant functions so that \( \mathcal{Z}(X, F) \) consists of precisely the sets \( \emptyset \) and
Thus, from Theorem 1.8 \( \mathcal{C}(X, F) \) cannot possess any non-constant function and therefore \( \mathcal{C}(X, F) \) must be a field.

Given a non-constant function \( f \), there surely exists two points \( x \) and \( y \) from \( X \) such that \( f(x) \neq f(y) \), i.e., as if \( f \) separates the two points. As far as the fact of separating some points is concerned, we can do with non-constant functions only; however, separating all pairs of distinct points is a separate issue which we would like to ensure. Indeed, we would like to ensure that points and closed subsets missing them are separated by functions from \( \mathcal{C}(X, F) \). It can be easily realized that this non-trivial criterion must require some regularity conditions to be imposed on the topological spaces themselves to which we shall revert in the next section. However, for the time being we shall concentrate on this central theme of separating points and closed sets by \( \mathcal{C}(X, F) \).

**Definition 1.4.** Two subsets \( A \) and \( B \) of the topological space \( X \) are called completely \( F \) separated, if and only if, there exists one \( f \in \mathcal{B}(X, F) \), \( 0 \leq f \leq 1 \), such that, \( A \subseteq \mathcal{Z}_{X,F}(f) \) and \( B \subseteq [f = 1] \). We also refer to the function \( f \) here on saying that \( f \) separates \( A \) and \( B \).

In other words, a pair of subsets of \( X \) is completely \( F \) separated, if and only if, there exists one continuous function in \( \mathcal{C}(X, F) \) that pulls
the two sets apart. Clearly then, it is easy to see that a pair of subsets $A$ and $B$ of a topological space $X$ is completely $F$ separated, if and only if, there exists a pair $s$ and $t$ from $F$ and one $f \in C(X, F)$ such that $t < s$ and that $A \subseteq [f \leq t]$ and $B \subseteq [f \geq s]$. Our next theorem provides some computationally efficient tools to check whether a pair of subsets is completely $F$ separated or not.

**Theorem 1.9.** For any two subsets $A$ and $B$ of $X$ the following statements are equivalent.

i. $A$ and $B$ are completely $F$ separated.

ii. $A$ and $B$ have disjoint zero set neighborhoods.

iii. $A$ and $B$ are contained in disjoint zero sets.

iv. There exists $s, t \in F$ with $t < s$ and one $f \in C(X, F)$ such that $A \subseteq [f \leq t]$ and that $B \subseteq [f \geq s]$.

Furthermore, if $A$ and $B$ are subsets of $X$ that are completely $F$ separated then there exists zero sets $Z_1$ and $Z_2$ of $X$ with respect to $F$ such that

$$A \subseteq X \setminus Z_1 \subseteq Z_2 \subseteq X \setminus B.$$  

**Proof.** We first show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii): Since $A$ and $B$ are subsets of $X$ that are completely $F$ separated, there exists one $f \in \mathcal{B}(X, F)$, $0 \leq f \leq 1$, so that $A \subseteq Z_{X,F}(f)$ and $B \subseteq [f = 1]$. Clearly then, $A \subseteq [f \leq \frac{1}{3}]$ and that $B \subseteq [f \geq \frac{2}{3}]$ proving (ii).

(ii) $\Rightarrow$ (iii): Immediate.

(iii) $\Rightarrow$ (iv): Let $h$ and $g$ be members from $\mathcal{C}(X, F)$ such that $A \subseteq Z_{X,F}(h)$, $B \subseteq Z_{X,F}(g)$ and $Z_{X,F}(h) \cap Z_{X,F}(g) = \emptyset$. Thus if $f : X \rightarrow \mathbb{R}$, where $f : x \mapsto \frac{h(x)^2}{h(x)^2 + g(x)^2}$, then $f \in \mathcal{B}(X, F)$, $0 \leq f \leq 1$, $A \subseteq Z_{X,F}(f) = Z_{X,F}(h) \subseteq [f \leq \frac{1}{3}]$ and $B \subseteq [f = 1] = Z_{X,F}(g) \subseteq [f \geq \frac{2}{3}]$.

(iv) $\Rightarrow$ (i): From Theorem 1.4(iii) both the sets $[f \leq t]$ and $[f \geq s]$ are zero sets in $X$ with respect to $F$ so that there exist $p, s \in \mathcal{C}(X, F)$ with $Z_{X,F}(p) = [f \leq t]$ and that $Z_{X,F}(s) = [f \geq s]$. Then on taking $f : X \rightarrow \mathbb{R}$ as $f : x \mapsto \frac{s(x)^2}{p(x)^2 + s(x)^2}$, the rest follows.
2. M. H. STONE'S THEOREM

For the second part, let \( f \in \mathcal{B}(X, F) \) such that \( 0 \leq f \leq 1 \) and \( A \subseteq Z_{X,F}(f), \ B \subseteq [f = 1] \). Then

\[
A \subseteq [f \leq \frac{1}{4}]
\]

\[
\subseteq [f < \frac{1}{2}]
\]

\[
=X \setminus [f \geq \frac{1}{2}]
\]

\[
\subseteq [f \leq \frac{1}{2}]
\]

\[
\subseteq [f < \frac{3}{4}]
\]

\[
=X \setminus [f \geq \frac{3}{4}]
\]

\[
\subseteq X \setminus B,
\]

so that we get

\[
A \subseteq X \setminus [f \geq \frac{1}{2}] \subseteq [f \leq \frac{1}{2}] \subseteq X \setminus B,
\]

and then on taking \( Z_1 = [f \geq \frac{1}{2}] \) and \( Z_2 = [f \leq \frac{1}{2}] \), the proposition stands proved.

2. Analogue of M. H. Stone's Theorem

We have already seen that the non-triviality of the function ring \( \mathcal{C}(X, F) \) is ensured when the topological space \( X \) turn out to be disconnected; we demand the existence of many more nice functions there.
One way to ensure such existence would be the method of complete $F$ separation as introduced in Definition 1.4. Indeed, we shall show in this section that we do not require any more than the following regularity condition to be satisfied by our topological spaces:

*Given any closed subset $A$ of the topological space $X$ and a point $x \in X \setminus A$, the point $x$ and the set $A$ should be completely $F$ separated.*

**Definition 2.1.** A topological space $X$ is said to be completely $F$ regular, if and only if, $\mathcal{C}(X, F)$ separates points and closed subsets of $X$ in the sense that for any closed subset $A \subseteq X$ and $x \in X \setminus A$ there exists one $f \in \mathcal{C}(X, F)$ such that $f$ separates the subsets $\{x\}$ and $A$, as in Definition 1.4.

It is plain that completely $\mathbb{R}$ regular spaces are precisely the familiar Tychonoff spaces. Given any ordered field $F$ there are plenty of completely $F$ regular spaces, as we are going to see, and the next theorem provides the first one.

**Theorem 2.1.** Every ordered field $F$ in its order topology is completely $F$ regular.
PROOF. Let $A \subseteq F$ be a closed subset of $F$ and let $x \in F \setminus A$. Since $A$ is a closed subset of $F$ there exists one $\epsilon \in F$ with $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subseteq F \setminus A$. Then the function $f : F \to F$ defined by $f : y \mapsto |y - x|$ is a member of $\mathcal{C}(X, F)$ and that $Z_{X,F}(f) = \{x\} \subseteq [f \leq \epsilon]$, while $A \subseteq [f \geq \epsilon]$, showing that $f$ separates $x$ and $A$. \hfill \Box$

The next theorem provides some other equivalent formulations of completely $F$ regular topological spaces.

**THEOREM 2.2.** The following statements are equivalent for a topological space $X$.

- $X$ is a completely $F$ regular.
- $\mathcal{B}(X, F)$ is a base for the closed subsets of $X$.
- $X$ has the weak topology induced by the set $\mathcal{C}(X, F)$.
- $X$ has the weak topology induced by a subset of $\mathcal{C}(X, F)$.

**PROOF.** Let $X$ be a topological space. We shall show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Follows from the definition of complete $F$ regularity as follows:

for any closed subset $Z$ of $X$ and any $x \not\in Z$, there exists by complete $F$ regularity of $X$ one $f \in \mathcal{C}(X, F)$ so
that $0 \leq f \leq 1$, $f(x) = 1$ and $Z \subseteq Z_{X,F}(f)$, proving
thereby that $\mathcal{J}(X,F)$ is a base for the closed subsets
of $X$.

(ii) $\Rightarrow$ (iii): Let $\mathcal{M}$ be the weak topology induced by $\mathcal{C}(X,F)$ on $X$ and $\mathcal{T}$ be the original topology on $X$. Since $\mathcal{M}$ is the smallest topology on $X$ making each of the functions in $\mathcal{C}(X,F)$ continuous, it follows that $\mathcal{M} \subseteq \mathcal{T}$. Thus, under (ii), it is enough to prove that the members of $\mathcal{J}(X,F)$ are closed subsets in the topology $\mathcal{M}$.

Since $F$ is equipped with its order topology the sets $\uparrow s$ and $\downarrow t$ for $s$ and $t$ from $F$ make a sub-base for the closed subsets of $F$, so that their pre-images under the maps in $\mathcal{C}(X,F)$, which are precisely the members of $\mathcal{J}(X,F)$ by Theorem 1.4(iii), make a sub-base for the closed subsets for the topology $\mathcal{M}$. In particular, therefore, these sets are closed in $\mathcal{M}$.

(iii) $\Rightarrow$ (iv): Immediate.

(iv) $\Rightarrow$ (i): Let $A$ be a closed subset of $X$ and $x \in X \setminus A$.

Since the topology on $X$ is the weak topology induced by a sub-family of $\mathcal{C}(X,F)$, it follows that the pre-images of the rays $\downarrow t$ and $\uparrow s$, for $s$ and $t$ in $F$, make a sub-base for the closed subsets of $X$. Hence $\mathcal{J}(X,F)$ makes a base for the closed subsets of $X$. 

as \(\mathcal{Z}(X, F)\) is closed under finite intersections by Theorem 1.4(v).

Thus, there exists one \(f \in \mathcal{C}(X, F)\) such that \(x \not\in \mathcal{Z}_{X,F}(f) \supseteq A\).

Consequently, taking \(g = \frac{1}{r}\), where \(r = f(x)\), it follows that \(g\) separates \(x\) and \(A\). Hence \(X\) is completely \(F\) regular.

\[\square\]

**Theorem 2.3.** For \(F \neq \mathbb{R}\) the completely \(F\) regular topological spaces are precisely the zero dimensional topological spaces.

In particular, all the completely \(F\) regular topological spaces are Tychonoff.

**Proof.** Let \(F \neq \mathbb{R}\), and assume that the topological space \(X\) is completely \(F\) regular. Then by Theorem 2.2, \(X\) has the weak topology induced by \(\mathcal{C}(X, F)\). Since \(F\) is zero dimensional, by Theorem 1.2.6, the clopen subsets of \(F\) make a base for the topology on \(F\) so that their pre-images under the maps in \(\mathcal{C}(X, F)\), which are themselves then clopen, make a sub-base for the topology on \(X\). Since finite unions and finite intersections of clopen sets are still clopen, it follows that the clopen subsets of \(X\) make a base for its topology. Hence \(X\) is zero dimensional.

Conversely, suppose that the topological space \(X\) is zero dimensional. Then \(X\) has a base of clopen sets. Since every clopen set is also
a zero set in $X$ with respect to $F$ it follows that $\mathcal{Z}(X, F)$ make a base for the closed subsets for $X$ so that from Theorem 2.2 it follows that $X$ is completely $F$ regular.

The second part is immediate from the fact that every zero dimensional (Hausdorff) topological space is Tychonoff.

Theorem 2.3 may be a little disturbing. In the first place it is seemingly new that the hitherto known topological property of zero dimensionality amidst topological spaces (i.e., Hausdorff topological spaces) is realized as a separation axiom; but this is not quite new in view of Weir [49], page 17 - 18, Example 3.4. In the second place the feeling of distinction between the various ordered fields and their classes of regularity that is ushered in by Definition 2.1 is simply erased the classes of complete $F$ regularity for ordered fields $F \neq \mathbb{R}$ are all same as the class of zero dimensional topological spaces. In other words the property of complete $F$ regularity depends on the dichotomy of $F \neq \mathbb{R}$ or $F = \mathbb{R}$; in the former it is precisely the class of zero dimensional topological spaces and in the latter it is precisely the class of Tychonoff spaces. Another fact needs to be highlighted here the family of completely $\mathbb{R}$ regular spaces is larger than the class of completely $F$ regular spaces for $F \neq \mathbb{R}$.
An immediate consequence of this is that complete $F$-regularity is a productive and hereditary topological property but not necessarily a continuous invariant.

**Theorem 2.4.** Given any topological space $X$ and any ordered field $F$, the following statements are equivalent.

i. $X$ is completely $F$-regular.

ii. $\mathcal{B}(X, F)$ separates points and closed subsets of $X$.

iii. $\mathcal{C}^*(X, F)$ separates points and closed subsets of $X$.

**Proof.** (1) $\Rightarrow$ (2): Follows from Definition 1.4.

(2) $\Rightarrow$ (3): For the case of $F = \mathbb{R}$, this is indeed true as then $\mathcal{C}^*(X, F) = \mathcal{B}(X, F)$. We consider the case when $F \neq \mathbb{R}$.

From the hypothesis it evidently follows that $\mathcal{C}(X, F)$ separates points and closed subsets of $X$, so that $X$ is completely $F$-regular. Then from Theorem 2.3 it follows that $X$ is a zero dimensional topological space. Consequently, if $A$ be a closed subset of $X$ and $x \not\in A$, then from zero dimensionality there exists one clopen set $L$ such that $x \in L \subseteq X \setminus A$. But as $L$ is clopen, the indicator function $\chi_L$ is a member of $\mathcal{C}^*(X, F')$ and separates the set $A$ from $x$.

(3) $\Rightarrow$ (1): Trivial.
Thus it follows that although we cannot assert, in general, whether \( \mathcal{C}(X, F) \) and \( \mathcal{C}^*(X, F) \) determines the same family of zero sets in \( X \) with respect to \( F \), as far as the criterion for separating points and closed sets is concerned they do have identical behavior.

With all these we have done the ground work necessary for the main theorem of this section.

**Theorem 2.5. Analogue of M. H. Stone's Theorem:** Given any topological space \( X \), there exists a completely \( F \)-regular topological space \( Y \) such that \( \mathcal{C}(X, F) \simeq \mathcal{C}(Y, F) \).

Furthermore, this isomorphism carry \( \mathcal{B}(X, F) \) onto \( \mathcal{B}(Y, F) \) and \( \mathcal{C}^*(X, F) \) onto \( \mathcal{C}^*(Y, F) \), respectively.

**Proof.** For \( x, y \in X \) define

\[
x \simeq y \iff (\forall f \in \mathcal{C}(X, F))(f(x) = f(y)).
\]

It is clear that \( \simeq \) is an equivalence relation on \( X \). Let us define some symbols which shall be used in the proof:

i. For \( x \in X \) let \( x/ \simeq = \{ y \in X : x \simeq y \} \).
ii. For any \( A \subseteq X \) let \( A/\sim = \{ x/\sim : x \in A \} \).

iii. \( \nu : X \to X/\sim \) defined by \( \nu : x \mapsto x/\sim \) is the usual quotient map.

iv. For any \( f \in \mathcal{C}(X, F) \) define the map \( f/\sim : X/\sim \to F \) by \( f/\sim : x/\sim \mapsto f(x) \); the definition of \( \sim \) for points of \( X \) ensures that \( f/\sim \) is well-defined.

v. Let \( s = \{ f/\sim : f \in \mathcal{C}(X, F) \} \).

\[
\begin{array}{ccc}
X & \xrightarrow{\nu} & X/\sim \\
\downarrow{f} & & \downarrow{f/\sim} \\
\uparrow{\mathcal{F}} & & \\
\end{array}
\]

Clearly then the commutativity of the triangle follows.

Now let the set \( X/\sim \) be equipped with the weak topology induced by \( s \). Then as for any \( f/\sim \in s \), with \( f \in \mathcal{C}(X, F) \) we have \( f = (f/\sim)_o\nu \), a continuous function, it follows that \( \nu \) is continuous. Furthermore, for any \( g \in \mathcal{C}(X/\sim, F') \) we have \( g_o\nu \in \mathcal{C}(X, F) \) and that for any \( x \in X \),

\[
g(x/\sim) = g(\nu(x))
\]

\[
= (g_o\nu)(x)
\]

\[
= ((g_o\nu)/\sim)_o\nu(x)
\]

\[
= ((g_o\nu)/\sim)(x/\sim),
\]
thereby showing that \( g = (g \circ \nu) / \simeq \) so that \( \mathcal{C}(X / \simeq, F) \subseteq \mathcal{S} \). Consequently, from this and the fact that \( X / \simeq \) has been endowed with the weak topology induced by \( \mathcal{S} \) we obtain \( \mathcal{C}(X / \simeq, F) = \mathcal{S} \).

We now assert that \( X / \simeq \) is a Hausdorff topological space. Given this, it shall then follow from above and Theorem 2.2 that \( X / \simeq \) is completely \( F \) regular.

To prove the Hausdorffness, let \( x / \simeq \) and \( y / \simeq \) be distinct elements in \( X / \simeq \). Then, there exists one \( f \in \mathcal{C}(X, F) \) such that \( f(x) \neq f(y) \), so that there exist \( U \in \mathcal{R}_{f(x)}^F \) and \( V \in \mathcal{R}_{f(y)}^F \) such that \( U \cap V = \emptyset \). Hence \( (f / \simeq)^{-1}(U) \cap (f / \simeq)^{-1}(V) = (f / \simeq)^{-1}(U \cap V) = \emptyset \), so that \( (f / \simeq)^{-1}(U) \in \mathcal{R}_{X / \simeq}^X \) and \( (f / \simeq)^{-1}(V) \in \mathcal{R}_{Y / \simeq}^Y \) are disjoint neighborhoods of the two points respectively. Thus \( X / \simeq \) is Hausdorff.

We shall now show that the rings \( \mathcal{C}(X, F) \) and \( \mathcal{C}(X / \simeq, F) \) are isomorphic. Consider the map \( \sigma : \mathcal{C}(X, F) \rightarrow \mathcal{C}(X / \simeq, F) \) given by \( \sigma : f \mapsto f / \simeq \). The factorization \( f = (f / \simeq) \circ \nu \) and the equality \( \mathcal{S} = \mathcal{C}(X / \simeq, F) \) suggests that \( \sigma \) is a bijection. Further, for \( f, g \in \mathcal{C}(X, F) \)
and $x \in X$ we have:

\[
(f/ \simeq + g/ \simeq)(x/ \simeq) = (f/ \simeq)(x/ \simeq) + (g/ \simeq)(x/ \simeq)
\]

\[
= f(x) + g(x)
\]

\[
= (f + g)(x)
\]

\[
= ((f + g)/ \simeq)(x/ \simeq),
\]

\[
(f/ \simeq g/ \simeq)(x/ \simeq) = (f/ \simeq)(x/ \simeq)(g/ \simeq)(x/ \simeq)
\]

\[
= f(x)g(x)
\]

\[
= (fg)(x)
\]

\[
= ((fg)/ \simeq)(x/ \simeq),
\]

and

\[
(f/ \simeq)(x/ \simeq) \leq (g/ \simeq)(x/ \simeq), \text{ if and only if, } f(x) \leq g(x).
\]

Thus $\sigma$ defines an order preserving isomorphism on $\mathcal{C}(X, F)$ onto $\mathcal{C}(X/ \simeq, F)$.

Finally, as for any $f \in \mathcal{C}(X, F)$ and for any $A \subseteq X$ we have $f(A) = (f/ \simeq)(A/ \simeq)$, we conclude easily from all that has been done that $\sigma$
takes $\mathcal{B}(X, F)$ isomorphically onto the ring $\mathcal{B}(X/\sim, F)$ and $\mathcal{C}^*(X, F)$ isomorphically onto the ring $\mathcal{C}^*(X/\sim, F)$.

The above theorem now gives us the power to restrict our attention to the class of completely $F$ regular topological spaces. Thus we shall concentrate on Tychonoff spaces, and in particular, if the range field is other than $\mathbb{R}$ on zero dimensional spaces. In this connection it shall not be quite unjust to recall the justification provided by Hewitt [25] for the choice of $\mathbb{R}$ for his range field in order to study $X$ through $\mathcal{C}(X, \mathbb{R})$:

It is natural to enquire why the rings $\mathcal{C}(X, \mathbb{R})$ and $\mathcal{C}^*(X, \mathbb{R})$ are chosen as instruments wherewith to study topological spaces $X$, to the exclusion of the rings $\mathcal{C}(X, \mathbb{C})$, $\mathcal{C}^*(X, \mathbb{C})$ and more generally $\mathcal{C}(X, T)$ where $T$ may be any topological ring. First, the limitation to topological fields offer considerable advantages in rings $\mathcal{C}(X, F)$ by virtue of simple characterizations of elements with inverse in such rings. Second, the topological field used should be connected and locally bicompact, in order to admit reasonable number of
continuous images of connected and bicom pact spaces respectively. By a celebrated theorem of Pontrajin these principles already limit us to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ the field of quaternions

Thus it is clear by now that Hewitt wished to study enough number of connected and locally compact spaces; however, if one generalizes his range field, he has to forego all such hopes — the spaces that he shall study are zero dimensional. Thus such a generalization, hopefully, may shed some light on the relevance of the structure of $\mathbb{R}$ in the study of topological spaces via function rings.

3. Prologue: Mrowka’s Classes of Complete Regularity

In this section we shall introduce the notion of $E$-complete regularity as conceived by Engelking and Mrowka [34], Weir [49], Chapter I, page 15. The main results are mentioned without any proof and it is shown that the class of $F$-complete regularity, when $F$ is an ordered field, is exactly the same as the class of completely $F$ regular topological spaces. However, in view of Theorem 2.3 nothing new is achieved
over here, but an alternate description of completely regular topological spaces in terms of an already known notion of complete regularity is provided.

**Definition 3.1.** Given a topological space $E$, a topological space $X$ is said to be $E$-completely regular, if and only if, $X$ is homeomorphic to a subspace of the topological power $E^\mathbb{N}$, for some cardinal number $\mathbb{N}$.

The class of $E$-completely regular spaces is denoted by the symbol $b(E)$ and a class $b$ of topological spaces is said to be a class of complete regularity, if and only if, there exists a topological space $E$ such that $b = b(E)$.

It is clear from the definition that $b([0,1]) = b(\mathbb{R})$ is precisely the class of all Tychonoff spaces.

The following result is an immediate consequence of the definition Definition 3.1.

**Theorem 3.1.** Let $X$ and $E$ be topological spaces and then the following statements are true.

1. The space $E$ is $E$-completely regular.
ii. $E$-complete regularity is hereditary, i.e., if $X \in b(E)$, and $Y \cong Z$ where $Z \subseteq X$, then $Y \in b(E)$.

iii. $E$-complete regularity is productive, i.e., the topological product of a set of $E$-completely regular topological spaces is $E$-completely regular.

iv. $b(X) \subseteq b(E) \iff X \in b(E)$.

v. For every cardinal number $\kappa$, $b(E) = b(E^\kappa)$.

The following theorem appears in Mrowka & Engelking [34] and is a characterization of $E$-completely regular topological spaces.

**THEOREM 3.2.** A topological space $X$ is $E$-completely regular, if and only if, the following two conditions are satisfied:

i. for every $p, q \in X$, $p \neq q$, there exists a continuous function $f : X \to E$ such that $f(p) \neq f(q)$;

ii. for every closed subset $A \subseteq X$ and $p \in X \setminus A$, there exists one $n \in \mathbb{N}$ and a continuous function $f : X \to E^n$ such that $f(p) \notin \text{cl}_{E^n}(f(A))$.

Now let $F$ be an ordered field equipped with its order topology. Then a topological space $X$ is completely $F$ regular, if and only if,
\( \mathcal{C}(X, F) \) separates points and closed subsets of \( X \), whence from Theorem 3.2, this implies that \( X \in b(F) \), in the sense of Mrowka.

Conversely, if \( X \) be \( F \)-completely regular, in the sense of Mrowka, then from definition it is homeomorphic to a subspace of some topological power \( F^\mathbb{N} \) of \( F \), so that as such spaces are themselves completely \( F \) regular it follows from the hereditary property of complete \( F \) regularity that \( X \) is completely \( F \) regular. Thus we have:

**Theorem 3.3.** For an ordered field \( F \), \( b(F) \) is precisely the class of completely \( F \) regular topological spaces.