CHAPTER - IV

(S, s) Inventory Policy for Single Replenishment Channel - III

INTRODUCTION

(S, s) inventory models for single replenishment chanel and one server and two serves in tandem have already been developed in the previous chapters. In this paper we have attempted to develop an 'S, s' Inventory model when there is a single replenishment and three operators have to be completed in series before the order can be fulfilled, which obtain in actual practice, like sending order, inspecting, transportation etc. For simplicity of computation we shall analyse in detail in this paper, the case where there are only three such operations in series to be performed. For our present purpose, we shall make the following assumption :-

1) Demand for the items takes place one at a time and the interdemand time is exponentially distributed with mean $\frac{1}{\lambda}$.

2) In accordance with the replenishment rule in the (S,s) inventory policy, orders for replenishment are triggered every time the inventory position touches the level s, and the inventory position is instantaneously raised to the level S by placing an order of size Q (Q = S - s) with the 'Supplier'. Consequently a replenishment order is placed every time that Q units are consumed (demanded). Thus the interarrival time of replenishment order of size Q to the 'Suppliers' will be governed by the Erlang distribution.

$$E_Q = \frac{\lambda(\lambda t)^{Q-1} e^{-\lambda t}}{(Q-1)!}$$

We shall further assume that the time required for processing the replenishment order in stages (1), (2) and (3) in series are exponentially distributed with means $\frac{1}{\mu_1}$, $\frac{1}{\mu_2}$ and $\frac{1}{\mu_3}$ respectively. This primary aim of this model is first to determine the stationary probabilities of net-inventories. The knowledge of these will be used to determine the following parameters of the system.

(i) Probability of stock out — P out

(ii) Expected on hand inventory — I

(iii) Expected net inventory — N

(iv) Expected size of back-order on the books at any point of time — B

(v) Expected size of the back-order on per unit time — E

The optimum values of S and s, and therefore the value of Q = S – s can then be determined on the basis of any one of the following criteria.

(i) Given small probability of stock-out

(ii) Minimising the sum of ordering inventory carrying and back-order costs per unit time.

Analysis of the Model – Let $P_i(t)$ be the probability of the net inventory being equal to i at time t. Again for convenience of analysis, we shall write $i = S - (k_1 + k_2 + k_3)Q - n,\quad k_1, k_2, k_3 \geq 0,\quad 0 \leq n \leq Q-1,$ where k_1 denotes the number of orders to be processed by the first server, k_2 the number of orders to be processed by the second server after having been processed by the first server and k_3 is the number of orders to be finally processed by the third server, all the server being in 'series' and n denotes the number of demand for the item that occur in the interval between two successive placements of replenishment orders with the 'supplier'. Again for the purpose of analysis we shall write $P_i(t)$ in the form $P_{k_1,k_2,k_3,n}$. The dynamic equations of the system will be as follows :

$$P_{k_1,k_2,k_3,n}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_2+\mu_3)\Delta t\} P_{k_1,k_2,k_3,n}(t)$$
$$+ P_{k_1,k_2,k_3,n-1}(t)(\lambda\Delta t) + P_{k_1+1,k_2-1,k_3,n}(t)(\mu_1\Delta t)$$
$$+ P_{k_1,k_2+1,k_3-1,n}(t)(\mu_2\Delta t) + P_{k_1,k_2,k_3+1,n}(t)(\mu_3\Delta t)$$
$$k_1, k_2, k_3 \geq 1;\ 1 \leq n \leq Q-1 \qquad 1(a)$$

$$P_{k_1,k_2,k_3,0}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_2+\mu_3)\Delta t\} P_{k_1,k_2,k_3,0}(t)$$
$$+ P_{k_1-1,k_2,k_3,Q-1}(t)(\lambda\Delta t) + P_{k_1+1,k_2-1,k_3,0}(t)(\mu_1\Delta t)$$
$$+ P_{k_1,k_2+1,k_3-1,0}(t)(\mu_2\Delta t) + P_{k_1,k_2,k_3+1,0}(t)(\mu_3\Delta t)$$
$$k_1, k_2, k_3 \geq 1 \qquad 1(b)$$

$$P_{k_1,k_2,0,n}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_2)\Delta t\} P_{k_1,k_2,0n}(t)$$
$$+ P_{k_1,k_2,0,n-1}(t)(\lambda\Delta t) + P_{k_1+1,k_2-1,0,n}(t)(\mu_1\Delta t)$$
$$+ P_{k_1,k_2,1,n}(t)(\mu_3\Delta t)\quad k_1, k_2 \geq 1;\ 0 \leq n \leq Q-1 \qquad 1(c)$$

$$P_{k_1,0,k_3,n}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_3)\Delta t\} P_{k_1,0,k_3,n}(t)$$
$$+ P_{k_1,0,k_3,n-1}(t)(\lambda\Delta t) + P_{k_1,1,k_3-1,n}(t)(\mu_2\Delta t)$$
$$+ P_{k_1,0,k_3+1,n}(t)(\mu_3\Delta t)\qquad k_1, k_3 \geq 1;\ \leq n \leq Q-1 \qquad 1(d)$$

$$P_{k_1,0,k_3,0}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_3)\Delta t\} P_{k_1,0,k_3,0}(t)$$

$$+ P_{k_1-1,0,k_3,Q-1}(t)(\lambda\Delta t)$$

$$+ P_{k_1,1,k_3-1,0}(t)(\mu_2\Delta t) + P_{k_1,0,k_3+1,0}(t)(\mu_3\Delta t)$$

$$k_1, k_3 \geq 1 \qquad 1(e)$$

$$P_{k_1,k_2,0,0}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_3)\Delta t\} P_{k_1,k_2,0,0}(t)$$

$$+ P_{k_1-1,k_2,0,Q-1}(t)(\lambda\Delta t) + P_{k_1+1,k_2-1,0,0}(t)(\mu_1\Delta t)$$

$$+ P_{k_1,k_2,1,0}(t)(\mu_3\Delta t)$$

$$k_1, k_2 \geq 1 \qquad 1(f)$$

$$P_{k_1,0,0,n}(t+\Delta t) = \{1-(\lambda+\mu_1)\Delta t\} P_{k_1,0,0,n}(t)$$

$$+ P_{k_1,0,0,n-1}(t)(\lambda\Delta t) + P_{k_1,0,1,n}(t)(\mu_3\Delta t)$$

$$k_1 \geq 1; \quad 1 \leq n \leq Q-1 \qquad 1(g)$$

$$P_{k_1,0,0,0}(t+\Delta t) = \{1-(\lambda+\mu_1)\Delta t\} P_{k_1,0,0,0}(t)$$

$$+ P_{k_1-1,0,0,Q-1}(t)(\lambda\Delta t) + P_{k_1,0,1,0}(t)(\mu_3\Delta t)$$

$$k_1 \geq 1 \qquad 1(h)$$

$$P_{0,k_2,k_3,n}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_2+\mu_3)\Delta t\} P_{0,k_2,k_3,n}(t)$$

$$+ P_{0,k_2,k_3,n-1}(t)(\lambda\Delta t) + P_{1,k_2-1,k_3,n}(t)(\mu_1\Delta t)$$

$$+ P_{0,k_2+1,k_3-1,n}(t)(\mu_2\Delta t) + P_{0,k_2,k_3+1,n}(t)(\mu_3\Delta t)$$

$$k_2, k_3 \geq 1; \quad 1 \leq n \leq Q-1 \qquad 1(i)$$

$$P_{0,k_2,0,n}(t+\Delta t) = \{1-(\lambda+\mu_2)\Delta t\} P_{0,k_2,0,n}(t)$$

$$+ P_{0,k_2,0,n-1}(t)(\lambda\Delta t) + P_{1,k_2-1,0,n}(t)(\mu_1\Delta t)$$

$$+ P_{0,k_2,1,n}(t)(\mu_3\Delta t)$$

$$k_2 \geq 1 \, ; \, 1 \leq n \leq Q-1 \qquad 1(j)$$

$$P_{0,0,k_3,n}(t+\Delta t) = \left\{1-(\lambda+\mu_3)\Delta t\right\} P_{0,0,k_3,n}(t) + P_{0,0,k_3,n-1}(t)(\lambda\Delta t)$$

$$+ P_{0,1,k_3-1,n}(t)(\mu_2\Delta t) + P_{0,0,k_3+1,n}(t)(\mu_3\Delta t)$$

$$k_3 \geq 1 \, ; \, 1 \leq n \leq Q-1 \qquad 1(k)$$

$$P_{0,0,0,n}(t+\Delta t) = \left\{1-\lambda\Delta t\right\} P_{0,0,0,n}(t) + P_{0,0,0,n-1}(t)(\lambda\Delta t)$$

$$+ P_{0,0,1,n}(t)(\mu_3\Delta t)$$

$$1 \leq n \leq Q-1 \qquad 1(l)$$

$$P_{0,k_2,k_3,0}(t+\Delta t) = \left\{1-(\lambda+\mu_2+\mu_3)\Delta t\right\} P_{0,k_2,k_3,0}(t)$$

$$+ P_{1,k_2-1,k_3,0}(t)(\mu_1\Delta t) + P_{0,k_2-1,k_3-1,0}(t)(\mu_2\Delta t)$$

$$+ P_{0,k_2,k_3+1,0}(t)(\mu_3\Delta t)$$

$$k_2, k_3 \geq 1 \qquad 1(m)$$

$$P_{0,k_2,0,0}(t+\Delta t) = \left\{1-(\lambda+\mu_2)\Delta t\right\} P_{0,k_2,0,0}(t)$$

$$+ P_{1,k_2-1,0,0}(t)(\mu_1\Delta t) + P_{0,k_2,1,0}(t)(\mu_3\Delta t)$$

$$k_2 \geq 1 \qquad 1(n)$$

$$P_{0,0,k_3,0}(t+\Delta t) = \left\{1-(\lambda+\mu_3)\Delta t\right\} P_{0,0,k_3,0}(t)$$

$$+ P_{0,1,k_3-1,0}(t)(\mu_2\Delta t) + P_{0,0,k_3+1,0}(t)(\mu_3\Delta t) \;\; k_3 \geq 1 \qquad 1(o)$$

$$P_{0,k_2,0,0}(t+\Delta t)=\{1-(\lambda+\mu_2)\Delta t\}P_{0,k_2,0,0}(t)$$

$$+P_{0,k_2-1,0,0}(t)(\mu_1\Delta t) \quad k_2\geq 1 \qquad 1(o)$$

The differential equations (boundary conditions) can be obtained as follows :

$$\frac{d}{dt}P_{k_1,k_2,k_3,n}(t)=-(\lambda+\mu_1+\mu_2+\mu_3)\ P_{k_1,k_2,k_3,n}(t)$$

$$+\lambda P_{k_1,k_2,k_3,n-1}(t)+\mu_1 P_{k_1+1,k_2-1,k_3,n}(t)$$

$$+\mu_2 P_{k_1,k_2+1,k_3-1,n}(t)+\mu_3 P_{k_1,k_2,k_3+1,n}(t)$$

$$k_1,k_2,k_3\geq 1,\quad 0<n\leq Q-1 \qquad \ldots\ 2(a)$$

$$\frac{d}{dt}P_{k_1,k_2,k_3,0}(t)=-(\lambda+\mu_1+\mu_2+\mu_3)\ P_{k_1,k_2,k_3,0}(t)$$

$$+\lambda P_{k_1-1,k_2,k_3,Q-1}(t)+\mu_1 P_{k_1+1,k_2-1,k_3,0}(t)$$

$$+\mu_2 P_{k_1,k_2+1,k_3-1,0}(t)+\mu_3 P_{k_1,k_2,k_3+1,0}(t)$$

$$k_1,k_2,k_3\geq 1 \qquad \ldots\ 2(b)$$

$$\frac{d}{dt}P_{k_1,k_2,0,n}(t)=-(\lambda+\mu_1+\mu_2)\ P_{k_1,k_2,0,n}(t)+\lambda P_{k_1,k_2,0,n-1}(t)$$

$$+\mu_1 P_{k_1+1,k_2-1,0,n}(t)+\mu_3 P_{k_1,k_2,1,n}(t)$$

$$k_1,k_2\geq 1,\quad 1\leq n\leq Q-1 \qquad \ldots\ 2(c)$$

$$\frac{d}{dt}P_{k_1,0,k_3,n}(t)=-(\lambda+\mu_1+\mu_3)\ P_{k_1,0,k_3,n}(t)+\lambda P_{k_1,0,k_3,n-1}(t)$$

$$+\mu_2 P_{k_1,1,k_3-1,n}(t)+\mu_3 P_{k_1,0,k_3+1,n}(t)$$

$$k_1,k_3\geq 1,\quad 1\leq n\leq Q-1 \qquad \ldots\ 2(d)$$

$$\frac{d}{dt} P_{k_1,0,k_3,0}(t) = -(\lambda + \mu_1 + \mu_3)\, P_{k_1,0,k_3,0}(t) + \lambda P_{k_1-1,0,k_3,Q-1}(t)$$

$$+ \mu_3 P_{k_1,0,k_3+1,0}(t) + \mu_2 P_{k_1,1,k_3-1,0}(t)$$

$$k_1, k_3 \geq 1 \qquad \ldots\ 2(e)$$

$$\frac{d}{dt} P_{k_1,k_2,0,0}(t) = -(\lambda + \mu_1 + \mu_2)\, P_{k_1,k_2,0,0}(t) + \lambda P_{k_1-1,k_2,0,Q-1}(t)$$

$$+ \mu_1 P_{k_1+1,k_2-1,0,0}(t) + \mu_3 P_{k_1,k_2,1,0}(t)$$

$$k_1, k_2 \geq 1 \qquad \ldots\ 2(f)$$

$$\frac{d}{dt} P_{k_1,0,0,n}(t) = -(\lambda + \mu_1)\, P_{k_1,0,0,n}(t) + \lambda P_{k_1,0,0,n-1}(t)$$

$$+ \mu_3 P_{k_1,0,1,n}(t)$$

$$k_1 \geq 1; \quad 1 \leq n \leq Q-1 \qquad \ldots\ 2(g)$$

$$\frac{d}{dt} P_{k_1,0,0,0}(t) = -(\lambda + \mu_1)\, P_{k_1,0,0,0}(t) + \lambda P_{k_1-1,0,0,Q-1}(t)$$

$$+ \mu_3 P_{k_1,0,1,0}(t)$$

$$k_1 \geq 1 \qquad \ldots\ 2(h)$$

$$\frac{d}{dt} P_{0,k_2,k_3,n}(t) = -(\lambda + \mu_2 + \mu_3)\, P_{0,k_2,k_3,n}(t) + \lambda P_{0,k_2,k_3,n-1}(t)$$

$$+ \mu_1 P_{1,k_2-1,k_3,n}(t) + \mu_2 P_{0,k_2+1,k_3-1,n}(t) + \mu_3 P_{0,k_2,k_3+1,n}(t)$$

$$k_2, k_3 \geq 1; \quad 1 \leq n \leq Q-1 \qquad \ldots\ 2(i)$$

$$\frac{d}{dt} P_{0,k_2,0,n}(t) = -(\lambda + \mu_2)\, P_{0,k_2,0,n}(t) + \lambda P_{0,k_2,0,n-1}(t)$$

$$+ \mu_1 P_{1,k_2-1,0,n}(t) + \mu_3 P_{0,k_2,1,n}(t)$$

$$k_2 \geq 1; \quad 1 \leq n \leq Q-1 \qquad \ldots\ 2(j)$$

$$\frac{d}{dt} P_{0,0,k_3,n}(t) = -(\lambda + \mu_3) P_{0,0,k_3,n}(t) + \lambda P_{0,0,k_3,n-1}(t)$$

$$+ \mu_2 P_{0,1,k_3-1,n}(t) + \mu_3 P_{0,0,k_3+1,n}(t)$$

$$k_3 \geq 1; \quad 1 \leq n \leq Q-1 \qquad \text{... 2(k)}$$

$$\frac{d}{dt} P_{0,0,0,n}(t) = -\lambda P_{0,0,0,n}(t) + \lambda P_{0,0,0,n-1}(t) + \mu_3 P_{0,0,1,n}(t)$$

$$1 \leq n \leq Q-1 \qquad \text{... 2(l)}$$

$$\frac{d}{dt} P_{0,k_2,k_3,0}(t) = -(\lambda + \mu_2 + \mu_3) P_{0,k_2,k_3,0}(t) + \mu_1 P_{1,k_2-1,k_3,0}(t)$$

$$+ \mu_2 P_{0,k_2+1,k_3-1,0}(t) + \mu_3 P_{0,k_2,k_3+1,0}(t)$$

$$k_2, k_3 \geq 1; \qquad \text{... 2(m)}$$

$$\frac{d}{dt} P_{0,0,0,0}(t) = -\lambda P_{0,0,0,0}(t) + \mu_3 P_{0,0,1,0}(t) \qquad \text{... 2(n)}$$

$$\frac{d}{dt} P_{0,0,k_3,0}(t) = -(\lambda + \mu_3) P_{0,0,k_3,0}(t) + \mu_2 P_{0,1,k_3-1,}(t) + \mu_3 P_{0,0,k_3+1,0}$$

$$k_3 \geq 1 \qquad \text{... 2(o)}$$

$$\frac{d}{dt} P_{0,k_2,0,0}(t) = -(\lambda + \mu_2) P_{0,k_2,0,0}(t) + \mu_1 P_{1,k_2-1,0,0,}(t)$$

$$k_2 \geq 1 \qquad \text{... 2(p)}$$

After putting $r_1 = \frac{\mu_1}{\lambda_1}$; $r_2 = \frac{\mu_2}{\lambda_2}$ and $r_3 = \frac{\mu_3}{\lambda_3}$ the equations for the stationary probabilities of the net inventories can be seen to satisfy the following difference equations.

$$P_{k_1,k_2,k_3,n-1} + r_1 P_{k_1+1,k_2-1,k_3,n} + r_2 P_{k_1,k_2+1,k_3-1,n}$$

$$+ r_3 P_{k_1,k_2,k_3+1,n} - (1 + r_1 + r_2 + r_3) P_{k_1,k_2,k_3,n} = 0$$

$$k_1, k_2, k_3 \geq 1 \quad 1 \leq n \leq Q-1 \qquad \text{... 3(a)}$$

$$P_{k_1-1,k_2,k_3,Q-1} + r_1 P_{k_1+1,k_2-1,k_3,0} + r_2 P_{k_1,k_2+1,k_3-1,0}$$

$$+ r_3 P_{k_1,k_2,k_3+1,0} - (1 + r_1 + r_2 + r_3) P_{k_1,k_2,k_3,0} = 0$$

$$k_1, k_2, k_3 \geq 1 \qquad \ldots 3(b)$$

$$P_{k_1,k_2,0,n-1} + r_1 P_{k_1+1,k_2-1,0,n} + r_3 P_{k_1,k_2,1,n}$$

$$-(1 + r_1 + r_2) P_{k_1,k_2,0,n} = 0$$

$$k_1, k_2, \geq 1; \quad 1 \leq n \leq Q-1 \qquad \ldots 3(c)$$

$$P_{k_1,0,k_3,n-1} + r_2 P_{k_1,1,k_3-1,n} + r_3 P_{k_1,0,k_3+1,n}$$

$$-(1 + r_1 + r_3) P_{k_1,0,k_3,n} = 0$$

$$k_1, k_3, \geq 1 \quad 1 \leq n \leq Q-1 \qquad \ldots 3(d)$$

$$P_{k_1-1,0,k_3,Q-1} + r_2 P_{k_1,1,k_3-1,0,} + r_3 P_{k_1,0,k_3+1,0}$$

$$-(1 + r_1 + r_3) P_{k_1,0,k_3,0} = 0$$

$$k_1, k_3, \geq 1 \qquad \ldots 3(e)$$

$$P_{k_1-1,k_2,0,Q-1} + r_1 P_{k_1+1,k_2-1,0,0,} + r_3 P_{k_1,k_2,1,0}$$

$$-(1 + r_1 + r_2) P_{k_1,k_2,0,0} = 0$$

$$k_1, k_2, \geq 1 \qquad \ldots 3(f)$$

$$P_{k_1,0,0,n-1} + r_3 P_{k_1,0,1,n} - (1 + r_1) P_{k_1,0,0,n} = 0$$

$$k_1 \geq 1; \quad 1 \leq n \leq Q-1 \qquad \ldots 3(g)$$

$$P_{k_1-1,0,0,Q-1} + r_3 P_{k_1,0,1,0} - (1 + r_1) P_{k_1,0,0,0} = 0$$

$$k_1 \geq 1 \qquad \ldots\ 3(h)$$

$$P_{0,k_2,k_3,n-1} + r_1 P_{1,k_2-1,k_3,n} + r_2 P_{0,k_2+1,k_3-1,n} + r_3 P_{0,k_2,k_3+1,n}$$

$$-(1 + r_2 + r_3)\, P_{0,k_2,k_3,n} = 0$$

$$k_2, k_3, \geq 1 \quad 1 \leq n \leq Q-1 \qquad \ldots\ 3(i)$$

$$P_{0,k_2,0,n-1} + r_1 P_{1,k_2-1,k_2-1,0,n} + r_3 P_{0,k_2,1,n}$$

$$-(1 + r_2)\, P_{0,k_2,0,n} = 0$$

$$k_2 \geq 1 \quad 1 \leq n \leq Q-1 \qquad \ldots\ 3(j)$$

$$P_{0,0,k_3,n-1} + r_2 P_{0,1,k_3-1,n} + r_3 P_{0,0,k_3+1,n}$$

$$-(1 + r_3)\, P_{0,0,k_3,n} = 0$$

$$k_3 \geq 1 \quad 1 \leq n \leq Q-1 \qquad \ldots\ 3(k)$$

$$P_{0,0,0,n-1} + r_3 P_{0,0,1,n} - P_{0,0,0,n} = 0$$

$$1 \leq n \leq Q-1 \qquad \ldots\ 3(l)$$

$$r_1 P_{1,k_2-1,k_3,0} + r_2 P_{0,k_2+1,k_3-1,0} + r_3 P_{0,k_2,k_3+1,0}$$

$$-(1 + r_2 + r_3)\, P_{0,k_2,k_3,0} = 0$$

$$k_2, k_3 \geq 1 \qquad \ldots\ 3(m)$$

$$r_3 P_{0,0,1,0} = P_{0,0,0,0} \qquad \ldots\ 3(n)$$

$$r_2 P_{0,1,k_3-1,0} + r_3 P_{0,0,k_3+1,0}$$

$$-(1 + r_3)\, P_{0,0,k_3,0} = 0$$

$k_3 \geq 1$ ... 3(o)

$$r_1 P_{1,k_2-1,0,0} + r_3 P_{0,k_2,1,0}$$

$$-(1+r_2) P_{0,k_2,0,0} = 0$$

$k_2 \geq 1$ ... 3(p)

The equation 3(a) represents the behaviour of the stationary probabilities $P_{k_1,k_2,k_3,n}$ in general and the rest of the equations from 3(b) to 3(p) are boundary conditions.

We shall assume the solution of the equation 3(a) in the form :-

$$P_{k_1,k_2,k_3,n} = C F_1(k_1) F_2(k_2) F_3(k_3) G(n) \qquad \dots (4)$$

$k_1 > 0, k_2 > 0, k_3 > 0, 1 \leq n \leq Q-1$, C being an arbitrary constant to be determined later.

Setting equation (4) in equation 3(a), and deviding throughout by $F_1(k_1) F_2(k_2) F_3(k_3) G(n)$, we obtain the following equation.

$$\frac{G(n-1)}{G(n)} + r_1 \frac{F_1(k_1+1) F_2(k_2-1)}{F_1(k_1) F_2(k_2)} + r_2 \frac{F_2(k_2+1) F_3(k_3-1)}{F_2(k_2) F_3(k_3)}$$

$$+ r_3 \frac{F_3(k_3+1)}{F_3(k_3)} - (1 + r_1 + r_2 + r_3) = 0$$

or $$\frac{G(n-1)}{G(n)} = (1 + r_1 + r_2 + r_3) - r_1 \frac{F_1(k_1+1) F_2(k_2-1)}{F_1(k_1) F_2(k_2)}$$

$$- r_2 \frac{F_2(k_2+1) F_3(k_3-1)}{F_2(k_2) F_3(k_3)} - r_3 \frac{F_3(k_3+1)}{F_3(k_3)} \qquad \dots (5)$$

Since the L.H.S of the equation (5) is a function of n only and the R.H.S is a function of k_1, k_2, k_3, each of them should be equal to a constant, ξ say. We then obtain the following equations :

$$\frac{G(n-1)}{G(n)} = \xi \text{ whene } G(n) = \alpha \frac{1}{\xi^n} \qquad \dots (6)$$

where α is an arbitary constant and

$$r_1 \frac{F_1(k_1+1)}{F_1(k_1)} \cdot \frac{F_2(k_2-1)}{F_2(k_2)} + r_2 \frac{F_2(k_2+1)}{F_2(k_2)} \cdot \frac{F_3(k_3-1)}{F_3(k_3)} + r_3 \frac{F_3(k_3+1)}{F_3(k_3)}$$

$$= 1 + r_1 + r_2 + r_3 - \xi \qquad \dots (7)$$

Since k_1, k_2 and k_3 are independent variables, we shall assure that :–

$$\frac{F_1(k_1+1)}{F_1(k_1)} = \eta_1, \quad \frac{F_2(k_2+1)}{F_2(k_2)} = \eta_2, \text{ and } \frac{F_3(k_3+1)}{F_3(k_3)} = \eta_3,$$

η_1, η_2 and η_3 being constants

whence it follows that,

$F_1(k_1) = \beta\, \eta_1^{k_1}$, $F_2(k_2) = \gamma\, \eta_2^{k_2}$ and

$F_3(k_3) = \delta\, \eta_3^{k_3}$, where β, γ, δ are arbitrary constants.

Hence, we have

$$P_{k_1, k_2, k_3, n} = \alpha \beta \gamma\, \eta_1^{k_1} \eta_2^{k_2} \eta_3^{k_3} \frac{1}{\xi^n}$$

Setting $\alpha \beta \gamma = C$, we have

$$P_{k_1, k_2, k_3, n} = C\, \eta_1^{k_1} \eta_2^{k_2} \eta_3^{k_3} \frac{1}{\xi^n} \qquad \dots (8)$$

$$k_1 > 0,\ k_2 > 0,\ k_3 > 0,\ 1 \le n \le Q - 1$$

Proposition (1) :– Equation (8) will be valid for $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, and $n = 0$

Proof :– setting $n = 1$ in equation 3(a), we obtain

$$P_{k_1,k_2,k_3,0} + r_1 P_{k_1+1,k_2-1,k_3,1} + r_2 P_{k_1,k_2+1,k_3-1,1}$$

$$r_3 P_{k_1,k_2,k_3+1,1} - (1 + r_1 + r_2 + r_3) P_{k_1,k_2,k_3,1}$$

$$\therefore\ P_{k_1,k_2,k_3,0} = C(1 + r_1 + r_2 + r_3)\,\eta_1^{k_1}\eta_2^{k_2}\eta_3^{k_3}\frac{1}{\xi}$$

$$- C r_1 \eta_1^{k_1+1}\eta_2^{k_2-1}\eta_3^{k_3}\frac{1}{\xi} - C r_2 \eta_1^{k_1}\eta_2^{k_2+1}\eta_3^{k_3-1}\frac{1}{\xi}$$

$$- C r_3 \eta_1^{k_1}\eta_2^{k_2}\eta_3^{k_3+1}\frac{1}{\xi}$$

$$= \frac{C\eta_1^{k_1}\eta_2^{k_2-1}\eta_3^{k_3-1}}{\xi}\left\{(1 + r_1 + r_2 + r_3)\eta_2\eta_3 - r_1\eta_1\eta_3 - r_2\eta_2^2 - r_3\eta_2\eta_3^2\right\} \quad \dots (9)$$

Now from equation (7), we have,

$$r_1\frac{\eta_1}{\eta_2} + r_2\frac{\eta_2}{\eta_3} + r_3\eta_3 = (1 + r_1 + r_2 + r_3) - \xi \quad \dots (10)$$

Setting the value of $(1 + r_1 + r_2 + r_3)$ from equation (10) in equation (9), we obtain,

$P_{k_1,k_2,k_3,0} = C\eta_1^{k_1}\eta_2^{k_2}\eta_3^{k_3}$, as was to be verified.

Proposition (2) :– The equation (8) is also valid for
$k_1 \geq 1,\ k_2 = 0,\ k_3 \geq 1,\ 1 \leq n \leq Q - 1$

Proof – Setting $k_2 = 1$, in equation 3(a) we obtain:–

$$P_{k_1 1 k_3 n-1} + r_1 P_{k_1+1,0,k_3,n-1} + r_2 P_{k_1,2,k_3-1,n} + r_3 P_{k_1,1,k_3+1,n}$$

$$= (1 + r_1 + r_2 + r_3) P_{k_1,1,k_3,n}$$

Hence

$$r_1 P_{k_1+1,0,k_3,n} = (1 + r_1 + r_2 + r_3) P_{k_1,1,k_3,n}$$

$$- P_{k_1,1,k_3,n-1} - r_2 P_{k_1,2,k_3-1,n} - r_3 P_{k_1,1,k_3+1,n}$$

$$k_1 \geq 1\ ,\ k_3 \geq 1\ ,\ 1 \leq n \leq Q-1$$

the rest of the proof will follows the sames lines as proposition (1)

Proposition (3) :– Equation (8) is also valid for $k_1 \geq 1$, $k_2 \geq 1$, $k_3 = 0$, $1 \leq n \leq Q-1$

Proof:– Setting $k_3 = 1$ in equation 3(a) we obtain

$$P_{k_1,k_2,1,n-1} + r_1 P_{k_1+1,k_2-1,1,n} + r_2 P_{k_1,k_2+1,0,n}$$

$$+ r_3 P_{k_1,k_2,2,n} - (1 + r_1 + r_2 + r_3) P_{k_1 k_2 1 n}$$

$$\therefore\ r_2 P_{k_1,k_2+1,0,n} = (1 + r_1 + r_2 + r_3) P_{k_1,k_2,1,n}$$

$$- r_1 P_{k_1+1,k_2-1,1,n} - r_3 P_{k_1,k_2,2,n}$$

or, $$= (1 + r_1 + r_2 + r_3) C \eta_1^{k_1} \eta_2^{k_2} \eta_3 \frac{1}{\xi^n}$$

$$- C r_1 \eta_1^{k_1+1} \eta_2^{k_2-1} \eta_3 \frac{1}{\xi^n} - C r_3 \eta_1^{k_1} \eta_2^{k_2} \eta_3^2 \frac{1}{\xi^n}$$

or, $$P_{k_1,k_2,1,n-1} = C \eta_1^{k_1} \eta_2^{k_2} \eta_3 \frac{1}{\xi^{n-1}}$$

The proof will be along the same lines as the previous propositions.

Now, using the results in Proportions (2) and (3) — we obtain from equation 3(c) —

$$r_1 \frac{\eta_1}{\eta_2} + r_3 \eta_3 = 1 + r_1 + r_2 - \xi \qquad \text{... (11)}$$

and from equation 3(d), we obtain

$$r_2 \frac{\eta_2}{\eta_3} + r_3 \eta_3 = 1 + r_2 + r_3 - \xi \qquad \text{... (12)}$$

Solving equations (10), (11) and (12) for η_1, η_2 and η_3, we obtain

$$\eta_1 = \frac{1 + r_1 - \xi}{r_1}$$

$$\eta_2 = \frac{1 + r_1 - \xi}{r_2} \qquad \text{... (13)}$$

$$\eta_3 = \frac{1 + r_1 - \xi}{r_3}$$

Proposition (4) — The equation (8) is also valid for

$k_1 \geq 1, \; k_2 \geq 1, \; k_3 = 0, \; n = 0$

Prof :- Setting $n = 1$, in equation 3(c) we obtain

$$P_{k_1, k_2, 0, 0} + r_1 P_{k_1+1, k_2-1, 0, 1} + r_3 P_{k_1, k_2, 1, 1}$$

$$- (1 + r_1 + r_2) P_{k_1, k_2, 0, 1} = 0$$

$$k_1 \geq 1, \; k_2 \geq 1$$

$$\therefore \; P_{k_1, k_2, 0, 0} = (1 + r_1 + r_2) P_{k_1, k_2, 0, 1} - r_1 P_{k_1+1, k_2-1, 0, 1}$$

$$- r_3 P_{k_1, k_2, 1, 1}$$

Now using the result of proposition (3) we can write the above equation as

$$P_{k_1, k_2, 0, 0} = c(1 + r_1 + r_2) \eta_1^{k_1} \eta_2^{k_2} \frac{1}{\xi} - c r_1 \eta_1^{k_1+1} \eta_2^{k_2-1} \frac{1}{\xi}$$

$$- c r_3 \eta_1^{k_1} \eta_2^{k_2} \eta_3 \frac{1}{\xi}$$

$$= \frac{c \eta_1^{k_1} \eta_2^{k_2-1}}{\xi} \left\{ (1 + r_1 + r_2) \eta_2 - r_1 \eta_1 - r_3 \eta_2 \eta_3 \right\}$$

Setting the value of $(1 + r_1 + r_2)$ from equation (11) — we obtain

$$P_{k_1,k_2,0,0} = \frac{c\,\eta_1^{k_1}\,\eta_2^{k_2-1}}{\xi}\left[\left(r_1\frac{\eta_1}{\eta_2} + r_3\,\eta_3 + \xi\right)\eta_2 - r_1\,\eta_1 - r_3\,\eta_2\,\eta_3\right]$$

$$= c\,\eta_1^{k_1}\,\eta_2^{k_2}$$

Hence proved.

Proposition 5:– The equation (8) is also valid for $k_1 \geq 1,\ k_2 = 0,\ k_3 \geq 1,\ n = 0$

Proof — setting $n = 1$ in equation 3

$$P_{k_1,0,k_3,0} + r_2\,P_{k_1,1,k_3-1,1} + r_3\,P_{k_1,0,k_3+1,1}$$

$$-(1 + r_1 + r_3)\,P_{k_1,0,k_3,1} = 0$$

$$\therefore\ P_{k_1,0,k_3,0} = (1 + r_1 + r_3)\,P_{k_1,0,k_3,1} - r_2\,P_{k_1,1,k_3-1,1}$$

$$- r_3\,P_{k_1,0,k_3+1,1}$$

Using proposition (2), we can write the above as,

$$P_{k_1,0,k_3,0} = (1 + r_1 + r_3)\,c\,\eta_1^{k_1}\,\eta_3^{k_3}\frac{1}{\xi} - r_2\,c\,\eta_1^{k_1}\,\eta_2\,\eta_3^{k_3-1}\frac{1}{\xi}$$

$$- r_3\,c\,\eta_1^{k_1}\,\eta_3^{k_3+1}\frac{1}{\xi}$$

$$= c\,\eta_1^{k_1}\,\eta_3^{k_3-1}\frac{1}{\xi}\left[(1 + r_1 + r_3)\,\eta_3 - r_2\,\eta_2 - r_3\,\eta_3^2\right]$$

Using equation (13) – and simple fying, we obtain

$P_{k_1,0,k_3,0} = c\,\eta_1^{k_1}\,\eta_3^{k_3}$, hence proved.

Propositions 6:— $P_{k_1,0,0,0} = c\,\eta_1^{k_1}, \quad k_1 \geq 1$

setting $n = 1$, in equation 3(g) we obtain

$$P_{k_1,0,0,0} + r_3 P_{k_1,0,1,1} - (1 + r_1) P_{k_1,0,0,1} = 0$$

Substituting the values of $P_{k_1,0,0,0}$, $P_{k_1,0,1,1}$ and $P_{k_1,0,0,1}$ we obtain

$$c\,\eta_1^{k_1} + c\,r_3\,\eta_1^{k_1}\,\eta_3\,\frac{1}{\xi} - (1 + r_1)\,c\,\eta_1^{k_1}\,\frac{1}{\xi}$$

$$= \frac{c\,\eta_1^{k_1}}{\xi}\left[\xi + r_3\,\eta_3 - (1 + r_1)\right] = 0 \qquad \text{[using equation, (13)]}$$

Hence $\quad P_{k_1,0,0,0} = c\,\eta_1^{k_1}$

Now combining the result of propositions 1 to 6, we finally obtain

$$P_{k_1,k_2,k_3,n} = c\,\eta_1^{k_1}\,\eta_2^{k_2}\,\eta_3^{k_3}\,\frac{1}{\xi\,n}$$

$$k_1 \geq 1,\; k_2 \geq 0,\; k_3 \geq 0,\; 0 \leq n \leq Q - 1 \qquad \text{... (14)}$$

We shall refer the above proparly of $P_{k_1,k_2,k_3,n}$ as the shielding effect of k_1.

We shall now proceed to compute

$$P_{0,k_2,k_3,n},\; k_2 \geq 0,\; k_3 \geq 0,\; 0 \leq n \leq Q - 1$$

Determination of $P_{0,k_2,k_3,0}$:- Writing the equation 3(m) in the form of difference equation in $P_{0,k_2,k_3,0}$, we have

$$\left[(1 + r_2 + r_3) - r_2\,E_2\,E_3^{-1} - r_3\,E_3\right] P_{0,k_2,k_3,0} = r_1\,P_{1,k_2-1,k_3,0}$$

$$= c\,r_1\,\eta_1\,\eta_2^{k_2-1}\,\eta_3^{k_3} \qquad \text{... (15)}$$

(Using the shilding effect of k_1)

when E_2, E_3 are shifting operators with respect to η_2 and η_3 respectively Equation (15) can also be written in the form

$$\left[(1 + r_2 + r_3)\, E_3 - r_2 E_2 - r_3 E_3^2 \right] P_{0,k_2,k_3,0}$$

$$= c\, r_1 \eta_1 \eta_2^{k_2 - 1} \eta_3^{k_3 + 1} \qquad \dots (16)$$

where E_3 is shifting operator with respect to k_3 and E_2 is the shifting operator with respect to k_2

Hence the complementory solution of equation (16) is

$$P_{0,k_2,k_3,0} = \left(\frac{1 + r_2 + r_3}{r_2} \right) \left(1 - \frac{r_3 E_3}{1 + r_2 + r_3} \right) E_3^{k_2} A(k_3)$$

where $A(k_3)$ is a function of k_3

and the P.I si given by

$$\frac{C\, r_1 \eta_1 \eta_2^{k_2 - 1} \eta_2^{k_3 + 1}}{(1 + r_2 + r_3)\, \eta_3 - r_2 \eta_2 - r_3 \eta_3^2} = \frac{C\, r_2 \eta_2^{k_2} \eta_3^{k_3}}{r_2 - r_1 + \xi}$$

Because of consideration for convergence — we shall take $A(k_3) = 0$.

Hence we obtain

$$P_{0,k_2,k_3,0} = \frac{C\, r_2 \eta_2^{k_2} \eta_3^{k_3}}{r_2 - r_1 + \xi} \qquad \dots (17)$$

Determination of $P_{0,k_2,k_3,1}$:- Setting $n = 1$ in equation 3(i), and writing the resulting equation in the form of a difference equation

$$\left[(1 + r_2 + r_3) - r_2 E_2 E_3^{-1} - r_2 E_3 \right] P_{0,k_2,k_3,1} = \frac{c\, r_2 \eta_2^{k_2} \eta_3^{k_3}}{r_2 - r_1 + \xi} + \frac{c\, r_2 \eta_2^{k_2} \eta_3^{k_3}}{\xi}$$

[using equation (17) and shielding effect]

or $$\left[(1 + r_2 + r_3)\, E_3 - r_2 E_2 - r_3 E_3^2 \right] P_{0,k_2,k_3,1} = \frac{c\, r_2 \eta_2^{k_2} \eta_3^{k_3 + 1}}{r_2 - r_1 + \xi} + \frac{c\, r_2 \eta_2^{k_2} \eta_3^{k_3 + 1}}{\xi}$$

Again discarding the complementary solution as befor, we have

$$P_{0,k_2,k_3,1} = C\, r_2 \eta_2^{k_2} \eta_3^{k_3} \left[\frac{1}{(r_2 - r_1 + \xi)^2} + \frac{1}{(r_2 - r_1 + \xi)\, \xi} \right]$$

By induction it can be easily verified that

$$P_{0,k_2,k_3,n} = C r_2 \eta_2^{k_2} \eta_3^{k_3} \left[\frac{1}{(r_2 - r_1 + \xi)^{n+1}} + \frac{1}{(r_2 - r_1 + \xi)^n \xi} + \cdot\cdot \frac{1}{r_2 - r_1 + \xi} \cdot \frac{1}{\xi n} \right]$$

$$= \frac{C r_2 \eta_2^{k_2} \eta_3^{k_3} \xi}{(r_2 - r_1)} \left[\frac{1}{\xi^{n+1}} - \frac{1}{(r_2 - r_1 + \xi)^{n+1}} \right] \qquad \ldots (18)$$

Determination of $P_{0,k_2,0,0}$:–

$$P_{0,k_2,0,n-1} + r_1 P_{1,k_2-1,0,n} + r_3 P_{0,k_2,1,n} - (1 + r_2) P_{0,k_2,0,n}$$

We shall write equation 3(j) in the form of a difference equation in $P_{0,k_2,0,n-1}$ follows :—

$$\left[(1 + r_2) E_4 - 1 \right] P_{0,k_2,0,n-1} = C r_1 \eta_1 \eta_2^{k_2-1} \frac{1}{\xi n} + \frac{C r_2 r_3 \eta_2^{k_2} \eta_3 \xi}{r_2 - r_1} \times \left[\frac{1}{\xi^{n+1}} - \frac{1}{(r_2 - r_1 + \xi)^{n+1}} \right]$$

$$\therefore\ P_{0,k_2,0,n-1} = \left(\frac{1}{1 + r_2} \right)^{n-1} A(k_2) + \frac{c r_1 \eta_1 \eta_2^{k_2-1}}{1 + r_2 - \xi} \frac{1}{\xi n - 1} + \frac{c r_2 r_3 \eta_2^{k_2} \eta_3 \xi}{r_2 - r_1} \times \left[\frac{1}{(1 + r_2 - \xi) \xi^n} - \frac{1}{(1 + r_1 - \xi)(r_2 - r_1 + \xi)^n} \right]$$

Setting $n = 1$, we have

$$P_{0,k_2,0,0} = A(k_2) + \frac{c r_2 \eta_2^{k_2}}{1 + r_2 - \xi} + \frac{c r_2^2 \eta_2^{k_2+1} \xi}{r_2 - r_1} \times \left[\frac{1}{(1 + r_2 - \xi)} \frac{1}{\xi} - \frac{1}{(1 + r_1 - \xi)(r_2 - r_1 + \xi)} \right]$$

Now from equation 3(p) it can be easily

Verified that $P_{0,k_2,0,0} = \dfrac{c\, r_2\, \eta_2^{k_2}}{r_2 - r_1 + \xi}$ ... (19)

Setting this value of $P_{0,k_2,0,0}$ in the preceeding equation, it can be easily varified that $A(k_2) = 0$.

Hence $$P_{0,k_2,0,n} = \frac{c\, r_2\, \eta_2^{k_2}}{1 + r_2 + \xi} \cdot \frac{1}{\xi^n} + \frac{c\, r_2^2\, \eta_2^{k_2+1}\, \xi}{(r_2 - r_1)}$$

$$\times \left[\frac{1}{(1 + r_2 - \xi)\, \xi^{n+1}} - \frac{1}{(1 + r_1 - \xi)(r_2 - r_1 + \xi)} \right] \quad \text{... (20)}$$

Determination of $P_{0,0,k_3,n}$:– For this purpose we shall write equation 3(k) in the form of a difference equation in $P_{0,0,k_3,n-1}$

$$\left[(1 + r_3) E_4 - r_3 E_3 E_4 - 1 \right] P_{0,0,k_3,n-1} = r_2 P_{0,1,k_3-1,n} \quad \text{... ..(21)}$$

whence

$$P_{0,0,k_3,n-1} = \left(\frac{1}{1 + r_3 - r_3 E_3} \right)^{n-1} A(K_3) +$$

$$\frac{c\, r_2^2\, \eta_2\, \eta_3^{k_3-1} \xi}{\{(1 + r_3) E_4 - r_3 E_3 E_4 - 1\}(r_2 - r_1)} \left[\frac{1}{\xi^{n+1}} - \frac{1}{(r_2 - r_1 + \xi)^{n+1}} \right]$$

$$= \left(\frac{1}{1 + r_3 - r_3 E_3} \right)^{n-1} A(k_3) + \frac{c\, r_2\, r_3\, \eta_3^{k_3}\, \xi}{(r_2 - r_1)} \left[\frac{1}{(r_3 - r_1)\, \xi^n (r_3 - r_2)(r_2 - r_1 + \xi)} \right]$$

where $A(k_3)$ is an arbitrary function of k_3 setting $n = 1$ in the above equation we obtain

$$P_{0,0,k_3,0} = A(k_3) + \frac{c\, r_2\, r_3\, \eta_3^{k_3}\, \xi}{(r_2 - r_1)} \left[\frac{1}{(r_3 - r_2 \xi)} - \frac{1}{(r_3 - r_2)(r_2 - r_1 + \xi)} \right]$$

Now from equation 3(o) we have

$$\left[r_3 E_3 - (1 + r_3) \right] P_{0,0,k_3,0} = - r_2 P_{0,1,k_3-1,0}$$

$$\therefore \quad P_{0,0,k_3,0} = \left(\frac{1 + r_3}{r_3} \right)^{k_3} A - \frac{c r_2^2 \eta_2 \eta_3^{k_3 - 1}}{\{ r_3 E_3 - (1 + r_3) \} (r_2 - r_1 + \xi)}$$

$$= \left(\frac{1 + r_3}{r_3} \right)^{k_3} A - \frac{c r_2^2 \eta_2 \eta_3^{k_3 - 1}}{\{ r_3 \eta_3 - (1 + r_3) \} (r_2 - r_1 + \xi)}$$

$$= \left(\frac{1 + r_3}{r_3} \right)^{k_3} A + \frac{c r_2 r_3 \eta_3^{k_3}}{(r_3 - r_1 + \xi)(r_2 - r_1 + \xi)} \qquad \cdots \qquad \cdots \quad (22)$$

for consideration of convergence we shall take A = 0

Hence finally

$$P_{0,0,k_3,0} = \frac{c r_2 r_3 \eta_3^{k_3}}{(r_3 - r_1 + \xi)(r_2 - r_1 + \xi)} \qquad \dots (23)$$

Setting equation (23) in equation (22) we obtain

$$A(k_3) = \frac{c r_2 r_3 \eta_3^{k_3} \xi}{(r_3 - r_1 + \xi)(r_3 - r_1)(r_3 - r_2)} \qquad \dots (24)$$

$$P_{0,0,k_3,n} = \frac{c r_2 r_3 \eta_3^{k_3} \xi}{(r_3 - r_1 + \xi)^{n+1} (r_3 - r_2)(r_3 - r_1)}$$

$$+ \frac{c r_2 r_3 \eta_3^{k_3} \xi}{(r_2 - r_1)} \left[\frac{1}{(r_3 - r_1) \xi^{n+1}} - \frac{1}{(r_3 - r_2)(r_2 - r_1 + \xi)^{n+1}} \right] \qquad \dots (25)$$

Determination of $P_{0,0,0,n}$:– Writing equation 3(e) in the form of the following difference equation

$$(E_4 - 1) P_{0,0,0,n-1} = r_3 P_{0,0,1,n}$$

$$P_{0,0,0,n-1} = A + \frac{1}{(E_4 - 1)} \frac{c r_2 r_3 \eta_3^{k_3} \xi}{(r_2 - r_1 + \xi)^{n+1} (r_3 - r_2)} \times (r_3 - r_1)$$

$$+ \frac{c r_2 r_3 \eta_3^{k_3} \xi}{(r_2 - r_1)} \left[\frac{1}{(r_3 - r_1) \xi^{n+1}} - \frac{1}{(r_3 - r_2)} \frac{1}{(r_2 - r_1 + \xi)^{n+1}} \right]$$

$$= A + c r_2 r_3 \eta_3^{k_3} \xi \left[\frac{1}{(r_2 - r_1 + \xi - 1) (r_2 - r_1 + \xi)^n (r_3 - r_2)} \right.$$

$$\left. + \left\{ \frac{1}{(r_2 - r_1)(r_3 - r_1)(1 - \xi) \xi^n} - \frac{1}{(r_2 - r_1)(r_3 - r_2)(r_2 - r_1 + \xi)^n (r_2 - r_1 + \xi - 1)} \right\} \right]$$

Now from equation (23), we get by setting $k_3 = 1$,

$$P_{0,0,1,0} = \frac{c r_2 r_3 \eta_3}{(r_3 - r_1 + \xi) (r_2 - r_1 + \xi)} \quad \text{... (27)}$$

And from boundary condition 3(n), we have from equation (27)

$$P_{0,0,0,0} = \frac{c r_2 r_3^2 \eta_3}{(r_3 - r_1 + \xi) (r_2 - r_1 + \xi)} \quad \text{... (28)}$$

Setting (28) in (26) after setting $n = 1$, it can be verified that

$$A = \frac{C (1 + r_1 - \xi) r_2 r_3 \xi}{(r_3 - r_1 + \xi - 1) (r_2 - r_1 + \xi - 1) (\xi - 1)} \quad \text{... (29)}$$

Determination of C :-

Let P^j be the probability that the inventory positions is

$S - j = s + Q - j, \ \ 0 \le j \le Q - 1$

Then

$$P^j = \sum_{k_3=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} P_{S-(k_1+k_2+k_3)Q-j} \qquad 0 \le j \le Q-1$$

$$= \sum_{k_3 \ge 0}^{\infty} \sum_{k_2 \ge 0}^{\infty} \sum_{k_1 \ge 0}^{\infty} P_{k_1,k_2,k_3,j} = P_{0,0,0,j} + \sum_{k_3 \ge 1}^{\infty} \sum_{k_2 \ge 0}^{\infty} P_{0,k_2,k_3,j} + \sum_{k_2 \ge 1}^{\infty} P_{0,k_2,0,j} + \sum_{k_3 = 1}^{\infty} P_{0,0,k_3,j} + \sum_{k_3 \ge 1}^{\infty} \sum_{k_2 \ge 1}^{\infty} \sum_{k_1 \ge 1}^{\infty} P_{k_1,k_2,k_3,j}$$

From the above, the inventory position varies from $s+1$ to $s+Q=S$, and each will have common probability $\frac{1}{Q}$ [36]

Hence $P^j = \frac{1}{Q}$, for $0 \le j \le Q-1$

Whence, it can be verified that we obtain

$$C = \frac{1}{Q} \times \frac{(r_2 - r_1 + \xi - 1)(r_3 - r_1 + \xi - 1)(\xi - 1)}{(1 + r_1 + \xi)\, r_2 r_3 \xi} \qquad \text{... (30)}$$

The characteristic equation for ξ :–

Setting equation (14) in equation 3(b), which represent a mixed boundary condition at $n=0$ and $n=Q-1$ and dividing throughout by $C\eta_1^{k_1}\eta_2^{k_2}\eta_3^{k_3}$, we obtain

$$1 - \eta_1 \xi^Q = 0$$

Setting the value of η_1 above from (13), we finally obtained the following equation for ξ,

$$\xi^{Q+1} - (1 + r_1)\xi^Q + r_1 = 0 \qquad \text{... (31)}$$

It can be verified that Equations 3(e) and 3(k) yield identical equation for ξ. It is interesting to note that the characteristic equation remains the same as in the case single server or two server in tanden. Thus the characteristic equation will be the same independed of the numbers of servers in tanden. We shall refer to equation (31) as the characteristic equation of the system.

The nature of the roots of the characteritic equation :–

From equation (14), and using equations (13), we obtain

$$P_{k_1, k_2, k_3, n} = C\left(\frac{1+r_1-\xi}{r_1}\right)^{k_1}\left(\frac{1+r_1-\xi}{r_2}\right)^{k_2}\left(\frac{1+r_1-\xi}{r_3}\right)^{k_3}\frac{1}{\xi n} \quad \text{... (32)}$$

$$k_1 \geq 1\,,\ k_2 \geq 0\,,\ 0 \leq n \leq Q-1$$

The parameter ξ in the equation (32) must satisfy the characteristic equation. Because of this reason, we shall now proceed to examine the nature of the root of the characteristic equation.

It can be easily seen that $\xi = 1$, is a root of the characteristic equation (31). However, we cannot use this root in equations (32), since then $P_{k_1, 0_2, k_3, n}$ will be independent of k_1 which is absurd.

Now equation (31) can be written as,

$$(\xi - 1)(\xi^Q - r_1\xi^{Q-1} - r_1\xi^{Q-2} \ldots . - r_1\xi - r_1) = 0 \quad \text{... (33)}$$

Since we have discarded the root $\xi = 1$ we shall have to find a positive root of the derived equation!

$$\xi^Q - r_1\xi^{Q-1} - r_1\xi^{Q-2} - \ldots . - r_1\xi - r_1 = 0 \quad \text{... (34)}$$

Proposition 6 :– Equation (34) has only one positive root ξ_1, where $1 < \xi_1 < 1 + r_1$ irespective of whether Q is odd or even.

Proof:– for convergence, we must have

$$0 < \frac{1 + r_1 - \xi}{r_1} < 1 \quad \text{... 35(a)}$$

$$0 < \frac{1 + r_1 - \xi}{r_2} < 1 \qquad \text{... 35(b)}$$

$$0 < \frac{1 + r_1 - \xi}{r_3} < 1 \qquad \text{... 35(c)}$$

from 35(a) $1 < \xi_1 < 1 + r_1$ . ... 35(d)

and

$$r_2 > 1 + r_1 - \xi \qquad \text{... 35(e)}$$

$$r_3 > 1 + r_1 - \xi \qquad \text{... 35(f)}$$

Thus the root ξ_1 of equation (34) mut satisfy the condition stated in equation 35(d) equations 35(e) and 35(f) lays down lower limits for r_2 and r_3 respectively

The equation (34) will possess only one positive root, irrespective of whether Q is even or odd, which had been proved in earlier Chapter II.

Propostion (7) :– "$\xi_1(Q)$ is an increasing function of Q" — is shown in Chapter III.

Proposition (8) :– It has been proved in Chapter III that $r_1 > 2^{\frac{1}{Q}} - 1$.

Again the probability of k order outstanding is given by

$$\Phi_k = \sum_{k_1 = 0}^{k} \sum_{k_2 = 0}^{k - k_1} \sum_{n = 0}^{Q - 1} P_{k_1, k_2, k - k_1 - k_2, n} \qquad \text{... (36)}$$

The probability of no order outstanding:–

$$\Phi_0 = \sum_{n=0}^{Q-1} P_{0,0,0,n} \qquad \ldots (37)$$

The expected number of orders outstanding will be given by

$$L = \sum_{k=0}^{\infty} k\,\Phi_k \qquad \ldots (38)$$

The probability of the first server idle will be given by

$$\Phi_0^1 = \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \sum_{n=0}^{Q-1} P_{0,k_2,k_3,n} \qquad \ldots (39)$$

The probability of 2nd server idle will be given by

$$\Phi_0^2 = \sum_{k_1=0}^{\infty} \sum_{k_3=0}^{\infty} \sum_{n=0}^{Q-1} P_{k_1,0,k_3,n} \qquad \ldots (40)$$

and the probability that the third server will be idle will be given by

$$\Phi_0^3 = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{n=0}^{Q-1} P_{k_1,k_2,0,n} \qquad \ldots (41)$$

The optimum values of S,s, and Q can be obtained either by putting the probability of stock-out equal to a specified small value or by minimising the sum of ordering, inventory carrying and back order costs.

We have

(i) $P_{out} = 1 - \sum_{j=0}^{S-1} P_{0,0,0,j}$ in the range $0 \le S \le Q-1$ ... 42(a) and in the range $l\,Q = S$

(ii) $$P_{out} = 1 - \sum_{k=0}^{l-1} \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} \sum_{j=0}^{Q-1} P_{k_1,k_2,k-(k_1+k_2),j} \qquad \text{... 42(b)}$$

and when $l\,Q < S < (\,l+1\,)\,Q$

(iii) $$P_{out} = 1 - \left[\sum_{k=0}^{l-1} \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} \sum_{j=0}^{Q-1} P_{k_1,k_2,k-(k_1+k_2),j} + \sum_{k_1=0}^{l} \sum_{k_2=0}^{l-k_1} \sum_{j=0}^{S-lQ-1} P_{k_1,k_2,l-(k_1+k_2),j} \right] \qquad \text{... 42(c)}$$

The expected net inventory N is given by

$$N = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \sum_{j=0}^{Q-1} (\,S - (\,k_1 + k_2 + k_3\,)\,Q - j\,)\,P_{k_1,k_2,k_3,j} \qquad \text{... 43}$$

The expected on hand inventory I is given by

(i) for $0 \le S < Q$

$$\mathrm{I} = \sum_{j=0}^{S-1} (\,S - j\,)\,P_{0,0,0,j} \qquad \text{... 44(a)}$$

(ii) in the range $l\,Q < S < (\,l+1)\,Q$

$$\mathrm{I} = \sum_{k=0}^{l-1} \sum_{k_1=0}^{k} \sum_{k_3=0}^{k-k_1} \sum_{j=0}^{Q-1} [\,S - k\,Q - j\,] \times P_{k_1,k_2,k-(k_1+k_2),j}$$

$$+\sum_{k_1=0}^{l}\sum_{k_2=0}^{l-k}\sum_{j=0}^{S-lQ-1}[S-lQ-j]\times P_{k_1,k_2,l-(k_1+k_2),j} \qquad \text{... 44(b)}$$

(iii) when $lQ = S$

$$I = \sum_{k=0}^{l-1}\sum_{k_1=0}^{k}\sum_{k_3=0}^{k-k_1}\sum_{j=0}^{Q-1}[S-kQ-j]\times P_{k_1,k_2,k-(k_1+k_2),j}$$

The expected number of back order B at any time t on the books is given by B = I – N ... (45)

and the expected number of back order per unit time

E is given by

$$E = \lambda P_{out} \qquad \text{... (46)}$$

Optimal policy of replenishment can either be based on (i) the specified value of P_{out} (Probability of Stock out) or (ii) the minimisation of the expected total cost per unit time.

$$K(S,Q) = \frac{A\lambda}{Q} + IC_1 + \Pi E_1 + \hat{\Pi} B + \lambda c \qquad \text{... (47)}$$

when A = the ordering cost

C_1 = the inventory carrying cost per unit per unit time.

Π = the cost of back order per unit

$\hat{\Pi}$ = the cost of back order per unit time, and

C = the price per unit of the item.