CHAPTER V

Spaces like $P$-spaces

In this chapter we shall investigate a class of Tychonoff spaces, called $P_F$-spaces for each ordered field $F$, which are very much akin to the familiar $P$-spaces, which in our terminology are the $P_{\mathbb{R}}$-spaces. In the first section we have developed some ancillary machinery that would be required for the main results of this chapter. In the second section the $P_F$-spaces are defined and their relationship with the $P$-spaces are investigated. In the process it has been shown that if $F$ be an ordered field with countable cofinal subset then every $P$-space is a $P_F$-space, Theorem 2.3 and then in Theorem 2.4 it is shown that if further $F$ be Cauchy complete then the class of $P$-spaces and the class of $P_F$-spaces are identical. Finally in the third section we have used these two theorems along with the construction of a countable cofinal Cauchy complete ordered field extension $G$ of a given ordered field $F$ to provide a number of constructions of the familiar $P$-spaces.
Throughout this section $F$ denotes an ordered field, equipped with its interval topology and that the topological spaces are all completely $F$ regular.

**Definition 1.1.** For any $p \in X$, define

$$O_p = \{ f \in \mathcal{C}(X, F) : Z_{X,F}(f) \subseteq \mathcal{M}_p \}$$

The calculus of zero sets in $X$ with respect to $F$ as developed in Theorem II.1.4, show that for each $p \in X$, $O_p$ is indeed an ideal of $\mathcal{C}(X, F)$ contained in the fixed maximal ideal $M_p$; furthermore, its description being completely provided in terms of the zero sets in $X$ with respect to $F$ forces $O_p$ to be a $Z_{F}$-ideal. These properties are collected together in the next theorem.

**Theorem 1.1.** Let $p \in X$.

i. $O_p$ is a $Z_{F}$-ideal of $\mathcal{C}(X, F)$ with $O_p \subseteq M_p$.

ii. $\bigcap Z_{X,F}(O_p) = \{p\}$; in particular, $O_p$ is fixed.

iii. $M_p$ is the unique maximal ideal in $\mathcal{C}(X, F)$ containing $O_p$. 
iv. Any ideal of \( \mathcal{C}(X, F) \) containing \( O_p \) can be extended to a unique maximal ideal.

v. If \( O_p \neq M_p \), then \( O_p \) is contained in a non-maximal prime ideal in \( \mathcal{C}(X, F) \).

vi. If \( P \) be a prime ideal of \( \mathcal{C}(X, F) \) and \( P \subseteq M_p \), then \( P \supseteq O_p \).

PROOF. i. Immediate.

ii. The second part follows from the first. Surely, \( p \in \bigcap Z_{X,F}(O_p) \); since \( X \) is completely \( F \) regular, every neighborhood of a point contains a zero-set neighborhood of the same point, and hence it follows that for any other \( q(\neq p) \in X \), \( q \not\in \bigcap Z_{X,F}(O_p) \), and thus the equality follows.

iii. Let \( O_p \subseteq M \subseteq \mathcal{M}_{X,F} \). If \( M \) be free, then there exists one \( f \in M \) such that \( p \not\in Z_{X,F}(f) \). Now, on using the complete \( F \) regularity of \( X \), it is possible to choose one \( g \in O_p \) such that \( Z_{X,F}(g) \subseteq X \setminus Z_{X,F}(f) \Rightarrow Z_{X,F}(g) \cap Z_{X,F}(f) = \emptyset \), contradicting the inclusion of \( O_p \) in \( M \). Hence, \( M \) must be fixed.

   Consequently, from Theorem III.1.3(i), if \( O_p \subseteq M = M_q \), for some \( q \in X \), and hence:

   \[ \{q\} = \bigcap Z_{X,F}(M_q) \subseteq \bigcap Z_{X,F}(O_p) = \{p\} \Rightarrow p = q. \]

iv. Follows from (iii).
v. From (i) as $O_p$ is a $\mathcal{F}$-ideal, and the result then follows from the fact that $O_p$ is then an intersection of prime ideals, Theorem III.1.2(i).

vi. Let $f \in O_p$, i.e., $\mathcal{Z}_{X,F}(f) \in \mathfrak{M}_p^X$.

Then there exists one $g \in \mathcal{C}(X,F)$ such that: $0 \leq g \leq 1$, $g(p) = 1$ and $X \setminus \text{int}_X \mathcal{Z}_{X,F}(f) \subseteq \mathcal{Z}_{X,F}(g)$. Consequently, $fg = 0 \in P$, and as $g \notin M_p \Rightarrow g \notin P$, it follows that $f \in P$.

\[ \Box \]

**Theorem 1.2.** Any pair of disjoint closed subsets of a topological space, of which at least one is compact, are completely $F$ separated.

**Proof.** Let $A$ and $B$ be disjoint closed subsets of the topological space $X$, and for the sake of definiteness, let $B$ be compact.

Since $X$ is completely $F$ regular, for each $x \in B$, there exists a pair of zero sets in $X$ with respect to $F$, $Z_x$ and $Z'_x$ such that $Z_x \in \mathfrak{M}_x^X$, $A \subseteq Z'_x$ and $Z_x \cap Z'_x = \emptyset$.

Surely, $\{\text{int}_X Z_x : x \in B\}$ is an open cover of the compact set $B$, which then admits of a finite subcover. Thus there exists finitely many points $x_1, x_2, \ldots, x_n \in B$ such that $B \subseteq \bigcup_{i=1}^n \text{int}_X Z_{x_i} \subseteq \bigcup_{i=1}^n Z_{x_i}$, which is a zero set in $X$ with respect to $F$. 
Then $\bigcap_{i=1}^{n} Z_{x_i}'$ becomes a zero set in $X$ with respect to $F$, containing $A$ and clearly $\left(\bigcap_{i=1}^{n} Z_{x_i}'\right) \cap \left(\bigcup_{i=1}^{n} Z_{x_i}\right) = \emptyset$.

Therefore $A$ and $B$ are contained in disjoint zero sets in $X$ with respect to $F$ and then from Theorem II.1.9 the sets $A$ and $B$ are completely $F$ separated. \qed

The next two theorems depend on a property of Cauchy completeness of an ordered field, which we define first.

**Definition 1.2.** i. A sequence $\{s_n\}_{n \geq 1}$ in $F$ is said to be a **Cauchy sequence**, if and only if, the following condition is satisfied:

$$(\forall \epsilon \in F_1)(\exists n_0 \in \mathbb{N})(\forall p, q \in \mathbb{N})(p, q > n_0 \Rightarrow |s_p - s_q| < \epsilon).$$

ii. $F$ is said to be **Cauchy complete**, if and only if, every Cauchy sequence in $F$ is convergent with respect to the topology induced by the order on $F$.

\Diamond

A natural question that can arise here is the existence of non-Archimedean ordered fields that are Cauchy complete. In the final
section of this chapter we do provide a generic technique to produce such fields.

**Theorem 1.3.** If $F$ be a Cauchy complete ordered field with $\text{cf}(F) = \aleph_0$ and $X$ a completely $F$ regular topological space then $\mathcal{Z}(X, F)$ is closed under countable intersections.

**Proof.** There are two major cases to be dealt with, namely when $F$ is Archimedean ordered and when $F$ is not Archimedean ordered. Since any Cauchy complete Archimedean ordered field is order isomorphic to $\mathbb{R}$, the field of real numbers, the first case follows from Gillman & Jerison [22], Theorem 1.14(a), page 16. Indeed in this case, if $\{f_n\}_{n>1}$ be any sequence of members from $\mathcal{C}(X, \mathbb{R})$ then the function $g : X \to \mathbb{R}$ defined by $g(x) = \sum_{n=1}^{\infty} (|f| \wedge \frac{1}{2^n}) (x)$ is also a member in $\mathcal{C}(X, \mathbb{R})$, as the series $\sum_{n=1}^{\infty} (f \wedge \frac{1}{2^n}) (x)$ is uniformly convergent on $X$ due to Weierstrass's $M$-test, and that $\mathcal{Z}_{X, F}(g) = \bigcap_{n=1}^{\infty} \mathcal{Z}_{X, \mathbb{R}}(f_n)$.

So we assume that the field $F$ is not Archimedean ordered. Then there exists an element $N$ in $F$ such that $N$ is infinitely large, i.e., for all $n \in \mathbb{N}$, $n < N$. Furthermore, since $\text{cf}(F)$ is $\aleph_0$, there exists a countable cofinal subset $\{t_1, t_2, \ldots, t_n, \ldots\}$ of $F_+$ which, aided with the Principle of Mathematical Induction, yields a strict monotonically decreasing sequence $\{s_n\}_{n>1}$ of $F_+$ converging to the limit 0.
1. PREPARATORY RESULTS

Let now $Z_n \in \mathcal{Z}(X, F), n \in \mathbb{N}$. Since by Theorem II.1.8 both $\mathcal{B}(X, F)$ and $\mathcal{C}(X, F)$ determine the same zero sets in $X$ with respect to $F$, for each $n \in \mathbb{N}$ there exists $g_n \in \mathcal{B}(X, F)$ such that $0 \leq g_n \leq 1$, and $Z_n = Z_{X,F}(g_n)$. Let $f_n = s_ng_n, n \in \mathbb{N}$.

Then for any two natural numbers $m, n(> m)$, we have:

$$|s_{m+1} + s_{m+2} + \cdots + s_n| \leq (n-m)s_{m+1} < ns_{m+1}$$

$$\rightarrow 0, \text{ as } m, n \rightarrow \infty,$$

whence $\sum_{n=1}^{\infty} s_n$ is an infinite series whose partial sums make a Cauchy sequence in $F$. As $F$ is Cauchy complete, this infinite series has a sum, say $s$ in $F$, i.e., $\sum_{n=1}^{\infty} s_n = s$.

Again for each $n \in \mathbb{N}$, $f_n = |f_n| = |s_ng_n| \leq |s_n| = s_n$, so that at each point $x \in X$, $|f_n(x)| \leq s_n$, which implies that for any two $m, n(> m) \in \mathbb{N}$, we have:

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)|$$

$$\leq |s_{m+1} + s_{m+2} + \cdots + s_n| \leq (n-m)s_{m+1} < ns_{m+1}$$

$$\rightarrow 0, \text{ as } m, n \rightarrow \infty.$$
Thus, for each \( x \in X \) the infinite series \( \sum_{n=1}^{\infty} f_n(x) \) in \( F \) has its partial sums to be a Cauchy sequence in \( F \), so that from the Cauchy completeness has a limit, say \( f(x) \), i.e., \( \sum_{n=1}^{\infty} f_n(x) = f(x), x \in X \). This defines a function \( f : X \to F \). Since each \( f_n \) is non-negative it follows that \( f \) is non-negative.

We assert that \( f \in C(X, F) \). \( \dagger \dagger \)

Given this assertion, it then follows that :

\[
Z_{X,F}(f) = \bigcap_{n=1}^{\infty} Z_{X,F}(f_n) = \bigcap_{n=1}^{\infty} Z_{X,F}(g_n),
\]

completing the proof.

PROOF. (Proof of the Assertion \( \dagger \dagger \) ) : It is shown that \( f \) is the uniform limit of the sequence of partial sums of the series \( \sum_{n=1}^{\infty} f_n(x) \).

Since an uniform limit of a sequence of continuous functions is a continuous function it shall follow that \( f \) is a continuous function, proving the assertion.

For each \( x \in X, n \in \mathbb{N} \), we have :

\[
|f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\
= \sum_{k=n+1}^{\infty} f_k(x) \leq \sum_{k=n+1}^{\infty} s_k.
\]
Since the series $\sum_{n=1}^{\infty} s_n$ is already convergent in $F$, the following statement is true:

$$(\forall \epsilon \in F_+)(\forall x \in X)(\exists m_0 \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq m_0 \Rightarrow |f_1(x) + f_2(x) + \cdots + f_n(x) - f(x)| < \epsilon),$$

with the choice of $m_0$ independent of the choice of the point $x \in X$, proving the uniform convergence of the series $\sum_{n=1}^{\infty} f_n$ to the function $f$ over the whole of $X$. □

**Theorem 1.4.** If $F$ be a Cauchy complete ordered field with cofinality character $\aleph_0$, $X$ a completely $F$ regular topological space, $V$ a $G_\delta$-subset of $X$ and $S \subseteq V$ a compact subset of $X$ then there exists one $Z \in \mathcal{Z}(X, F)$ such that $S \subseteq Z \subseteq V$.

In particular, every compact $G_\delta$-subset of $X$ is a zero set in $X$ with respect to $F$.

**Proof.** The second statement follows from the first on taking $S = V$. Thus it remains to prove the first statement only.
Since $V$ is a $G_\delta$-set, there exist open sets $U_n, n \in \mathbb{N}$, such that $V = \bigcap_{n \geq 1} U_n$.

Since $S \subseteq V$ is a compact subset, by Theorem 1.2, $S$ and each of the sets $X \setminus U_n, n \in \mathbb{N}$, are completely $F$ separated and thus there exists for each $n \in \mathbb{N}$ zero sets in $X$ with respect to $F$, $Z_n$, such that $S \subseteq Z_n \subseteq U_n$.

Then by Theorem 1.3, $Z = \bigcap_{n \geq 1} Z_n$ is a zero set in $X$ with respect to $F$ and that $S \subseteq Z \subseteq \bigcap_{n \geq 1} U_n = V$, completing the proof of the first statement. \hfill \Box

2. $P_F$-spaces

We start with the definition.

**Definition 2.1.** Given an ordered field $F$, a completely $F$ regular topological space $X$ is said to be a $P_F$-space, if and only if, every prime ideal of $\mathcal{C}(X, F)$ is maximal.

It is clear that the $P_\mathbb{R}$-spaces are the familiar $P$-spaces introduced by L. Gillman and M. Henriksen, [19], and also in Gillman & Jerison [22], Problem 4J. The algebraic invariance of prime and the maximal ideals in a ring do provide us with the following theorem.
THEOREM 2.1. i. If $F$ be an ordered field, $X$ and $Y$ be completely $F$ regular topological spaces, $X$ be a $P_F$-space and $\mathcal{C}(X, F)$ is isomorphic (as lattice ordered commutative rings with unity) to $\mathcal{C}(Y, F)$ then $Y$ is also a $P_F$-space.

ii. If $F$ and $G$ are isomorphic ordered fields and $X$ be a $P_F$-space then $X$ is a $P_G$-space.

PROOF. Immediate. 

The next theorem provides us with a number of equivalent descriptions of a $P_F$-space. This may be compared with the classical case of $F = \mathbb{R}$, as in Gillman & Jerison [22], Problem 4J.

THEOREM 2.2. Given an ordered field $F$ and a completely $F$ regular topological space $X$ the following are all equivalent.

i. $X$ is a $P_F$-space.

ii. $(\forall p \in X)(M_p = O_p)$.

iii. Every member of $\mathcal{J}(X, F)$ is open.

iv. Every ideal of $\mathcal{C}(X, F)$ is a $J_F$-ideal.

v. $(\forall f, g \in \mathcal{C}(X, F))((f, g) = (f^2 + g^2))$, where $(f, g)$ and $(f^2 + g^2)$ are the ideals generated by the pair $f$ and $g$ and the function $f^2 + g^2$ respectively.
vi. \( C(X, F) \) is a regular ring, i.e.,

\[(\forall f \in C(X, F))(\exists g \in C(X, F))(f = f^2 g).\]

vii. Every principal ideal of \( C(X, F) \) is generated by an idempotent.

viii. Every proper ideal of \( C(X, F) \) is an intersection of prime ideals.

ix. Every proper ideal of \( C(X, F) \) is an intersection of maximal ideals.

x. Given \( S \), a cozero set in \( X \) with respect to \( F \), there exists a \( g \in C(X, F) \) such that \( f = g \upharpoonright S \).

PROOF. The proof shall be provided in the following steps.

\[ \alpha : (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i). \]

\[ \beta : (vi) \Rightarrow (ix) \Rightarrow (viii) \Rightarrow (vi). \]

\[ \gamma : (iii) \Rightarrow (x) \Rightarrow (vi). \]

\[ \delta : (vi) \Leftrightarrow (vii). \]

\[ \alpha : (i) \Rightarrow (ii) : \text{From (i), there does not exist any non maximal prime ideal in } C(X, F), \text{ so that from Theorem 1.1(v) it follows that for each } p \in X, O_p = M_p, \text{ proving the implication.} \]

\[ (ii) \Rightarrow (iii) : \text{Choose and fix any } Z \in \mathfrak{Z}(X, F). \text{ Then from definition, there exists one } f \in C(X, F) \text{ such that } Z = Z_{X,F}(f). \text{ Consequently, for any } p \in Z = Z_{X,F}(f), \text{ we have from (ii)} \]
that \( f \in M_p = O_p \Rightarrow Z \in \mathcal{N}_p \), so that \( Z \) is a neighborhood of \( p \). Hence \( Z \) is open in \( X \).

(iii) \( \Rightarrow \) (iv): Let \( I \) be a proper ideal of \( \mathcal{C}(X,F) \) and let \( f \in \mathcal{C}(X,F) \) such that \( Z_{X,F}(f) \in Z_{X,F}(I) \).

Then there exists one \( g \in I \) so that \( Z_{X,F}(f) = Z_{X,F}(g) \).

Since from (iii) every zero set in \( X \) with respect to \( F \) is open it follows that \( Z_{X,F}(f) \) is a zero set neighborhood of \( Z_{X,F}(g) \), so that from Theorem II.1.7(i) we have \( g|f \). This entails that \( f \in (g) = \) the principal ideal in \( \mathcal{C}(X,F) \) generated by \( g \).

Since \( g \in I \), then we have \( f \in (g) \subseteq I \Rightarrow f \in I \), showing that \( I \) is a \( \mathfrak{p}_F \)-ideal.

(iv) \( \Rightarrow \) (v): Choose and fix \( f, g \in \mathcal{C}(X,F) \).

Surely, \( f^2 + g^2 \in (f,g) \Rightarrow (f^2 + g^2) \subseteq (f,g) \).

But as \( Z_{X,F}(f), Z_{X,F}(g) \supseteq Z_{X,F}(f^2 + g^2) \), and by (iv) \( (f^2 + g^2) \) is a \( \mathfrak{p}_F \)-ideal, we have then that \( f, g \in (f^2 + g^2) \Rightarrow (f,g) \subseteq (f^2 + g^2) \), proving the equality \( (f,g) = (f^2 + g^2) \).

(v) \( \Rightarrow \) (vi): From (v), for each \( f \in \mathcal{C}(X,F) \), \( (f) = (f^2) \), and the rest follows.

(vi) \( \Rightarrow \) (i): Choose and fix any prime ideal \( P \) of \( \mathcal{C}(X,F) \) and let \( f \in \mathcal{C}(X,F) \setminus P \).
By (vi) there exists one \( g \in \mathcal{C}(X, F) \), such that \( f = gf^2 \).

Therefore, \( f - gf^2 = 0 \Leftrightarrow f(1 - fg) = 0 \Rightarrow 1 - fg \in P \), as \( f \notin P \) and \( P \) is a prime ideal. Consequently, \((P, f) = \) the ideal generated by the set \( P \cup \{f\} = \mathcal{C}(X, F) \) as \( 1 = (1 - fg) + fg \). Hence \( P \) is a maximal ideal.

\( \beta : \) (vi) \( \Rightarrow \) (ix): From the part in \((\alpha)\) it follows that (vi) is equivalent to (i), and that if \( X \) be a \( P_F \)-space then every ideal of \( X \) is a \( \mathfrak{z}_F \)-ideal, (from (iv)), so that the from Theorem III.1.2(i) it follows that every proper ideal of \( \mathcal{C}(X, F) \) is an intersection of prime (= maximal, from assumption) ideals of \( \mathcal{C}(X, F) \).

(ix) \( \Rightarrow \) (viii): Immediate, as in any commutative ring with unity every maximal ideal is prime.

(viii) \( \Rightarrow \) (vi): The conclusion of (vi) is always true for the units of \( \mathcal{C}(X, F) \), so that it remains to prove its truth for the non-units under the assumption of (viii). Since every proper ideal is an intersection of prime ideals, so is \((f^2)\), for any non-unit \( f \in \mathcal{C}(X, F) \). Then \( f \in (f^2) \), as \( f \) and \( f^2 \) always belong to the same prime ideal. This entails that \((f) = (f^2)\), proving thereby that \( \mathcal{C}(X, F) \) is a regular ring.
γ:  (iii) ⇒ (x): Let $S$ be a cozero set in $X$ with respect to $F$ and $f : S \to F$ be continuous. By (iii) then, $S$ is clopen so that the function $g : X \to F$, where $g(x) = f(x)$, if $x \in S$, and $= 0$, if $x \notin S$, is a member of $\mathcal{C}(X, F)$ and extends $f$, i.e., $f = g |_S$.

(x) ⇒ (vi): Choose and fix any $f \in \mathcal{C}(X, F)$. We shall show that $(f) \subseteq (f^2)$. As the other inclusion is always true, this shall prove the equality, and therefore the regularity of the ring $\mathcal{C}(X, F)$.

Now by (x), the continuous function $\frac{1}{f} : X \setminus Z_{X,F}(f) \to F$ defined by $x \mapsto \frac{1}{f(x)}$ has an extension to a continuous $g \in \mathcal{C}(X, F)$. Consequently, $f = gf^2$, proving the necessary inclusion.

δ:  (vi) ⇒ (vii): Let $(g)$ be the principal ideal generated by the element $g \in \mathcal{C}(X, F)$. From (vi) it then follows that there shall exist one $g_0 \in \mathcal{C}(X, F)$ such that $g = g_0 g^2$.

Then, $(gg_0)^2 = g^2 g_0^2 = (g^2 g_0)g_0 = gg_0$, shows that $gg_0$ is an idempotent.

Finally, since $g = g_0 g^2 = g(gg_0) \Rightarrow gg_0 | g$, and then $(g) \subseteq (gg_0) \subseteq (g)$, (the last one being always true) so that $(g) = (gg_0)$, i.e., $(g)$ is generated by an idempotent.
(vii) ⇒ (vi): Choose and fix any $f \in \mathcal{C}(X, F)$. By (vii) the principal ideal $(f)$ generated by $f$ is generated by an idempotent $g \in \mathcal{C}(X, F)$. Thus $g^2 = g$ and that $(f) = (g)$ ⇒ ($\exists s, t \in \mathcal{C}(X, F))(f = gs$ and $g = ft$).

Then, $f = gs = g^2 s = f^2 t^2 s = f^2 (st^2)$, proving the regularity of the ring $\mathcal{C}(X, F)$.

\[\square\]

It is to be noted that the proof of the equivalence of the conditions (vi) and (vii) in Theorem 2.2 is true in any commutative ring with identity and concerns nothing special about the function rings $\mathcal{C}(X, F)$.

The alternative characterizations of a $P_F$-space in Theorem 2.2 help us to provide examples of $P_F$-spaces for each ordered field $F$. If $F = \mathbb{R}$, then the familiar $P$-spaces provide necessary examples. If $F \neq \mathbb{R}$, then as $F$ is zero-dimensional and $\mathbb{R}$ is connected, $\mathcal{C}(\mathbb{R}, F)$ is a field, and thus a regular ring. From Theorem II.2.5 it follows that there exists one completely $F$ regular topological space $X$ such that $\mathcal{C}(X, F) \simeq \mathcal{C}(\mathbb{R}, F)$, so that in particular, $\mathcal{C}(X, F)$ is a regular ring. Hence, $X$ is a $P_F$-space.

However, it would have been nice if some answer could be found to the question: given any zero-dimensional space $X$ does there exist an ordered field $F$ such that $X$ is a $P_F$-space?
The $P_F$-spaces in the classical case with $F = \mathbb{R}$ had a nice topological description, viz., every $G_\delta$-set is open. However, such a description can well be attributed to the fact that in a completely regular topological space whenever a compact set is contained in a $G_\delta$-set, there always exists a zero set with respect to $\mathbb{R}$ in between. Indeed to construct the $P_\ast$ spaces we shall require to restrict ourselves to the class of ordered fields that are Cauchy complete with countable cofinality character, as Theorem 1.4 rightly suggests.

Before doing so, we shall be using this nice topological description of $P_\ast$-spaces to provide some more examples. Since for a $P_F$-space $X$ every zero set in $X$ with respect to $F$ is open, it follows that none of the ordered fields $F$, considered as a topological space, is a $P_F$-space, as every singleton in $F$ is a zero set in $F$ with respect to $F$, but none of these are isolated, so that they are not open. However, if we consider any $\eta_\alpha$-field $F$, for $\alpha \geq 1$, then $F$ is a $P_\ast$-space, as every $G_\delta$-set in $F$ is open there. This shows that there does exist examples of $P_\ast$-spaces that are not $P_F$-spaces, for some suitable $F$. However it would have been nice to provide an example of a $P_F$-space for some suitable $F$, which is not a $P_\ast$-space.
The following two theorems however indicate a class of ordered fields $F$ for which the class of $P$-spaces and that of $P_f$-spaces turn out to be identical.

**Theorem 2.3.** If $F$ be an ordered field with $\text{cf}(F) = \aleph_0$, then every $P$-space is a $P_f$-space.

**Proof.** Let $X$ be a $P$-space and let $Z \in \mathcal{Z}(X, F)$. Since $Z$ is a closed $G_\delta$-set, by Theorem II.1.5, and as $X$ is a $P$-space, it follows that $Z$ is open. Consequently, every member of $\mathcal{Z}(X, F)$ is open, so that from Theorem 2.2 we have $X$ to be a $P_f$-space. \qed

**Theorem 2.4.** If $F$ be a Cauchy complete ordered field with countable cofinality character, then every $P_f$-space is a $P$-space.

**Proof.** Let $X$ be a $P_f$-space and $V$ be a $G_\delta$-set in $X$. Then for any $x \in V$ there exists, by Theorem 1.4, one $Z_x \in \mathcal{Z}(X, F)$ such that $x \in Z_x \subseteq V$. But then by Theorem 2.2 as $Z_x$ is open we have $V \in \mathcal{N}_x^X$. Thus $V$ is a neighborhood of all its points, so that $V$ is open. Thus every $G_\delta$-set in $X$ is open, implying thereby that $X$ is a $P$-space. \qed

Thus it follows from Theorem 2.3 and Theorem 2.4 that for any Cauchy complete ordered field $F$ with $\text{cf}(F) = \aleph_0$ the class of all $P$-spaces and the class of all $P_f$-spaces are identical. In other words,
Theorem 2.4 provides a construction of $P^2$ spaces through a good number of ordered fields $F$; that there exists many examples of such ordered fields is shown in the next section.

3. Cauchy Complete Countable Cofinal Ordered Fields

**Theorem 3.1.** Given any ordered field $F$ there always exists an ordered field extension of $F$ which is Cauchy complete and with countable cofinality character.

**Proof.** Let $\theta$ be an indeterminate over the field $F$ and consider the polynomial ring $F[\theta]$. Let:

$$P = \{ f \in F[\theta] : \text{the coefficient of the leading term of } f \text{ is non-negative} \}.$$

Then clearly,

$$P + P \subseteq P$$

$$P \cdot P \subseteq P$$

$$P \cup (-P) = F[\theta]$$

$$P \cap (-P) = \{0\}.$$
Thus for \( f, g \in F[\theta] \) if we do put:

\[
f \leq g \iff g - f \in P,
\]

then it is clear that \( \leq \) defines a linear order on \( F[\theta] \).

Let \( F(\theta) \) be the field of fractions of the ring \( F[\theta] \), i.e., each and every element of \( F(\theta) \) can be expressed in the form \( \frac{f}{g} \) where \( f, g \in F[\theta] \) and that \( g > 0 \) = the zero polynomial. Let the order be extended as usual, i.e., for \( f, g, h, k \in F[\theta] \) with \( h, k > 0 \), we shall say that \( \frac{f}{h} \leq \frac{g}{k} \iff fk \leq gh \). Then it is quite clear that \( F(\theta) \) contains an order isomorphic copy of the field \( F \) — the constant polynomials, and that

\[
1 \ll \theta \ll \theta^2 \ll \ldots \ll \theta^n \ll \theta^{n+1} \ll \ldots,
\]

where, in an ordered ring \( \mathcal{R} \), for \( x, y \in \mathcal{R} \), we write \( x \ll y \) for the statement that \( (\forall n \in \mathbb{N})(y > nx) \), and say that \( y \) is infinitely larger than \( x \) or that \( x \) is infinitely smaller than \( y \).

Further it is clear that the set \( \{1, \theta, \theta^2, \theta^n, \theta^{n+1}, \ldots\} \) is cofinal in \( F(\theta) \), so that \( \text{cf}(F(\theta)) = \aleph_0 \).

Let \( F^\# \) be the Cauchy completion of the field \( F(\theta) \). Then since \( \text{cf}(F(\theta)) = \text{cf}(F^\#) \), it follows that \( F^\# \) is a required extension of the field \( F \).

\[ \square \]
Apart from offering a plethora of linearly ordered non-Archimedean fields by which the class of \( P \)-spaces could be realized, the above construction also highlights an amusing feature of ordered fields. It is quite possible to have an ordered field for which there exists a subfield with a larger true \( \eta \)-character, and thus a larger cofinality character than the parent field — e.g., if we take \( F = ^*\mathbb{R} \), the field of non-standard reals due to Abraham Robinson and apply the construction in Theorem 3.1 to it, the field \(^*\mathbb{R}^\#\) is a countable cofinal \( \eta_0 \)-field, although \(^*\mathbb{R}\) is an \( \eta_1 \)-field with cofinality character at least \( \aleph_1 \).
V. SPACES LIKE $P$-SPACES