1. Introduction.

The d.l.v.p. derivative (cf. Section 2 of Chapter 0) of even or odd orders are originated from the second or first symmetric difference e.g., \( f(x+t) + f(x-t) - 2f(x) \) or, \( f(x+t) - f(x-t) \). Analogously, we have introduced in this chapter a generalized derivative originating from the generalized Riemann derivative (cf. Section 5 of Chapter 0) and studied the measurability of this generalized derivative and approximate generalized derivative and corresponding derivates.

2. Definitions and Notations.

Definition 2.1. Let \( k \in \mathbb{N}^+ \) and let \( f : \mathbb{R} \to \mathbb{R} \). Let

\[
\Delta_k^-(x,h;f) = \alpha_{k,0} f(x) + \sum_{i=1}^{k} \alpha_{k,i} f(x+2^{i-1}h)
\]

where \( \alpha_{k,i} \)'s are given by the \( k+1 \) equations

\[
\alpha_{k,k} = 1; \sum_{i=0}^{k} \alpha_{k,i} = 0; \sum_{i=1}^{k} 2^i \alpha_{k,i} = 0,
\]

for \( s = 1,2,\ldots,k-1 \).

The limiting value of \( \alpha_{k} \Delta_k^-(x,h;f)/h^k \) as \( h \to 0 \), if exists ,
is defined to be the derivative $D_k f(x)$ of $f$ at $x$ of order $k$, where

$$a_k = k! \left[ \sum_{i=1}^{k} \alpha_{k,i} \left(2^{i-1}\right)^k \right]^{-1}.$$  

This derivative is considered in [28] and it is more general than the Peano derivative of order $k$ in the sense that if the Peano derivative $f^{(k)}(x)$ exists then $D_k f(x)$ exists and equals $f^{(k)}(x)$. Considering one sided upper and lower limits we get the definitions of the four derivates $D_k^+ f(x)$, $D_k^- f(x)$, $D_k^* f(x)$ and $D_k^\ast f(x)$ of $f$ at $x$ of order $k$. The approximate derivative $D_k^\ast f(x)$ and the approximate derivates $D_k^\ast f(x)$, etc. are defined analogously.

The motivation of the next definition is obtained from the definition of d.l.V.P. derivative (cf. Section 2 of Chapter 0). Just as d.l.V.P. derivative of orders $2k$ and $2k+1$ correspond to symmetric Riemann derivative (cf. Section 3 of Chapter 0) of order $2$ and $1$ respectively, the generalized derivative introduced below corresponds to generalized Riemann derivative (cf. Section 5 of Chapter 0) of order $k$.

Definition 2.2. Let

$$A = \{ b_0, b_1, \ldots, b_{k+1}; a_0, a_1, \ldots, a_{k+1} \}$$

be a system in (5.1) of Chapter 0 satisfying (5.2) of that chapter. For $p \in \mathbb{N}$, define
(2.4) \[ C_p = C_{k,p} = C^l_{k,p}(A) = \sum_{i=0}^{k+l} a_i b_i^p. \]

Then

\[ C_p = 0, \text{ if } 0 \leq p \leq k-1 \]
\[ = L, \text{ if } p = k. \]

Let \( \nu_0 = 0 \) and \( \nu_1 \) be the smallest positive integer such that \( C_{k+\nu_1} \neq 0 \) and \( \nu_2 \) be the smallest positive integer greater than \( \nu_1 \) such that \( C_{k+\nu_2} \neq 0 \) and in general \( \nu_i \) be the smallest positive integer greater than \( \nu_{i-1} \) such that \( C_{k+\nu_i} \neq 0 \). Thus to each system \( A \) there is a sequence \( \{\nu_i\} \) constructed as above.

Let \( n \in \mathbb{N} \). Then \( f \) is said to have generalized (resp. approximate generalized) Riemann derivative at \( x \) of order \( (k,\nu_n) \) with respect to the system \( A \) if there are real numbers \( \alpha_{k+\nu_1}, \ldots, \alpha_{k+\nu_n} \), independent of \( h \), such that

\[
\sum_{i=0}^{k+l} a_i f(x+b_i h) = \sum_{j=0}^{k+\nu_j} C_{k+\nu_j} h^j \frac{\alpha_{k+\nu_j}}{(k+\nu_j)!} + h^n \varepsilon_{k,n}(x,h;f)
\]

where

\[
\lim_{h \to 0} \varepsilon_{k,n}(x,h;f) = 0 \quad \text{(resp. } \lim_{h \to 0} \varepsilon_{k,n}(x,h;f) = 0 \text{ )}
\]

and the number \( \alpha_{k+\nu_n} \) is called the generalized (resp. approximate generalized) Riemann derivative of \( f \) at \( x \) of order \( (k,\nu_n) \) with
respect to the system A and is denoted by $\text{GRD}_{k,\nu_n} f(x) = \text{GRD}_{k,\nu_n} f(x, A)$ (resp. $\text{GRD}_{k,\nu_n} f(x) = \text{GRD}_{k,\nu_n}^\text{ap} f(x, A)$).

It is clear from the above definition that if $\text{GRD}_{k,\nu_n} f(x)$ exists then $\text{GRD}_{k,\nu_j} f(x)$ exists for $0 \leq j \leq n$. For $n = 0$ we have, since $\nu_0 = 0$, $\text{GRD}_{k,\nu_0} f(x) = \text{GRD}_{k} f(x)$ (cf. (5.3) of Chapter 0). Similar remarks hold for the approximate case.

It is clear that if the Peano derivative $f_{(x+\nu_n)}(x)$ exists then $\text{GRD}_{k,\nu_n} f(x)$ exists and equals $f_{(x+\nu_n)}(x)$.

Suppose $\text{GRD}_{k,\nu_n} f(x)$ exists. Then the right hand upper generalized Riemann derivate of $f$ at $x$ of order $(k,\nu_{n+1})$ with respect to the system A denoted by $\text{GRD}_{k,\nu_n}^+ f(x) = \text{GRD}_{k,\nu_n}^+ f(x, A)$ is defined to be the right hand upper limit of

$$\left[ \frac{(k+\nu_{n+1})!}{(C_{k+\nu_{n+1}} h^{n+1})} \right] x^{k+1} \left[ \sum_{i=0}^{k+1} \frac{f(x+b_i h) - \sum_{j=0}^{k+1} C_{k+\nu_j} h^j \text{GRD}_{k,\nu_j} f(x)/(k+\nu_j)!}{i} \right].$$

The other three generalized Riemann derivates are defined similarly. If all of them are equal then the common value is the generalized Riemann derivative $\text{GRD}_{k,\nu_{n+1}} f(x)$ of $f$ at $x$ of order $(k,\nu_{n+1})$, (possibly infinite).

Considering approximate limits we get the definitions of approximate generalized Riemann derivatives and derivates of $f$ at
x of order \((k, \nu_{n+1})\).

In what follows we always consider that \(a_i, b_i, i = 0, 1, \ldots, k+l\), are real numbers in (5.1) of Chapter 0 satisfying (5.2) of that chapter.


Lemma 3.1. Let \(f: \mathbb{R} \to \mathbb{R}\) be approximately continuous and let \(p > 0\). Let, for \(n \in \mathbb{N}^+\),

\[
g_n(x) = \sup_{0 < u < \frac{1}{n}} \left(1/u^p\right) \sum_{i=0}^{k+l} a_i f(x + b_i u).
\]

Then \(g_n\) is measurable.

Proof. Let \(x_0 \in \mathbb{R}\) and \(\varepsilon > 0\) be arbitrary. We first suppose that \(g_n(x_0)\) is finite. Then there is \(\xi, 0 < \xi < 1/n\), such that

\[
g_n(x_0) - \varepsilon < \left(1/\xi^p\right) \sum_{i=0}^{k+l} a_i f(x_0 + b_i \xi).
\]

Since \(f\) is approximately continuous, for each \(i\), \(f(x + b_i \xi) \to f(x_0 + b_i \xi)\) as \(x \to x_0\) through a set \(G_i\) having \(x_0\) as a point of density. Let \(G = \bigcap_{i=0}^{k+l} G_i\). Then \(x_0\) is a point of density of \(G\) and

\[
limit_{x \to x_0 \atop x \in G} \left(1/\xi^p\right) \sum_{i=0}^{k+l} a_i f(x + b_i \xi) = \left(1/\xi^p\right) \sum_{i=0}^{k+l} a_i f(x_0 + b_i \xi).
\]

So there is \(\delta > 0\) such that

\[
\left|\left(1/\xi^p\right) \sum_{i=0}^{k+l} a_i f(x + b_i \xi) - \left(1/\xi^p\right) \sum_{i=0}^{k+l} a_i f(x_0 + b_i \xi)\right| < \varepsilon
\]

for \(x \in G \cap (x_0 - \delta, x_0 + \delta)\).
From (3.1) and (3.2)

\begin{equation}
(3.3) \quad g_n(x_0) - 2\varepsilon < \left(\frac{1}{\xi^p}\right)^{k+1} \sum_{i=0}^{k+l} a_i f(x+b_i \xi) \leq g_n(x)
\end{equation}

for \( x \in G \cap (x_0 - \delta, x_0 + \delta) \).

Therefore since \( \varepsilon \) is arbitrary,

\[ g_n(x_0) \leq \liminf_{x \to x_0} \text{ap } g_n(x). \]

So, \( g_n \) is approximately lower semicontinuous at \( x_0 \).

Now suppose \( g_n(x_0) = \infty \). Let \( N \in \mathbb{N}^+ \) be arbitrarily large. Then there is \( \xi, 0 < \xi < 1/n \) such that (3.1) holds with its left hand side replaced by \( N \) and proceeding as above we get the relation (3.3) with its left hand side replaced by \( N-\varepsilon \) i.e.,

\[ N-\varepsilon < \left(\frac{1}{\xi^p}\right)^{k+1} \sum_{i=0}^{k+l} a_i f(x+b_i \xi) \leq g_n(x) \]

for all \( x \in G \cap (x_0 - \delta, x_0 + \delta) \),

which gives

\begin{equation}
(3.4) \quad \liminf_{x \to x_0} \text{ap } g_n(x) \geq N-\varepsilon.
\end{equation}

Since \( N \) is arbitrary, the left hand side of (3.4) is \( \infty \) showing that \( g_n \) is approximately lower semicontinuous at \( x_0 \).

If \( g_n(x_0) = -\infty \) the approximate lower semicontinuity of \( g_n \) at \( x_0 \) is obvious. Since \( x_0 \in \mathbb{R} \) is arbitrary, \( g_n \) is approximately lower semicontinuous on \( \mathbb{R} \). Thus, \( g_n \) is measurable.
Theorem 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be approximately continuous. Then $\overline{\text{GRD}_k}^+ f$ etc. are measurable.

Proof. Putting $p = k$ we have by Lemma 3.1 $g_n$ is measurable, for each $n \in \mathbb{N}^+$, where

$$g_n(x) = \sup_{0 < u < 1/n} \left( \frac{1}{u} \right) \sum_{i=0}^{k+1} a_i f(x + b_i u).$$

Since $\lim_{n \to \infty} g_n(x) = (L/k!) \overline{\text{GRD}_k}^+ f(x)$, $\overline{\text{GRD}_k}^+ f$ is measurable. Similarly the other three derivates are measurable.

The condition of approximate continuity in Lemma 3.1 and hence in Theorem 3.2 may not be redundant in view of the examples given in [35]. However in the following theorem we show that if $\text{GRD}_{k,v} f$ exists then the approximate continuity is not required.

Theorem 3.3. Let $n \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be measurable. If $E \subset \mathbb{R}$ is the set of all points $x$ such that $\text{GRD}_{k,v} f(x)$ exists (possibly infinite) then $E$ is measurable and $\text{GRD}_{k,v} f$ is measurable on $E$.

Proof. Let $n = 0$. Then for $x \in E$ we have

$$\text{GRD}_k f(x) = \text{GRD}_{k,0} f(x) = (k!/L) \lim_{n \to \infty} n \sum_{i=0}^{k+1} a_i f(x + b_i/n).$$

and since $f$ is measurable, the result follows in this case. We suppose that the result is true for all $r < n$, $n \in \mathbb{N}^+$ and let $E \subset \mathbb{R}$.
be the set of all points \( x \) such that \( \text{GRD}_{k,v} f(x) \) exists (possibly infinite). Then by definition \( \text{GRD}_{k,v} f(x) \) exists and is finite on \( E \) for \( r = 0,1,\ldots,n-1 \). Since the result is true for all \( r < n \), \( E \) is measurable and \( \text{GRD}_{k,v} f \) is measurable on \( E \) for all \( r < n \).

Therefore, since

\[
\text{GRD}_{k,v} f(x) = \left[ \frac{(k+v)!,}{C_{k+v}^n} \right] \lim_{m \to \infty} \sum_{i=0}^{k+v} \left( \sum_{j=0}^{k+v} \text{GRD}_{k,v} f(x) / ((k+v)_j)! m^j \right),
\]

\( \text{GRD}_{k,v} f \) is measurable on \( E \).

**Theorem 3.4.** Let \( n \in \mathbb{N}^+ \) and let \( f: \mathbb{R} \to \mathbb{R} \) be approximately continuous and let

\[ E_n = \{ x : \text{GRD}_{k,v} f(x) \text{ exists finitely} \}. \]

Then \( \text{GRD}_{k,v} f \) etc. are measurable on \( E_n \).

**Proof.** For \( m \in \mathbb{N}^+ \), \( x \in E_n \), let

\[ g_m(x) = \sup_{0 < u < 1/m} B(x,u;f) \]

where

\[ B(x,u;f) = \left[ \frac{(k+v)!,}{C_{k+v}^n} u^{n+1} \right] \lim_{m \to \infty} \sum_{i=0}^{k+v} \left( \sum_{j=0}^{k+v} \text{GRD}_{k,v} f(x) / ((k+v)_j)! m^j \right), \]

\[ P(x,u;f) = \sum_{j=0}^{k+v} C_{k+v}^u \text{GRD}_{k,v} f(x) / ((k+v)_j)! . \]

Let \( r \in \mathbb{N}^+ \). Let \( \epsilon > 0 \) be arbitrary. Since \( \text{GRD}_{k,v} f \) is measurable
on $E_n$ for $i = 0, 1, \ldots, n$, there is a perfect set $Q_r \subseteq E_n$ such that $\mu(E_n - Q_r) < 1/r$ and $\text{GRD}_{k,0} f$ is continuous on $Q_r$ relative to $Q_r$ for $i = 0, 1, \ldots, n$. Let $Q^0_r$ be the set of all points of $Q_r$ which are points of density of $Q_r$. Let $x_0 \in Q^0_r$.

We first suppose that $g_m(x_0)$ is finite. Then there exists $\xi$, $0 < \xi - x_0 < 1/m$ such that

$$ g_m(x_0) - \varepsilon < B(x_0, \xi - x_0; f). \tag{3.5} $$

Since $f$ is approximately continuous, there are measurable sets $G_i$ such that $x_0$ is a point of density of $G_i$ for $i = 0, 1, \ldots, k+l$ and

$$ \lim_{x \to x_0} f(x + b_i(\xi - x)) = f(x_0 + b_i(\xi - x_0)) \quad \text{for } i = 0, 1, \ldots, k+l. $$

Let $G = \bigcap_{i=0}^{k+l} G_i$. Then $G$ is measurable and $x_0$ is a point of density of $G$ and

$$ \lim_{x \to x_0} \sum_{i=0}^{k+l} a_i f(x + b_i(\xi - x)) = \sum_{i=0}^{k+l} a_i f(x_0 + b_i(\xi - x_0)). \tag{3.6} $$

Also since $P(x, \xi - x; f)$ is continuous (as a function of $x$) on $Q_r$ relative to $Q_r$ and $x_0 \in Q^0_r \subseteq Q_r$,

$$ \lim_{x \to x_0} P(x, \xi - x; f) = P(x_0, \xi - x_0; f). \tag{3.7} $$
From (3.6) and (3.7)

\[
\lim_{x \to x_0} B(x, \xi - x; f) = B(x_0, \xi - x_0; f) \quad \text{for } x \in G \cap Q^0_r
\]

So, there is \( \delta, \ 0 < \delta < \min [\xi - x_0, 1/m - \xi + x_0] \) such that

(3.8) \[ |B(x, \xi - x; f) - B(x_0, \xi - x_0; f)| < \varepsilon \]

whenever \( x \in G \cap Q^0_r \cap (x_0 - \delta, x_0 + \delta) \).

From (3.5) and (3.8) we get,

(3.9) \[ g_m(x_0) - 2\varepsilon < B(x, \xi - x; f) \]

for \( x \in G \cap Q^0_r \cap (x_0 - \delta, x_0 + \delta) \).

Now, if \( x \in (x_0 - \delta, x_0 + \delta) \) then \( 0 < \xi - x < 1/m \) and so (3.9) gives

(3.10) \[ g_m(x_0) - 2\varepsilon < g_m(x) \] \text{ for } x \in G \cap Q^0_r \cap (x_0 - \delta, x_0 + \delta) \).

From (3.10) since \( x_0 \) is a point of density of \( G \cap Q^0_r \) and \( \varepsilon > 0 \) is arbitrary,

\[
g_m(x) \leq \liminf_{x \to x_0} g_m(x).
\]

So, in this case \( g_m \) is approximately lower semicontinuous at \( x_0 \).

Now suppose \( g_m(x_0) = \infty \). Let \( N \in \mathbb{N}^+ \) be arbitrarily large. Then there is \( \xi, \ 0 < \xi - x_0 < 1/m \) such that (3.5) holds with its left hand side replaced by \( N \) and proceeding as above we get the relation
(3.10) with its left hand side replaced by $N - \varepsilon$ which gives

$$(3.11) \quad \liminf_{x \to x_0} \text{ap} \ g_m(x) \geq N - \varepsilon.$$  

Since $N$ is arbitrary, the left hand side of (3.11) is $\infty$ showing that $g_m$ is approximately lower semicontinuous at $x_0$. If $g_m(x_0) = -\infty$ the approximate lower semicontinuity of $g_m$ at $x_0$ is obvious.

Thus in either case $g_m$ is approximately lower semicontinuous at $x = x_0$ and therefore on $Q^0_r$, for all $r \in \mathbb{N}^+$. Therefore $g_m$ is measurable on $Q^0_r$, for all $r \in \mathbb{N}^+$. Let $Q^0 = \bigcup_{r=1}^{\infty} Q^0_r$. Then $g_m$ is measurable on $Q^0$. Also $\mu(E_n - Q^0_r) \leq \mu(E_n - Q^0_r) = \mu(E_n - Q^0) < 1/r$, for all $r \in \mathbb{N}^+$. Therefore $\mu(E_n - Q^0) = 0$. So, $g_m$ is measurable on $E_n$. Since

$$\overline{GRD}_{k, \nu}^+ f(x) = \lim_{m \to \infty} g_m(x), \ x \in E_n,$$

$\overline{GRD}_{k, \nu}^+ f$ is measurable on $E_n$. Similarly other three derivates are measurable on $E_n$. This completes the proof.

Remark 3.5. For the measurability of the first symmetric derivates and Peano derivates only measurability of $f$ is needed (see [41; p.255, Theorem 7.9] and [30] respectively). Therefore proving the above result for measurable functions remains open. It may however be noted that some caution is necessary to
tackle the symmetric cases (see [35; 19; 41, p. 136]).


Theorem 4.1. Let \( f : \mathbb{R} \to \mathbb{R} \) be measurable and \( n \in \mathbb{N} \). Let \( a, b \in \mathbb{R} \), \( a < b \). Let

\[
\begin{align*}
E_0 &= E_0(f) = \{ x \in (a, b) : \text{GRD}_{k, ap}^k f(x) \text{ exists finitely} \} \\
E_i &= E_i(E_0(f) = \{ x \in E_{i-1} : \text{GRD}_{k, \nu_i, ap}^k f(x) \text{ exists finitely} \},
\end{align*}
\]

for \( i = 1, 2, \ldots, n \).

Suppose \( E \) is measurable and \( \text{GRD}_{k, \nu_i, ap}^k f \) is measurable on \( E_i \) for \( i = 0, 1, \ldots, n \). Then \( \text{GRD}_{k, \nu_{n+1}, ap}^k f \) etc. are measurable on \( E_n \).

To prove the theorem we need few lemmas. We suppose in the following lemmas except Lemma 4.2 that the hypotheses of the theorem hold.

For \( x \in E_n \) and \( 0 < h < 1 \) define

\[
(4.1) \quad m(x, h) = m(x, h; f) = \mu(\{ t : t \in (0, h) ; W(x, t; f) \geq 0 \})
\]

where

\[
W(x, t; f) = \left[ (k+\nu_{n+1})! / C_{k+\nu_{n+1}} \right] x^{k+1} \left[ \sum_{i=0}^{n} a_i f(x+b_i t) - \sum_{j=0}^{k+\nu} C_{k+\nu_j} t^j \text{GRD}_{k, \nu_j, ap}^k f(x) / (k+\nu_j)! \right].
\]

Since \( f \) is measurable, the right hand side of (4.1) is well defined for fixed \( x \in E_n \).
Lemma 4.2. Let $f$ and $F$ be two real valued measurable functions defined on $\mathbb{R}$ and let $Q$ be a measurable set of positive measure such that $F = f$ on $Q$. Let $x_0 \in Q$ be a point of density of $Q$ such that $\text{GRD}_{k,\nu_r,\text{ap}} f(x_0)$ exists. Then $\text{GRD}_{k,\nu_r,\text{ap}} F(x_0)$ exists and equals $\text{GRD}_{k,\nu_r,\text{ap}} f(x_0)$, $0 \leq r \leq n$.

Proof. By definition there is a set $H$ having 0 as a point of density such that

$$\text{(4.2)} \quad \text{GRD}_{k,\nu_r,\text{ap}} f(x_0) = \lim_{h \to 0} \frac{k!}{k} \sum_{i=0}^{k} a_i f(x_0 + b_i h),$$

Let for $i$, $0 \leq i \leq k+1$,

$$Q_i = \{ h : x_0 + b_i h \in Q \},$$

$$P = H \cap \bigcap_{i=0}^{k+1} Q_i.$$ Then 0 is a point of density of $P$ and from (4.2)

$$\text{GRD}_{k,\nu_r,\text{ap}} f(x_0) = \lim_{h \to 0} \frac{k!}{k} \sum_{i=0}^{k} a_i f(x_0 + b_i h),$$

$$h \in P$$

$$= \lim_{h \to 0} \frac{k!}{k} \sum_{i=0}^{k+1} a_i F(x_0 + b_i h) = \text{GRD}_{k,\nu_r,\text{ap}} F(x_0).$$

Repeating this argument we get this result.

Lemma 4.3. The function $m$ in (4.1) is measurable on $E_n$ for fixed $h$, $0 < h < 1$.

Proof. We first prove the lemma for continuous $f$. Let
h be fixed. Let \( r \in \mathbb{N}^+ \). Since \( \text{GRD}_{k, \nu_i, \text{ap}} f \) is measurable on \( E^n \) for \( i = 0, 1, \ldots, n \) there is a perfect set \( Q_r \subset E^n \) such that \( \mu(E^n - Q_r) < 1/r \) and for all \( i = 0, 1, \ldots, n \) \( \text{GRD}_{k, \nu_i, \text{ap}} f \) is continuous on \( Q_r \) relative to \( Q_r \). Let \( \lambda \in \mathbb{R} \) and

\[
H_r = \{ x : x \in Q_r, m(x, h) < \lambda \}.
\]

We show that \( H_r \) is measurable. If \( \lambda \leq 0 \) then \( H_r = \emptyset \) and if \( \lambda > h \) then \( H_r = Q_r \). So we suppose \( 0 < \lambda \leq h \). Let

\[
S = \{ \xi : \mu(\{ t : t \in (0, h) ; W(x, t; f) \geq \xi \}) \geq \lambda \}.
\]

Clearly

\[
(4.3) \quad \xi_1 < \xi_2 \text{ and } \xi_2 \in S \text{ imply } \xi_1 \in S \text{ and } \sup S \in S.
\]

Let

\[
\phi(x, h, \lambda) = \sup S, \quad x \in Q_r.
\]

Then

\[
(4.4) \quad H_r = \{ x : x \in Q_r, \phi(x, h, \lambda) < 0 \}.
\]

Let \( \epsilon > 0 \) be arbitrary. Let \( x_0 \in Q_r \) and let \( \phi(x_0, h, \lambda) = \xi_0 \). Then

\[
(4.5) \quad \mu(\{ t : t \in (0, h) ; W(x_0, t; f) \geq \xi_0 + \epsilon \}) < \lambda.
\]

Let

\[
p = \max \{ |b_0|, |b_1|, \ldots, |b_{k+1}| \}, \quad I = [a-\text{ph}, b+\text{ph}].
\]

Since \( f \) is continuous on \( I \), it is uniformly continuous there and so there is \( \delta' > 0 \) such that

\[
|f(x_1) - f(x_2)| < \epsilon \text{ for } |x_1 - x_2| < \delta', \quad x_1, x_2 \in I.
\]

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Also since $GRD_{k,v,ap}^f$ are all continuous on $Q_r$ relative to $Q_r$, they are uniformly continuous there and so there are $\delta_i$, $i = 0, 1, \ldots, n$ such that

$$|GRD_{k,v,ap}^f(x_1) - GRD_{k,v,ap}^f(x_2)| < \varepsilon$$

for $|x_1 - x_2| < \delta_i$, $x_1, x_2 \in Q_r$, $i = 0, 1, \ldots, n$.

Let $|x - x_0| < \delta = \min \{\delta^i, \delta_0, \delta_1, \ldots, \delta_n\}$ and $x \in Q_r$. Then

\begin{equation}
(4.7) \quad \mu(\{t : t \in (0, h); W(x, t; f) \geq \xi + (L' + 1)\varepsilon\}) < \lambda
\end{equation}

where

$$L' = \left\{ (k+\nu'_{n+1})! / \left[ C_{k+\nu'_{n+1}} \right] \left[ \sum_{i=0}^{k+l} a_i + \sum_{j=0}^{n} C_{k+\nu'} / (k+\nu')! \right] \right\}.$$

For, the conditions $|x-x_0| < \delta$, $x \in Q_r$ and $0 < t < h$ imply

$$|W(x, t; f) - W(x_0, t; f)| < L'\varepsilon$$

and therefore

$$\{t : t \in (0, h); W(x, t; f) \geq \xi + (L' + 1)\varepsilon\} \subset \{t : t \in (0, h); W(x_0, t; f) \geq \xi + \varepsilon\}.$$

Therefore (4.6) implies (4.7). Now (4.7) and (4.4) imply

$$\phi(x, h, \lambda) < \xi + (L' + 1)\varepsilon = \phi(x_0, h, \lambda) + (L' + 1)\varepsilon$$

for $x \in Q_r \cap (x_0 - \delta, x_0 + \delta)$.

Therefore $\phi$ is upper semicontinuous at $x_0$ relative to $Q_r$ for fixed $\lambda$. Since $x_0 \in Q_r$ be arbitrary, $\phi$ is upper semicontinuous on $Q_r \cap (x_0 - \delta, x_0 + \delta)$. Hence from (4.5) $H_r$ is measurable and so from
(4.3) \( m \) is measurable on \( Q_r \). Since \( \mu(\{ E_n - \bigcup_{r=1}^{\infty} Q_r \}) = 0 \), \( m \) is measurable on \( E_n \) for fixed \( h \), \( 0 < h < 1 \).

Now we prove Lemma 4.3 for measurable function \( f \). For all \( r \in \mathbb{N}^+ \) there is a closed set \( P_r \subset I \) such that \( \mu(I - P_r) < 1/r \) and \( f/P_r \) is continuous on \( P_r \). Let \( Q_r = \bigcup_{i=1}^r P_i \). Then \( Q_r \) is closed, \( Q_r \subset I \), \( \mu(I - Q_r) < 1/r \), \( f/Q_r \) is continuous on \( Q_r \) and \( Q_r \subset Q_{r+1} \). Let \( F_r(x) = f(x) \), for \( x \in \{ a-\phi \} \cup Q_r \cup \{ b+\phi \} \) and \( F_r \) is linear in the closure of each contiguous intervals of \( Q_r \) in \( I \). Let \( F_r(x) = f(b+\phi) \) if \( x > b+\phi \) and \( F_r(x) = f(a-\phi) \) if \( x < a-\phi \). Then \( F_r \) is continuous in \( I \). Let \( Q_r^0 = \{ x: x \in Q_r ; x \text{ is a point of density of } Q_r \} \).

Then \( Q_r^0 \subset Q_r^0 \). Let \( Q_r^0 = \bigcup_{r=1}^{\infty} Q_r^0 \). Then \( \mu(I - Q_r^0) \leq \mu(I - Q_r) = \mu(I - Q_r) < 1/r \), for all \( r \in \mathbb{N}^+ \). Therefore \( \mu(I - Q_r^0) = 0 \). Let \( x \in Q_r^0 \cap E_n \). Let \( \varepsilon > 0 \) be arbitrary. Let \( N(x) = \inf \{ r: x \in Q_r^0 \} \), \( M = \sum_{i=0}^{k+1} 1/|b_i| \), \( r \geq \gamma(x) = \max \{ M/\varepsilon, N(x) \} \)

where the summation excludes that \( i = i_o \), if there is any, such that \( b_i = 0 \). For each \( i \), such that \( b_i \neq 0 \), let

\[
G_r(x) = \{ t \leq (0,h); x + b_t t \in I - Q_r \}
\]

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\[ G_r(x) = \bigcup_{i=0}^{k+1} G_{r_i}(x) . \]

Then \( \mu(G_r(x)) \leq \sum_{i=0}^{k+1} \left( \frac{1}{|b_t|} \right) \cdot \left( \frac{1}{r} \right) = \frac{M}{r} \leq \varepsilon . \) If \( t \in (0,h) - G_r(x) \) then

\[ x + b_t \in Q_r^0 , \ i = 0,1,\ldots,k+1 \]

and so

\[ \sum_{i=0}^{k+1} a_i f(x + b_t) = \sum_{i=0}^{k+1} a_i F(x + b_t) . \]

Also since \( x \in Q_r^0 \cap E_n \), by Lemma 4.2,

\[ \text{GRD}_{k,\nu_i,ap} f(x) = \text{GRD}_{k,\nu_i,ap} r \]

So

\[ \{ t : t \in (0,h) ; W(x,t;f) \geq 0 \} - \{ t : t \in (0,h) ; W(x,t;F_r) \geq 0 \} \subset G_r(x) . \]

Hence

\[ m(x,h;f) - m(x,h;F_r) \]

\[ = \mu(\{ t : t \in (0,h) ; W(x,t;f) \geq 0 \}) \]

\[ - \mu(\{ t : t \in (0,h) ; W(x,t;F_r) \geq 0 \}) \]

\[ \leq \mu(\{ t : t \in (0,h) ; W(x,t;f) \geq 0 \} - \{ t : t \in (0,h) ; W(x,t;F_r) \geq 0 \}) \]

\[ \leq \mu(G_r(x)) \leq \varepsilon . \]

Interchanging the role of \( f \) and \( F_r \) we get a relation similar to above and finally get

\[ | m(x,h;f) - m(x,h;F_r) | \leq \varepsilon , \text{ for all } r \geq \gamma(x) . \]
Therefore
\[ \lim_{r \to \infty} m(x,h;F_r) = m(x,h;f) . \]

Since \( x \in Q^0_n \cap E_n(f) \) is arbitrary and \( \mu(I-Q^0_n) = 0 \),

(4.8) \[ \lim_{r \to \infty} m(x,h;F_r) = m(x,h;f) \text{ a.e. on } E_n(f) . \]

If \( x \in Q^0_r \cap E_i(f) \) then by Lemma 4.2 \( GRD_{k,v_i,ap} F(x) \) exists and
equals \( GRD_{k,v_i,ap} f(x) \). Since \( (a,b) - Q_r \) is the union of a

countable collection of open intervals on each of which \( F_r \) is
linear and hence \( GRD_{k,v_i,ap} F \) exists, \( E_i(F_r) \) and \( ((a,b) - Q_r) \cup \)
\( (Q^0_r \cap E_i(f)) \) differ by a set of measure zero, where \( E_i(F_r) \) is the
set defined in Theorem 4.1 with \( f \) replaced by \( F_r \). Hence \( E_i(F_r) \) is
measurable. Also \( GRD_{k,v_i,ap} F \) is measurable on \( (a,b) - Q_r \) and on
\( Q^0_r \cap E_i(f) \) and hence \( GRD_{k,v_i,ap} F \) is measurable on \( E_i(F_r) \). This
being true for all \( i = 0,1,\ldots,n \), the hypotheses of Theorem 4.1 are
satisfied for \( F_r \). Moreover \( F_r \) is continuous. Hence by the
special case \( m(x,h;F_r) \) is measurable on \( E_n(F_r) \) and hence measurable
on \( ((a,b) - Q_r) \cup (Q^0_r \cap E_n(f)) \) and therefore measurable on
\( Q^0_r \cap E_n(f) \). Since \( \{Q^0_r\} \) is an expanding sequence of measurable sets,

\[ \lim_{r \to \infty} m(x,h;F_r) \text{ is measurable on } \bigcup_{r=1}^{\infty} Q^0_r \cap E_n(f) , \text{ i.e., on } E_n(f) . \]

Hence by (4.8) \( m(x,h;f) \) is measurable on \( E_n(f) \).
Lemma 4.4. The functions $\alpha$ and $\beta$ where

$$\alpha(x) = \liminf_{h \to 0^+} \frac{m(x,h)}{h}; \quad \beta(x) = \limsup_{h \to 0^+} \frac{m(x,h)}{h}$$

are measurable on $E_n$.

Proof. Let

$$A(x,h) = \inf_{0<u<h} \frac{m(x,u)}{u}, \quad 0 < h < 1.$$  \hspace{1cm} (4.9)

Since $m(x,u)/u$ is a continuous function of $u$ in $(0,h)$, the infimum in (4.9) is the same if only rational values of $u$ are considered. By Lemma 4.3, $m(x,u)/u$ is measurable on $E_n$ for fixed $u \in (0,h)$ and so $A(x,h)$ is a measurable function of $x$ on $E_n$ for fixed $h$. Now as $h$ decreases $A(x,h)$ increases and $\alpha(x) = \lim_{h \to 0^+} A(x,h)$ and therefore $\alpha$ is measurable on $E_n$. Similarly $\beta$ is measurable on $E_n$.

Lemma 4.5. For $x \in E_n$ and $\lambda \in \mathbb{R}$ let

$$S = S_\lambda = \{ t : t > 0; \ W(x,t;f) \geq \lambda \ t \}_{n+1}^{k+\nu}$$

$$\alpha_\lambda(x) = d(S,0), \quad \beta_\lambda(x) = d(S,0)$$

where $d(S,0)$ and $d(S,0)$ denote respectively the lower and upper right densities of $S$ at 0. Then $\alpha_\lambda$ and $\beta_\lambda$ are measurable on $E_n$.

Proof. Let

$$F(x) = f(x) - \lambda \ x^{n+1}/(k+\nu)_{n+1}!.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} k+\nu

Then $F$ satisfies the hypotheses of Theorem 4.1. Also

$$W(x,t;F) = W(x,t;f) - \lambda \ t^{n+1}.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} k+\nu
Therefore
\[ S = \{ t : t > 0; W(x, t; F) \geq 0 \} . \]

Now
\[ \alpha^\lambda(x) = \liminf_{h \to 0+} \frac{\mu(S \cap (0, h))}{h} = \liminf_{h \to 0+} \frac{m(x, h; F)}{h} . \]

So by Lemma 4.4 \( \alpha^\lambda \) is measurable. Similarly \( \beta^\lambda \) is measurable.

**Proof of Theorem 4.1.** Let \( \lambda \in \mathbb{R} \) and \{\( \lambda \_ i \)\} be a strictly decreasing sequence converging to \( \lambda \). Let for all \( \_ i \in \mathbb{N}^+ \)
\[ T_\_ i = \{ x \in E ; d (\{ t : t > 0; W(x, t; f) \geq \_ i \}, 0) = 0 \} . \]

Then
\[ T_\_ i = \{ x : x \in E ; \beta^\lambda(x) = 0 \} . \]

Therefore by Lemma 4.5, \( T_\_ i \) is measurable for all \( \_ i \in \mathbb{N}^+ \). Now it can be easily verified that
\[ \{ x : x \in E ; \text{GRD}^+_{k, n+1, \text{ap}} f(x) \leq \lambda \} = \bigcap_{j=1}^{\infty} T_\_ j \]
which shows that \( \text{GRD}^+_{k, n+1, \text{ap}} f \) is measurable on \( E_n \). Similarly the other three derivatives are measurable on \( E_n \). This completes the proof.

Now we come to prove the main results of this section

**Theorem 4.6.** Let \( f : \mathbb{R} \to \mathbb{R} \) be measurable. Then \( \text{GRD}^+_{k, \text{ap}} f \)

etc. are measurable.

**Proof.** If \( W(x, t; f) \) is replaced by \( \sum_{i=0}^{k} a_i f(x + b_i t) \) throughout
in the above arguments which are used to prove Theorem 4.1 we get that for any measurable \( f : \mathbb{R} \to \mathbb{R} \) and for \( a, b \in \mathbb{R}, a < b \), \( \text{GRD}^+_k, \text{ap}^f \) etc. are measurable on \((a, b)\). Hence the proof is clear.

**Corollary 4.7.** If \( f : \mathbb{R} \to \mathbb{R} \) is measurable then the set \( E \) of points \( x \) where \( \text{GRD}^+_k, \text{ap}^f \) exists (possibly infinite) is measurable and \( \text{GRD}^+_k, \text{ap}^f \) is measurable on \( E \).

**Theorem 4.8.** Let \( f : \mathbb{R} \to \mathbb{R} \) be measurable and \( n \in \mathbb{N}^+ \). Then the set \( E \) of points \( x \) for which \( \text{GRD}^+_n, \nu^p, \text{ap}^f \) exists finitely is measurable and \( \text{GRD}^+_n, \nu^p, \text{ap}^f \) is measurable on \( E \). Further \( \text{GRD}^+_k, \nu^p, \text{ap}^f \) etc. are measurable on \( E \).

**Proof.** By Theorem 4.6, \( E_0 \) is measurable and \( \text{GRD}^+_k, \text{ap}^f \) measurable on \( E_0 \). So by Theorem 4.1 \( \text{GRD}^+_k, \nu^p, \text{ap}^f \) etc. are measurable on \( E_0 \). Therefore the set

\[
E_1 = \{ x \in E_0 ; -\infty < \text{GRD}^+_k, \nu^p, \text{ap}^f(x) \leq \infty \}
\]

is measurable and \( \text{GRD}^+_k, \nu^p, \text{ap}^f \) is measurable on \( E \). Hence by Theorem 4.1 \( \text{GRD}^+_k, \nu^p, \text{ap}^f \) etc. are measurable on \( E_1 \). Therefore the set

\[
E_2 = \{ x \in E_1 ; -\infty < \text{GRD}^+_k, \nu^p, \text{ap}^f(x) \leq \infty \}
\]

is measurable and \( \text{GRD}^+_k, \nu^p, \text{ap}^f \) etc. are measurable on \( E_2 \). Therefore the set

\[
E_3 = \{ x \in E_2 ; -\infty < \text{GRD}^+_k, \nu^p, \text{ap}^f(x) \leq \infty \}
\]
is measurable and \( \text{GRD}_{k,v_1,\lambda} f(x) \) is measurable on \( E_2 \).

Proceeding in this way finally we get that the set

\[
E_n = \{ x \in E_{n-1} : -\infty < \text{GRD}_{k,v,n,\lambda} f(x) = \text{GRD}_{k,v,n,\lambda}^+ f(x) = \text{GRD}_{k,v,n,\lambda}^- f(x) < \infty \}
\]

is measurable and \( \text{GRD}_{k,v,n,\lambda} f \) is measurable on \( E_n \). Therefore by Theorem 4.1 \( \text{GRD}_{k,v,n+1,\lambda} f \) etc. are measurable on \( E_n \). This completes the proof.

**Corollary 4.9.** Let \( f : \mathbb{R} \to \mathbb{R} \) be measurable and \( n \in \mathbb{N}^+ \). Then the set \( E_n \) of points \( x \) for which \( \text{GRD}_{k,v,n,\lambda} f(x) \) exists (possibly infinite) is measurable and \( \text{GRD}_{k,v,n,\lambda} f \) is measurable on \( E_n \).

**5. Special Cases.**

We now suppose that \( l = 0 \) and \( L = k! \). In this case the numbers \( b_0, b_1, \ldots, b_k \) determine uniquely the numbers \( a_0, a_1, \ldots, a_k \) by the relation (5.2) of Chapter 0, from which we get

\[
(5.1) \quad a_i = k! \left[ \prod_{j \neq i} (b_i - b_j) \right]^{-1}, \quad i = 0, 1, \ldots, k.
\]

**Case I.** Let \( b_i = i - k/2 \), \( i = 0, 1, \ldots, k \). Then from (5.1)

\[
a_i = (-1)^{k-1} \binom{k}{i} \text{ and so from (3.1) of Chapter 0}
\]

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Thus for this system the derivative $GRD_k f$ is the $k$th symmetric Riemann derivative $RD_k f$. To investigate the generalized derivative $GRD_{k,\nu_n} f$ with respect to this system we are to determine the sequence $\{\nu_i\}$ and constants $C_{k,k+\nu_i}$. In this case (cf. relation (2.4))

$$C_p = C_{k,p} = \sum_{i=0}^{k} (-1)^{i-k} \binom{k}{i} (i-k/2)^p, \quad p \in \mathbb{N}^+.$$ 

Lemma 5.1.

(5.2) $C_{k,p} = 0$ if $p+k$ is odd

(5.3) $C_{1,2n+1} = 1/2^{2n}, \quad C_{2,2n+2} = 2$ for $n \in \mathbb{N}$

(5.4) $C_{2m,2n} = 2 \sum_{j=m-1}^{n-1} \binom{2n}{2j} C_{2m-2,2j}, \quad n \geq m \geq 2, \quad m,n \in \mathbb{N}^+$

(5.5) $C_{2m+1,2n+1} = 2 \sum_{j=m-1}^{n-1} \binom{2n+1}{2j+1} C_{2m-1,2j+1}, \quad n \geq m \geq 1, \quad m,n \in \mathbb{N}^+$

(5.6) $C_{k,p} > 0$ for all $k,p$ such that $p \geq k$ and $k+p$ is even.

Proof. If $k = 2m, \quad m \in \mathbb{N}^+$, then

$$C_{k,p} = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} (i-m)^p$$

$$= \sum_{r=0}^{m-1} (-1)^r \binom{2m}{r} (m-r)^p [1 + (-1)^p] = 0,$$

when $p$ is odd, and if $k = 2m+1, \quad m \in \mathbb{N}$, then
\[ c_{k,p} = \sum_{i=0}^{2m+1} (-1)^{2m+1-i} (2m+1\choose i) (i - (2m+1)/2)^p \]

\[ = \sum_{r=0}^{m} (-1)^r (2m+1\choose r) ((2m+1)/2 - r)^p [1-(-1)^p] = 0 , \]

when \( p \) is even and so (5.2) is proved. Now we prove (5.4) since (5.3) is easy. We have when \( s \in \mathbb{N}^+ \)

\[ \Delta_2(x,h;x^{2s}) = (x+h)^{2s} + (x-h)^{2s} - 2x^{2s} \]

\[ = 2\sum_{i=1}^{s} \binom{2s}{i} x^{2s-2i} h^i \]

and by (5.2)

\[ \Delta_{2m}(x,h;x^{2n}) = \sum_{i=0}^{2m} (-1)^i (2m\choose i) (x + ih - mh)^{2n} \]

\[ = \sum_{i=0}^{2m} (-1)^i (2m\choose i) \sum_{j=0}^{2n} (2n\choose j) x^{2n-j} (i-m)^j h^j \]

\[ = \sum_{j=0}^{2n} (2n\choose j) x^{2n-j} h^j \sum_{i=0}^{2m} (-1)^i (2m\choose i) (i-m)^j \]

\[ = \sum_{j=0}^{2n} (2n\choose j) x^{2n-j} h^j \binom{2m}{2m-j} \]

\[ = \sum_{l=m}^{n} \binom{2n}{2l} x^{2n-2l} h^j . \]

Also using relations (5.2), (5.7) and (5.8)

\[ \Delta_{2m}(x,h;x^{2n}) = \Delta_2 [x,h;\Delta_{2m-2}(x,h;x^{2n})] \]

\[ = \Delta_2 [x,h;\sum_{j=m-1}^{n} \binom{2n}{2m-2,2j} x^{2n-2j} h^{2j}] \]

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Equating the coefficient of $h^{2n}$ from the right hand sides of (5.8) and (5.9) we get the relation (5.4). By similar arguments we obtain

\[\Delta_{2m+1}(x,h;x^{2n+1}) = \sum_{l=m}^{n} C_{2m+1,2l+1}(2n+1)_{2l+1} x^{2n-2l} h^{2l+1}.\] (5.10)

\[\Delta_{2m+1}(x,h;x^{2n+1}) = 2 \sum_{j=m-1}^{n-1} \sum_{r=1}^{n-j} (2n-2j)_{2r} C_{2m-2,2j} x^{2n-2j-2r} h^{2r+2j}.\] (5.11)

and equating the coefficient of $h^{2n+1}$ of right hand sides of (5.10) and (5.11) we get the relation (5.5) . Then the relation (5.6) follows from (5.3), (5.4) and (5.5) . This completes the proof.

Now it follows from Lemma 5.1 that corresponding to the system $\Delta_{k}$, the sequence $\{\nu\}$ such that $C_{k,k+\nu} \neq 0$ is given by $\nu = 2i$ and the generalized Riemann derivative of $f$ at $x$ of order $(k,\nu) = (k,2n)$, denoted by $RD_{k,2n}f(x)$, if it exists, is given by

\[\Delta_{k}(x,h;f) = \sum_{i=0}^{n} C_{k,k+2i} h^{k+2i} RD_{k,2i} f(x)/(k+2i)! + o(h^{k+2n})\]
with similar definition of approximate derivatives and the
derivates. (Note that in this case left hand and right hand
derivates are the same).

In particular putting \( k = 1 \) and \( 2 \) respectively we get from
\((5.3)\)

\[
\frac{[f(x+h) - f(x-h)]}{2} = \sum_{i=0}^{n} \frac{2i+1}{h^{2i+1}} f(x) + o(h^{2n+1})
\]
and

\[
\frac{[f(x+h) + f(x-h)]}{2} = f(x) + \sum_{i=0}^{n} \frac{2i+2}{h^{2i+2}} f(x) + o(h^{2n+2})
\]

Thus the derivatives \( \text{RD}_{1,2n} f(x) \) and \( \text{RD}_{2,2n} f(x) \) are respectively the
d.l.v.p. derivatives of order \( 2n+1 \) and \( 2n+2 \) of \( f \) at \( x \).

The following theorem has some interest.

**Theorem 5.2.** If \( d_{2m} f(x) \) exists, \( m \in \mathbb{N}^+ \), then \( \text{RD}_{2r,2m-2r} f(x) \)
exists and equals \( d_{2m} f(x) \) for \( r = m, m-1, \ldots, 1 \); the converse holds
only for \( r = 1 \). If \( d_{2m+1} f(x) \) exists, \( m \in \mathbb{N}^+ \), then \( \text{RD}_{2r+1,2m-2r} f(x) \)
exists and equals \( d_{2m+1} f(x) \) for \( r = m, m-1, \ldots, 0 \); the converse holds
only for \( r = 0 \).

**Proof.** Suppose \( d_{2m} f(x) \) exists. Then writing \( d_0 f(x) = f(x) \)
we have

\[
\frac{[f(x+h) + f(x-h)]}{2}
\]
and so for \( 1 \leq r \leq m \), since \( c_{k,p} = 0 \) if \( p < k \)

\[
\Delta_{2r}(x,h;f) = \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} f(x + jh - rh) \]

\[
= \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} \left[ f(x + jh - rh) + f(x - jh + rh) \right] / 2 \]

\[
= \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} \left( \sum_{i=0}^{m} (j-r)^2 i \cdot 2^i d_{2i} f(x)/(2i)! + o(h^{2m}) \right) \]

\[
= \sum_{i=0}^{m} h^{2i} d_{2i} f(x) C_{2r,2i}/(2i)! + o(h^{2m}) \]

\[
= \sum_{j=0}^{m-r} C_{2r,2r+2j} h^{2r+2j} d_{2r+2j} f(x)/(2r+2j)! + o(h^{2m}) \]

Hence \( \Delta_{2r,2m-2r} f(x) \) exists and equals \( d_{2m} f(x) \).

Conversely, let \( r = 1 \) and let \( \Delta_{2,2m-2} f(x) \) exist. Then

\[
\Delta_{2}(x,h;f) = \sum_{j=0}^{m-1} C_{2,2j+2j} h^{2+2j} \Delta_{2r,2j} f(x)/(2j)! + o(h^{2m}) \]

and hence \( d_{2m} f(x) \) exists and equals \( \Delta_{2,2m-2} f(x) \) by (5.3).

To complete the proof we are to show that if \( s > 2 \) then

\( \Delta_{s,2n} f(x) \) may exist without existing \( d_{s+2n} f(x) \). Let

\[
f(x) = x^{s-2} \quad \text{if } x \text{ is rational} \]

\[
= 0 \quad \text{if } x \text{ is irrational}. \]

Then since \((s - s/2)h \) is rational or irrational according as \( h \) is rational or irrational,
\[ \Delta_s(0, h; f) = \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} f((i-s/2)h) = 0, \quad \text{for all } h \]

and hence \(R_{s, 2n}^s f(0) = 0\) for all \(n \in \mathbb{N}\). But

\[
[f(h) + (-1)^{s-2} f(-h)]/2 = h^{s-2}, \quad \text{if } h \text{ is rational}
\]

\[= 0, \quad \text{if } h \text{ is irrational} \]

and hence \(d_{s-2} f(0)\) does not exist.

The odd case is similar.

**Case II.** Let \(b_i = i, \ i = 0, 1, \ldots, k\). Then from (5.1)

\[ a_i = (-1)^{k-i} \binom{k}{i} \]

and so from (4.1) of Chapter \(s\)

\[ \sum_{i=0}^{k} a_i f(x+b_i h) = \Delta^*_k(x, h; f). \]

Thus for this system the derivative \(GRD_k^s f\) is the \(k\)th unsymmetric Riemann derivative which will be denoted here by \(RD_k^s f\). To investigate the generalized Riemann derivative with respect to this system we are to determine the sequence \(\{\nu_i\}\) and the constants

\[ C_{k, k+\nu_i}^*. \quad \text{Here} \]

\[ C_p^* = C_{k, p}^* = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^p, \quad p \in \mathbb{N}. \]

**Lemma 5.3.**

(5.12) \[ C_{k, p}^* = \sum_{j=k-1}^{p-1} \binom{p}{j} C_{k-1, j}^* \quad \text{for } p \geq k \geq 2, \ p \in \mathbb{N}^+ \]

(5.13) \[ C_{k, p}^* > 0 \quad \text{for all } k, p, \ p \geq k. \]
Proof. To prove (5.12) we have

\[ \Delta_k^*(x, h; x^p) = \sum_{i=0}^{k} (-1)^{k-i}(\frac{k}{i}) (x^i h^i)^p \]

\[ = \sum_{i=0}^{k} (-1)^{k-i}(\frac{k}{i}) \sum_{j=0}^{p} (\frac{p}{j}) x^{p-j} h^j \]

\[ = \sum_{j=0}^{p} (\frac{p}{j}) x^{p-j} h^j \sum_{i=0}^{k} (-1)^{k-i}(\frac{k}{i}) t^j \]

\[ = \sum_{j=0}^{p} (\frac{p}{j}) x^{p-j} h^j \sum_{i=0}^{k} (-1)^{k-i}(\frac{k}{i}) t^j \]

\[ = \sum_{j=0}^{p} (\frac{p}{j}) x^{p-j} h^j \sum_{i=0}^{k} (-1)^{k-i}(\frac{k}{i}) t^j \]

Again

\[ \Delta_k^*(x, h; x^p) = \Delta_1^* [x, h; \Delta_{k-1}^*(x, h; x^p)] \]

\[ = \Delta_1^* [x, h; \sum_{j=k-1}^{p} (\frac{p}{j}) x^{p-j} h^j] \]

\[ = \sum_{j=k-1}^{p-1} (\frac{p}{j}) x^{p-j} h^j \sum_{i=1}^{p-j} (\frac{p-j}{i}) x^{p-j-i} h^i \]

Equating coefficient of \( h^p \) of right hand sides of (5.14) and (5.15)

we get the relation (5.12). Since \( C_{i,p}^* = 1 \) for all \( p \in \mathbb{N}^+ \), the

relation (5.13) follows from (5.12). This completes the proof.

From Lemma 5.3 it follows that corresponding to the system

\( \Delta_k^* \), the sequence \( \{v_i\} \) for which \( C_{k,k+v_i}^* \neq 0 \) is given by \( v_i = i \) and
the generalized Riemann derivative of \( f \) of order \((k, n) = (k, n)\), denoted by \( RD_{k,n}^* f(x) \), if it exists, is given by

\[
\Delta_{k}^*(x, h; f) = \sum_{i=0}^{n} \frac{\partial^i}{\partial x^i} \left( f(x) \right) + o(h^n).
\]

In particular, if \( k = 1 \), then since \( C_{1,1+i} = 1 \) for all \( i \), we have

\[
f(x+h) - f(x) = \sum_{i=0}^{n} h^{i+1} RD_{i,i}^* f(x)/(i+1)! + o(h^{i+1})
\]

and hence \( RD_{i,i}^* f(x) \) gives the Peano derivative \( f_{(n+1)}(x) \) of order \( n+1 \) of \( f \) at \( x \).

**Theorem 5.4.** If the Peano derivative \( f_{(m)}(x), m \in \mathbb{N}^+ \), exists then \( RD_{r,m-r}^* f(x) \) exists and equals \( f_{(m)}(x) \) for \( r = m, m-1, \ldots, 1 \). The converse holds only for \( r = 1 \).

**Proof.** Let \( f_{(m)}(x) \) exist. Then writing \( f_{(0)}(x) = f(x) \)

\[
\Delta_{r}^*(x, h; f) = \sum_{i=0}^{r} \frac{(-1)^{r-i}}{i!} \left( f(x+ih) \right)
\]

\[
= \sum_{i=0}^{r} \frac{(-1)^{r-i}}{i!} \left( \sum_{j=0}^{m} (ih)^j f_{(j)}(x)/(j)! + o(h^m) \right)
\]

\[
= \sum_{j=0}^{m} h^j f_{(j)}(x) C_{r,j}^* /j! + o(h^m)
\]

\[
= \sum_{l=0}^{m-r} C_{r,r+l}^* h^{r+1} f_{(r+l)}(x)/(r+l)! + o(h^m).
\]

Hence \( RD_{r,m-r}^* f(x) \) exists and equals \( f_{(m)}(x) \).

For \( r = 1 \) let \( RD_{1,m-1}^* f(x) \) exist then

\[
\Delta_{1}^*(x, h; f) = f(x+h) - f(x)
\]
Therefore \( f_{(m)}(x) \) exists and equals \( R^*_d, m-1 f(x) \).

For \( r > 1 \) \( R^*_d, n f(x) \), \( n \in \mathbb{N} \), may exist without \( f_{(r+n)}(x) \) existing. For, consider \( f(x) = |x| \) then \( f_{(1)}(0) \) does not exist but since \( \Delta^*(0, h; f) = 0 \) for all \( r \geq 2 \), \( R^*_d, f(0) = 0 \) for all \( r \geq 2 \) and all \( n \in \mathbb{N} \).

Case III. We now consider a case which is more general than that required to tackle (2.2). Let \( q \) be any real number different from 0,± 1. Let \( b_0 = 0, b_i = q^{i-1} \) for \( i = 1, 2, \ldots, k \). From (5.1)

\[
(5.16) \quad a_k = k! \left[ q^{k-1} \prod_{j=1}^{k-1} (q^{k-1} - q^{j-1}) \right]^{-1} \neq 0, \text{ for } k \geq 2,
\]

and we can determine \( a_0, a_1, \ldots, a_k \) from the equations

\[
(5.17) \quad \sum_{i=0}^{k} a_i = 0; \quad \sum_{i=1}^{k} a_i q^{i-1} = 0 \quad \text{for } j = 1, 2, \ldots, k-1;
\]

\[
\sum_{i=1}^{k} a_i (q^{i-1})^k = k!
\]

and so putting \( a_i / a_k = \alpha_{k,i} \), we have

\[
\sum_{i=0}^{k} a_i f(x+b h) = a_k \Delta^*_{k,k,q} (x, h; f)
\]

where
(5.18) \[ \Delta_{k,q}(x,h;f) = \alpha_{k,0} f(x) + \sum_{i=1}^{k-1} \alpha_{k,i} f(x + q^{i-1}h) + f(x + q^{k-1}h). \]

To investigate the generalized derivative with respect to this system we are to determine the sequence \( \{\nu_i\} \) and the constants \( c_{k,k+\nu_i} \). Here

\[ c_{k,p} = c_{k,p}(q) = \sum_{i=1}^{k} a_i (q^{-i})^p, \quad p \in \mathbb{N}^+ \]

where \( a_i \)'s are given by (5.17).

Lemma 5.5.

\[ c_{k,p} = k! \prod_{i=1}^{k-1} (q^p - q^i)/(q^k - q^i), \quad \text{for } p \geq k \geq 2 \text{ and } \]

\[ c_{1,p} = 1, \quad p \in \mathbb{N}^+. \]

Proof. Let \( \beta_{k,p} = \sum_{i=1}^{k} \alpha_{k,i} (q^{-i})^p \). Then

\[ c_{k,p} = a_k \sum_{i=1}^{k} \alpha_{k,i} (q^{-i})^p = a_k \beta_{k,p}. \]

Now from (5.18)

(5.19) \[ \Delta_{k,q}(x,h;x^p) = \alpha_{k,0} x^p + \sum_{i=1}^{k} \alpha_{k,i} (x + q^{i-1}h)^p \]

\[ = \alpha_{k,0} x^p + \sum_{i=1}^{k} \alpha_{k,i} \sum_{j=0}^{p} \binom{p}{j} x^{p-j}(q^{i-1}h)^j \]

\[ = \alpha_{k,0} x^p + \sum_{j=0}^{p} \binom{p}{j} x^{p-j} \sum_{i=1}^{k} \alpha_{k,i} (q^{-i})^j \]

\[ = \alpha_{k,0} x^p + \sum_{j=0}^{p} \beta_{k,j} \binom{p}{j} x^{p-j} h^j. \]

Again it can be verified from (5.18) that for \( k \geq 2 \),
\[ \tilde{\Delta}_{k,q}(x,h;f) = \tilde{\Delta}_{k-1,q}(x,\sqrt{qh};f) - q^{k-1} \tilde{\Delta}_{k-1,q}(x,h;f) \]

and so we have

\begin{equation}
(5.20) \quad \Delta_{k,q}(x,h;x^p) = \tilde{\Delta}_{k-1,q}(x,\sqrt{qh};x^p) - q^{k-1} \tilde{\Delta}_{k-1,q}(x,h;x^p) \\
= \alpha_{k-1,0} x^p + \sum_{j=0}^{p} \beta_{k-1,j}(P)^{x^{p-j}}(qh)^j \\
- q^{k-1} \left[ \alpha_{k-1,0} x^p + \sum_{j=0}^{p} \beta_{k-1,j}(P)^{x^{p-j}}h^j \right].
\end{equation}

Equating coefficient of \( h^p \) from the right hand sides of (5.19) and (5.20) we have

\[ \beta_{k,p} = (q^p - q^{k-1}) \beta_{k-1,p}, \quad \text{for } p \geq k > 2. \]

Now since \( \beta_{1,p} = \alpha_{1,1} = 1 \), for all \( p \in \mathbb{N}^+ \),

\[ \beta_{k,p} = \prod_{i=1}^{k-1} (q^p - q^i), \quad p \geq k \geq 2. \]

Also by (5.16)

\[ a_k = k! \left[ \prod_{i=1}^{k-1} (q^k - q^i) \right]^{-1}, \quad k \geq 2. \]

Hence

\[ C_{k,p} = a_k \beta_{k,p} = k! \prod_{i=1}^{k-1} \left[ (q^p - q^i)/(q^k - q^i) \right], \quad k \geq 2. \]

Also for \( k = 1 \), \( a_1 = 1 \) and hence

\[ C_{1,p} = a_1 \beta_{1,p} = 1 \text{ for all } p \in \mathbb{N}^+. \]

This completes the proof.

From Lemma 5.5 it follows that corresponding to this system
the sequence \( \{ \nu_i \} \) for which \( C_{k, k + \nu_i} \neq 0 \) is given by \( \nu_i = i \) and the generalized Riemann derivative of \( f \) at \( x \) of order \( (k, \nu_n) = (k, n) \), denoted here by \( D_{k, n}f(x) \), if it exists, is given by

\[
\alpha_k^{n, q}(x, h; f) = \sum_{i=0}^{n} C_{k, k + i, \nu_n} (q) h^{k+i} D_{k, i}f(x)/(k+i)! + o(h^{k+n}).
\]

In particular, if \( k = 1 \) then \( D_{1, n}f(x) \) also gives the Peano derivative \( f_{(n+1)}(x) \).

Theorem 5.6. If the Peano derivative \( f_{(m)}(x), m \in \mathbb{N}^+ \), exists then \( D_{r, m-r}f(x) \) exists and equals \( f_{(m)}(x) \) for \( r = m, m-1, \ldots, 1 \). The converse holds only for \( r = 1 \).

Proof. Let \( f_{(m)}(x) \) exist. Then writing \( f_{(0)}(x) = f(x) \)

\[
\alpha_{r, r, q}^{n}(x, h; f) = \sum_{i=1}^{r} \alpha_i f(x+q^{-1}h)\]

\[
= \alpha_0 f(x) + \sum_{i=1}^{r} \alpha_i \left[ \sum_{j=0}^{m} \left( q^{-1}h \right)^j f_{(j)}(x)/j! + o(h^m) \right] \sum_{i=1}^{r} \alpha_i (q^{-1})^j + o(h^m) \]

\[
= \sum_{j=0}^{m} \left( h^j f_{(j)}(x)/j! \right) \alpha_{r, j} + o(h^m)
\]

(since \( \alpha_{r, 0} = \sum_{i=0}^{r} \alpha_i = 0 \))

\[
= \sum_{l=0}^{m-r} \left( h^{r+l} f_{(r+l)}(x)/(r+l)! + o(h^m) \right).
\]

Hence \( D_{r, m-r}f(x) \) exists and equals \( f_{(m)}(x) \). For \( r = 1 \) let \( D_{1, m-1}f(x) \) exist. Then
\[ a \Delta_{1,1, q}^X (x, h; f) = f(x+h) - f(x) \]
\[ = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} f^{(j)}(x) + o(h^m) \]
\[ = \sum_{l=1}^{m} h^l D^l_{1, l-1} f(x)/l! + o(h^m) . \]

Therefore \( f^{(m)}(x) \) exists and equals \( D^m_{1, m-1} f(x) \).

For \( r > 1 \), \( D^r_{r, n} f(x) \) may exist without existing \( f^{(r+n)}(x) \) for \( n \in \mathbb{N}^+ \). This completes the proof.

Putting \( q = 2 \) in the above we get the expression in (2.1) and \( D^2_{k, 0} f(x) = D^2_k f(x) \).

Remarks. From the previous theorems, the derivatives \( RD_{k, n}^f \), \( RD_{k, n}^f \) and \( D_{k, n} f \) and their corresponding derivate and approximate derivatives are all measurable when \( f \) is approximately continuous. Their approximate analogues are also measurable when \( f \) is measurable. If in particular \( n = 0 \) then these derivatives are the derivatives \( RD_k^f \), \( RD_k^f \) and \( D_k f \) respectively and hence the corresponding derivatives and approximate derivatives are all measurable.

The difference \( \Delta_{k, q} \) considered in Case III is obtained from the difference considered by Ash [2; p.490] by putting all the \( b_i \)'s equal to \( q \) and hence Case III is applicable to the generalized derivative of Ash discussed in [2; p.492].

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