CHAPTER - I

MEASURABILITY OF PEANO DERIVATES AND APPROXIMATE PEANO DERIVATES

1. Introduction.

The measurability of Peano derivates of order \( k \geq 2 \) of a measurable functions does not seem to be covered in the literature, although some authors sometimes used this while proving related results. Since the measurability of the Peano derivates of measurable functions is not automatic it is desirable that it is proved somewhere. In this chapter we prove that Peano derivates and approximate Peano derivates of measurable functions are measurable.


Theorem 2.1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be measurable and let \( k \in \mathbb{N}^+ \). Then the set \( E_k \subset \mathbb{R} \) of points \( x \) such that \( f(k)(x) \) exists finitely is measurable and \( f(k)(x) \) is measurable on \( E_k \). Further \( f^+(k+1)(x), f^-(k+1)(x) \) and \( f^-(k+1)(x) \) are all measurable on \( E_k \).

Proof. For each \( n \in \mathbb{N}^+ \), let

\[
F_n(x) = \frac{k}{n} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+i/n).
\]

Since \( f \) is measurable \( F_n \) is measurable. Also

\[
f(k)(x) = \lim_{t \to 0} \left[ \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+it) \right] t^k = \lim_{t \to 0} \lim_{n \to \infty} F_n(x).
\]
for $x \in E_k$. Hence the first part is clear. We prove the second part. For $n \in \mathbb{N}^+$, define

$$g_n(x) = \sup_{0 < t < 1/n} \gamma_{k+1}(x, t), \quad x \in E_k,$$

where $\gamma_{k+1}(x, t)$ is defined in (1.3) of Chapter 0.

Let $\eta, \varepsilon$ be arbitrarily small positive numbers. Since by the above, $f, f^{(1)}, \ldots, f^{(k)}$ are measurable on $E_k$, there is a perfect set $P = P \subset E_k$ such that $\mu(E_k - P) < \eta$ and the functions $f, f^{(1)}, \ldots, f^{(k)}$ are continuous on $P$ relative to $P$. Let $x_0 \in P$.

We first suppose that $g_n(x_0)$ is finite. Then there is $\xi$, $0 < \xi - x_0 < 1/n$ such that

$$g_n(x_0) - \varepsilon < \gamma_{k+1}(x_0, \xi - x_0).$$

Since $\gamma_{k+1}(x, \xi - x)$, as a function of $x$, is continuous at $x_0$ relative to $P$, there is $\delta$, $0 < \delta < \min [\xi - x_0, 1/n - \xi + x_0]$, such that

$$|\gamma_{k+1}(x, \xi - x) - \gamma_{k+1}(x_0, \xi - x_0)| < \varepsilon \text{ for } x \in P \cap (x_0 - \delta, x_0 + \delta).$$

From (2.1) and (2.2)

$$g_n(x) - 2\varepsilon < \gamma_{k+1}(x, \xi - x) \text{ for } x \in P \cap (x_0 - \delta, x_0 + \delta).$$

Now, if $x \in P \cap (x_0 - \delta, x_0 + \delta)$ then

$$\xi - x = \xi - x_0 + x_0 - x < \xi - x_0 + \delta < \xi - x_0 + 1/n - \xi + x_0 = 1/n$$

and

$$\xi - x = \xi - x_0 + x_0 - x > \delta + x_0 - x > 0.$$

Hence from (2.3)
(2.4) \( g_n(x_o) - 2\epsilon < g_n(x) \) for \( x \in P \cap (x_o - \delta, x_o + \delta) \).

So, \( g_n \) is lower semicontinuous at \( x_o \) relative to \( P \).

Now suppose \( g_n(x_o) = \infty \). Let \( N \in \mathbb{N}^+ \) be arbitrarily large. Then there is \( \xi, 0 < \xi - x_o < 1/n \) such that (2.1) holds with its left hand side replaced by \( N \) and proceeding as above we get the relation (2.3) with its left hand side replaced by \( N - \epsilon \) and applying the same procedure we get instead of (2.4)

(2.5) \( N - \epsilon < g_n(x) \) for \( x \in P \cap (x_o - \delta, x_o + \delta) \)

which gives

(2.6) \( \liminf_{x \to x_o} g_n(x) \geq N - \epsilon \).

Since \( N \) is arbitrary, the left hand side of (2.6) is \( \infty \) showing that \( g_n \) is lower semicontinuous at \( x_o \) relative to \( P \).

If \( g_n(x_o) = -\infty \) the lower semicontinuity of \( g_n \) at \( x_o \) is obvious.

Since \( x_o \) is any point of \( P \), \( g_n \) is lower semicontinuous on \( P \) relative to \( P \). Thus \( g_n \) is measurable on \( P \). Since \( \eta \) is arbitrary, it follows that for each \( \nu \in \mathbb{N}^+ \), there is a perfect set \( P_\nu \subset E \) such that \( \mu(E_k - P_\nu) < 1/\nu \) and \( g_n \) is measurable on \( P_\nu \). Hence \( g_n \) is measurable on \( \bigcup_{\nu=1}^{\infty} P_\nu \). Since \( \mu(E_k - \bigcup_{\nu=1}^{\infty} P_\nu) = \mu(\bigcap_{\nu=1}^{\infty} (E_k - P_\nu)) \leq \mu(E_k - P_\nu) < 1/\nu \) for all \( \nu \in \mathbb{N}^+ \), \( \mu(E_k - \bigcup_{\nu=1}^{\infty} P_\nu) = 0 \) and so \( g_n \) is measurable on \( E_k \). Since

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\[ \overline{f}^{(k+1)}(x) = \lim_{n \to \infty} \frac{g_n(x)}{k+l(x)} , \quad x \in E_k , \]

it follows that \( \overline{f}^{(k+1)} \) is measurable on \( E_k \). Similar arguments hold for \( f^{+ (k+1)}(x) \), \( \overline{f}^{(k+1)}(x) \), and \( f^{- (k+1)}(x) \).

**Corollary 2.2.** Let \( f: \mathbb{R} \to \mathbb{R} \) be measurable. Then the set \( E \subset \mathbb{R} \) of points \( x \) such that \( f(k)(x) \) exists (possibly infinite) is measurable and \( f(k)(x) \) is measurable on \( E \).

**Proof.** Since \( f(k) = \overline{f}^{(k)} = f^{+ (k)} = \overline{f}^{- (k)} = f^{- (k)} \) whenever \( f(k) \) exists (possibly infinite) the proof follows from the above theorem.

3. **Remarks**

1. Since the Cesaro derivatives of order \( k \) (for definition see [11]) are Peano derivatives of order \( k+1 \) of continuous function [4], the Cesaro derivatives are also measurable.

2. In [8,p.54] the authors proved that a finite \( n \)-th Peano derivative of a measurable function \( f \) is measurable and remarked that similar argument would give the measurability of \( \overline{f}^{(n+1)} \) and \( f^{- (n+1)} \) which is not true [18, p.20]. Now from Theorem 2.1 the measurability of \( \overline{f}^{(n+1)} \) and \( f^{- (n+1)} \) follows and the results obtained there remain valid. (We take this opportunity to mention that the defect mentioned in [18,pp.19-20] has been corrected in two ways)
3. It may be of some interest to note that the set
\[ G_{mn} = \{ x : x \in E_k ; |\gamma_{k+1}(x,t)| \leq m \text{ for } 0 < |t| < 1/n \} \]
is measurable. In fact, from the second part of the above proof, the function
\[ h_n(x) = \sup_{0 < |t| < 1/n} |\gamma_{k+1}(x,t)|, \quad x \in E_k \]
is measurable. Since \( G_{mn} = \{ x : x \in E_k ; h_n(x) \leq m \} \) the result follows. Similarly for every \( \varepsilon > 0 \) the set
\[ H_{\varepsilon n} = \{ x : x \in E_k ; |c_k(x,t,f)| \leq \varepsilon \text{ for } 0 < |t| < 1/n \} \]
is measurable, where \( c_k(x,t,f) \) is given by (1.1) of Chapter 0. The measurability of the sets \( G_{mn} \) and \( H_{\varepsilon n} \) are used in various cases before.

4. Measurability of approximate Peano derivates

Lemma 4.1. Let \( Q \subset \mathbb{R} \) be a measurable set and let \( f : Q \rightarrow \mathbb{R} \) be measurable. Let \( k \in \mathbb{N}^+ \). Let

\[ E_o = Q, \quad E_i = \{ x : x \in E_{i-1} ; f(i),_{ap}(x) \text{ exists finitely } \}, \quad 1 \leq i \leq k. \]

Suppose that \( E_i \) is measurable and \( f(i),_{ap} \) is measurable on \( E_i \) for \( 1 \leq i \leq k \). Then \( f^+(k+1),_{ap}(x), \quad f^-(k+1),_{ap}(x) \), \( f^+(k+1),_{ap}(x), \quad f^-(k+1),_{ap}(x) \) are all measurable on \( E_k \).

Proof. We first suppose, as a special case, that \( Q \) is
bounded and closed, $Q = E_k$ and $f, f(1), ap, \ldots, f(k), ap$ are all continuous on $Q$ relative to $Q$. Let

$$P = \{ x : x \in Q ; f^+(k+1), ap(x) \leq a \}$$

where $a \in \mathbb{R}$. Let $D$ be the set of all points of $Q$ which are also points of density of $Q$. For each $x \in Q$ and $n \in \mathbb{N}^+$, let

$$G_n(x) = \{ t : t \geq x, t \in Q; f(t) - \sum_{i=0}^{k} (t-x)^i f(i), ap(x)/i! \leq (a + 1/n)(t-x)^{k+1}/(k+1)! \}.$$

Then $G_n(x)$ is measurable. For $l, m, n \in \mathbb{N}^+$, let

$$Q_{l,m,n} = \{ x : x \in Q; \mu(G_n(x) \cap [x, x+h]) \geq (1-1/m)h \} \text{ for } 0 \leq h \leq 1/l \}.$$

Then

$$Q_{l,m,n} = \{ x : \mu(G_n(x) \cap [x, x+h]) \geq (1-1/m)h \} \text{ for } 0 \leq h \leq 1/l \}.$$

Then

$$P \cap D \subset \bigcap_{n,m} \bigcup_{l} Q_{l,m,n} \subset P.$$

To see this let $x \in P \cap D$. Then $f^+(k+1), ap(x) \leq a$ and $x$ is a point of density of $Q$. So for each $n \in \mathbb{N}^+$, $x$ is a point of right density of $G_n(x)$. Hence for any $m \in \mathbb{N}^+$ there is $l \in \mathbb{N}^+$ such that $x \in Q_{l,m,n}$.

This proves the first inclusion in (4.3). Next, let $x \in Q_{l,m,n}$ for each $m,n \in \mathbb{N}^+$ and some $l \in \mathbb{N}^+$. Then $x$ is a point of right density of $G_n(x)$ for each $n \in \mathbb{N}^+$. Hence $x \in P$, which proves the second inclusion in (4.3).

Let $l,m,n$ be fixed. We show that $Q_{l,m,n}$ is closed. Let $\{x_i\}$ be
any sequence in \( Q_{i,m,n} \) which converges to \( x_0 \). Then since \( Q \) is closed, \( x_0 \in Q \). Let \( 0 \leq h \leq 1/l \). Let

\[
(4.4) \quad H_i = G_n(x_i) \cap [x_i, x_i + h], \quad i \in \mathbb{N}.
\]

Then from (4.2)

\[
(4.5) \quad \mu(H_i) \geq (1-1/m)h \quad \text{for } i \in \mathbb{N}^+.
\]

Let \( \xi \in H \) for infinite values of \( i \). So there is a sequence \( \{i_\nu\} \subset \mathbb{N}^+ \) such that \( \xi \in H \) for all \( \nu \in \mathbb{N}^+ \). Hence \( \xi \in [x_0, x_0 + h] \).

Since \( \xi \in G_n(x_\nu) \) for all \( \nu \in \mathbb{N}^+ \), we have from (4.1) \( \xi \geq x_\nu, \xi \in Q \) and

\[
f(\xi) - \sum_{j=0}^{k} (\xi - x_\nu)^j f_{(j),ap}(x_\nu) / j! \leq (a+1/n)(\xi - x_\nu)^{k+1}/(k+1)!
\]

for all \( \nu \in \mathbb{N}^+ \). Since \( f, f_{(1),ap}, \ldots, f_{(k),ap} \) are continuous on \( Q \), \( \xi \geq x_0, \xi \in Q \) and

\[
f(\xi) - \sum_{j=0}^{k} (\xi - x_0)^j f_{(j),ap}(x_0) / j! \leq (a+1/n)(\xi - x_0)^{k+1}/(k+1)!
\]

and hence \( \xi \in G_n(x_0) \). So \( \xi \in H_0 \). Thus

\[
\bigcap_{r=1}^{\infty} \bigcup_{i=r}^{\infty} H_i \subset H_0.
\]

Since \( F_r = \bigcup_{i=r}^{\infty} H_i \) is decreasing and \( F_r \) is a bounded set,

\[
\lim_{r \to \infty} \mu(F_r) = \mu\left( \bigcap_{r=1}^{\infty} F_r \right) \leq \mu(H_0).
\]

Since \( \mu(H_\nu) \leq \mu(F_r) \) for \( i \geq r \) we have

\[
\limsup_{i \to \infty} \mu(H_\nu) \leq \lim_{r \to \infty} \mu(F_r) \leq \mu(H_0).
\]
Hence from (4.5) \((1-1/m)h \leq \mu(H_0)\). Since \(h\) is arbitrary, this gives, using (4.4) and (4.2), that \(x_0 \in Q,m,n\). So, \(Q,m,n\) is closed. Hence \(\bigcap_{n} \bigcap_{m} \bigcup_{l} Q_{l,m,n}\) is measurable. Since \(P\) and \(P \cap D\) differ by a set of measure zero, \(P\) is measurable.

Now we come to the proof of the general case. Let \(Q\) be bounded. Let \(\varepsilon > 0\) be arbitrary. Since \(E_k\) is measurable and \(f,f_{(1)},ap, \ldots, f_{(k)},ap\) are measurable on \(E_k\), there is a perfect set \(Q_0 \subset E_k\), such that \(\mu(E_k - Q_0) < \varepsilon\) and \(f,f_{(1)},ap, \ldots, f_{(k)},ap\) are continuous on \(Q_0\) relative to \(Q_0\). Then, by the above special case, \(f^+_{(k+1)},ap\) is measurable on \(Q_0\). Since \(\varepsilon\) is arbitrary, \(f^+_{(k+1)},ap\) is measurable on \(E_k\). If \(Q\) is unbounded, we apply the argument on \(Q_n = Q \cap [-n,n], n \in \mathbb{N}^+\). Since \(Q = \bigcup Q_n\), \(f^+_{(k+1)},ap\) is measurable on \(E_k\). Similarly \(f^+_{(k+1)},ap(x), f^-_{(k+1)},ap(x)\) and \(f^-_{(k+1)},ap(x)\) are all measurable on \(E_k\). This completes the proof.

Theorem 4.2. Let \(Q\) be a measurable set and let \(f : Q \to \mathbb{R}\) be measurable. Let \(k \in \mathbb{N}^+\). Let \(E_k\) be the set of points \(x \in Q\) such that \(f(k),ap(x)\) exists and is finite. Then \(E_k\) is measurable and \(f(k),ap\) is measurable on \(E_k\). Further \(f^+_{(k+1)},ap(x), f^-_{(k+1)},ap(x)\), \(f^+_{(k+1)},ap(x)\) and \(f^-_{(k+1)},ap(x)\) are all measurable on \(E_k\).

Proof. Since \(f\) is measurable, the first derivates \(f^+_{(1)},ap\),
etc. are all measurable [37,p.299;13]. Hence the set
\[ E_1 = \{ x : x \in \mathbb{Q} ; -\infty < f_1^+(x) = \alpha_1^+(x) \}
= \{ f_1^-(x) = \alpha_1^-(x) < \infty \} \]
is measurable and \( f_1^+, \alpha_1^+ \) is measurable on \( E_1 \). Putting \( k = 1 \) in the lemma we have that \( f_2^+, \alpha_2^+ \), \( f_2^-, \alpha_2^- \) and \( f_2^-, \alpha_2^- \) are all measurable on \( E_1 \). Hence the set
\[ E_2 = \{ x : x \in E_1 ; -\infty < f_2^+(x) = \alpha_2^+(x) \}
= \{ f_2^-(x) = \alpha_2^-(x) < \infty \} \]
is measurable and \( f_2^+, \alpha_2^+ \) is measurable on \( E_2 \). Putting \( k = 2 \) in the lemma \( f_3^+, \alpha_3^+ \) etc., are all measurable on \( E_2 \). Proceeding inductively the set
\[ E_k = \{ x : x \in E_{k-1} ; -\infty < f_k^+(x) = \alpha_k^+(x) \}
= \{ f_k^-(x) = \alpha_k^-(x) < \infty \} \]
is measurable and \( f_k^+, \alpha_k^+ \) is measurable on \( E_k \). By the above lemma \( f_{k+1}^+, \alpha_{k+1}^+ \) etc., are all measurable on \( E_k \). This completes the proof.

Corollary 4.3. Let \( f : \mathbb{Q} \rightarrow \mathbb{R} \) be measurable. Then the set \( E \subset \mathbb{Q} \) of points \( x \) such that \( f_k^+(x), \alpha_k^+(x) \) exists (possibly infinite) is measurable and \( f_k^+, \alpha_k^+ \) is measurable on \( E \).

The proof is similar to that of Corollary 2.2.