CHAPTER – III

(S, s) Inventory Policy for Single Replenishment Channel – II

INTRODUCTION

An (S,s) inventory model has been developed with the assumption that there are an 'infinite' number of replenishment channels and a single stage service to be rendered for replenishment [10]. A model for an (S,s) inventory policy with single replenishment channel and a single stage of service to be performed for replenishment has been developed by the authors [ ].

In actual practice a number of operations (services) has to be performed in series before replenishment order can be fulfilled, like sending the order, documenting, packaging, transportation etc. For simplicity of computation we shall analyse in detail in this paper, the case where there are only two such operations in series to be performed. For our present purpose, we shall make the following assumptions :–

1) Demand for the item takes place one at a time and the interdemand time is exponentially distributed with mean $\frac{1}{\lambda}$.

2) In accordance with the replenishment rule in the (S, s) inventory policy, orders for replenishment are triggered every time the inventory position touches the level s, and the inventory position is instantaneously raised to the level S by placing an order of size Q (Q = S – s) with the "Supplier". Consequently a replenishment order is placed every time that Q unit are consumed (demanded). Thus the inter-arrival time of replenishment order of size Q to the 'Supplier' will be governed

by the Erlang distribution

$$E_Q = \frac{\lambda(\lambda t)^{Q-1} e^{-\lambda t}}{(Q-1)!}$$

We shall further assume that the time required for processing the replenishment order in stages (1) and (2) in series are exponentially distributed with means $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ respectively.

The primary aim of this model is first to determine the stationary probabilities of net-inventories. The knowledge of these will be used to determine the following parameters of the system.

(i) Probability of stockout – P_{out}

(ii) Expected on hand inventory – I

(iii) Expected net inventory – N

(iv) Expected size of back-order on the books at any point of time – B

(v) Expected size of the back-order on per unit time – E

The optimum values of S and s and therefore the value of Q = S–s can then be determined on the basis of any one of the following criteria

(i) Given small probability of stockout;

(ii) Minimising the sum of ordering, inventory carrying and back-order costs per unit time.

Analysis of the Model

Let $P_i(t)$ be the probability of the net-inventory being equal to i at time t. For convenience of analysis we shall write $i = S - (k_1 + k_2)Q - n$; $k_1, k_2 \geq 0$ $0 \leq n \leq Q-1$, where k_1 denotes the number of orders in the queue to be processed by the first server and k_2 be the number of orders to be finally processed by the second server in tandem in the system, and n denotes the number of demands for the item that have occurred in the interval between two successive placement of replenishment orders with the supplier.

Thus $P_i(t) = P_{S-(k_1+k_2)Q-n};\ k_1, k_2, \geq 0$ and $0 \leq n < Q-1$.

Again for the purpose of analysis, we shall write $P_i(t)$ in the form $P_{k_1,k_2,n}(t)$

The dynamic equations for the system will be as follows:

$$P_{k_1,k_2,n}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_2)\Delta t\} P_{k_1,k_2 n}(t)$$

$$+ P_{k_1,k_2,n-1}(t)(\lambda \Delta t) + P_{k_1+1,k_2-1,n}(t)(\mu_1, \Delta t)$$

$$+ P_{k_1,k_2+1,n}(t)(\mu_2 \Delta t). \qquad k_1 > 0, k_2 > 0, 0 < n \leq Q-1$$

$$P_{k_1,k_2,0}(t+\Delta t) = \{1-(\lambda+\mu_1+\mu_2)\Delta t\} P_{k_1,k_2,0}(t)$$

$$+ P_{k_1-1,k_2,Q-1}(t)(\lambda \Delta t) + P_{k_1+1,k_2-1,0}(t)(\mu_1 \Delta t)$$

$$+ P_{k_1,k_2+1,0}(t)(\mu_2 \Delta t)$$

$$k_1,\ k_2 \geq 1,\ \ 1 \leq n \leq Q-1$$

$$P_{k_1,0,n}(t+\Delta t) = \{1-(\lambda+\mu_1)\Delta t\} P_{k_1,0,n}(t)$$

$$+ P_{k_1,0,n-1}(t)(\lambda \Delta t) + P_{k_1,1,n}(t)(\mu_2 \Delta t)$$

$$k_1 \geq 1,\ \ 1 \leq n \leq Q-1$$

$$P_{k_1,0,0}(t+\Delta t) = \left\{1-(\lambda+\mu_1)\Delta t\right\} P_{k_1,0,0}(t)$$

$$+ P_{k_1-1,0,Q-1}(t)(\lambda\Delta t) + P_{k_1,1,0}(t)(\mu_2\Delta t) \quad k_1 \geq 1$$

$$P_{0,k_2,n}(t+\Delta t) = \left\{1-(\lambda+\mu_2)\Delta t\right\} P_{0,k_2,n}(t)$$

$$+ P_{0,k_2,n-1}(t)(\lambda\Delta t) + P_{1,k_2-1,n}(t)(\mu_1\Delta t)$$

$$+ P_{0,k_2+1,n}(t)(\mu_2\Delta t)$$

$$k_1 \geq 1, \quad 1 \leq n \leq Q-1$$

$$P_{0,0,n}(t+\Delta t) = \left\{1-(\lambda\Delta t)\right\} P_{0,0,n}(t)$$

$$+ P_{0,0,n-1}(t)(\lambda\Delta t) + P_{0,1,n}(t)(\mu_2\Delta t)$$

$$1 \leq n \leq Q-1$$

$$P_{0,k_2,0}(t+\Delta t) = \left\{1-(\lambda+\mu_2)\Delta t\right\} P_{0,k_2,0}(t)$$

$$+ P_{1,k_2-1,0}(t)(\mu_1\Delta t) + P_{0,k_2+1,0}(t)(\mu_2\Delta t) \quad k_2 \geq 1$$

From the above, we obtain the following differential equation.

$$\frac{d\,P_{k_1,k_2,n}(t)}{dt} = -(\lambda+\mu_1+\mu_2)P_{k_1,k_2,n}(t) + \lambda P_{k_1,k_2,n-1}(t)$$

$$+ \mu_1 P_{k_1+1,k_2-1,n}(t) + \mu_2 P_{k_1,k_2+1,n}(t)$$

$$k_1 > 0, k_2 > 0, 0 < n \leq Q-1 \qquad 1(a)$$

The remaining defferential equations (boundary conditions), can be obtained as follows:

$$\frac{d}{dt} P_{k_1,k_2,0}(t) = -(\lambda + \mu_1 + \mu_2)\, P_{k_1,k_2,0}(t) + \lambda P_{k_1-1,k_2,Q-1}(t)$$

$$+ \mu_1 P_{k_1+1,k_2-1,0}(t) + \mu_2 P_{k_1,k_2+1,0}(t) \qquad 1(b)$$

$$k_1, k_2 \geq 1$$

$$\frac{d}{dt} P_{k_1,0,n}(t) = -(\lambda + \mu_1) P_{k_1,0,n}(t)$$

$$+ \lambda P_{k_1,0,n-1}(t) + \mu_2 P_{k_1,1,n}(t)$$

$$k_1 \geq 1, \; 1 \leq n \leq Q-1 \qquad 1(c)$$

$$\frac{d}{dt} P_{k_1,0,0}(t) = -(\lambda + \mu_1) P_{k_1,0,0}(t)$$

$$+ \lambda P_{k_1-1,0,Q-1}(t) + \mu_2 P_{k_1,1,0}(t) \quad k_1 \geq 1 \qquad 1(d)$$

$$\frac{d}{dt} P_{0,k_2,n}(t) = -(\lambda + \mu_2) P_{0,k_2,n} + \lambda P_{0,k_2,n-1}(t)$$

$$+ \mu_1 P_{1,k_2-1,n}(t) + \mu_2 P_{0,k_2+1,n}(t) \qquad 1(e)$$

$$k_2 \geq 1, \; 1 \leq n \leq Q-1$$

$$\frac{d}{dt} P_{0,0,n}(t) = -\lambda P_{0,0,n}(t) + \lambda P_{0,0,n-1}(t) + \mu_2 P_{0,1,n}(t)$$

$$1 \leq n \leq Q-1 \qquad 1(f)$$

$$\frac{d}{dt} P_{0,k_2,0}(t) = -(\lambda + \mu_2) P_{0,k_2,0} + \mu_1 P_{1,k_2-1,0}$$

$$+ \mu_2 P_{0,k_2+1,0}(t) \qquad k_2 \geq 1 \qquad 1(g)$$

$$\frac{d}{dt} P_{0,0,0}(t) = -\lambda P_{0,0,0}(t) + \mu_2 P_{0,1,0}(t) \qquad 1(h)$$

After putting $r_1 = \frac{\mu_1}{\lambda}$, and $r_2 = \frac{\mu_2}{\lambda}$, the equations for the stationary probabilities of the net inventories can be seen to satisfy the following difference equations.

$$P_{k_1,k_2,n-1} + r_1 P_{k_1+1,k_2-1,n} + r_2 P_{k_1,k_2+1,n} - (1 + r_1 + r_2) P_{k_1,k_2,n} = 0$$

$$k_1, k_2 \geq 0, \ 0 \leq n \leq Q-1 \qquad 2(a)$$

$$P_{k_1-1,k_2,Q-1} + r_1 P_{k_1+1,k_2-1,0} + r_2 P_{k_1,k_2+1,0} - (1 + r_1 + r_2) P_{k_1,k_2,0} = 0$$

$$k_1, k_2 \geq 1 \qquad 2(b)$$

$$P_{0,k_2,n-1} + r_1 P_{1\,k_2-1,n} + r_2 P_{0,k_2+1,n} - (1 + r_2) P_{0,k_2,n} = 0 \qquad k_2 \geq 1, 1 \leq n \leq Q-1 \qquad 2(c)$$

$$P_{k_1,0,n-1} + r_2 P_{k_1,1,n} - (1 + r_1) P_{k_1,0,n} = 0$$

$$k_1 \geq 1 \ 1 \leq n \leq Q-1 \qquad 2(d)$$

$$P_{k_1-1,0,Q-1} + r_2 P_{k_1,1,0} - (1 + r_1) P_{k_1,0,0} = 0 \quad k_1 \geq 1 \qquad 2(e)$$

$$P_{0,0,n-1} + r_2 P_{0,1,n} - P_{0,0,n} = 0 \quad 1 \leq n \leq Q-1 \qquad 2(f)$$

$$r_1 P_{1,k_2-1,0} + r_2 P_{0,k_2+1,0} - (1 + r_2) P_{0,k_2,0} = 0 \quad k_2 \geq 1 \qquad 2(g)$$

$$r_2 P_{0,1,0} = P_{0,0,0} \qquad 2(h)$$

We shall solve the equation 2(a); the other equations may be considered as boundary conditions. We shall assume that the solution of equation 2(a) will be of the form

$$P_{k_1, u_2, n} = C F_1(k_1) F_2(k_2) G(n) \tag{3}$$

$k_1 \geq 1$, $k_2 \geq 1$, $1 \leq n \leq Q-1$, and C is an arbitrary constant to be determined later. Setting equation (3) in equation 2(a) and deviding throughout by $F_1(k_1) F_2(k_2) G(n)$, we obtain the following equation.

$$\frac{G(n-1)}{G(n)} + r_1 \frac{F_1(k_1+1)}{F_1(k_1)} \frac{F_2(k_2-1)}{F_2(k_2)} + r_2 \frac{F_2(k_2+1)}{F_2(k_2)} - (1 + r_1 + r_2) = 0$$

$$\frac{G(n-1)}{G(n)} = (1 + r_1 + r_2) - r_1 . \frac{F_1(k_1+1)}{F_1(k_1)} \frac{F_2(k_2-1)}{F_2(k_2)} - r_2 \frac{F_2(k_2+1)}{F_2(k_2)} \tag{4}$$

Since the L.H.S. of equation (4) is a function of n only and the R.H.S is a function of k_1 and k_2, so each of them should be equal to a constant. Denoting this constant by ξ, we have from equation (4).

$$\frac{G(n-1)}{G(n)} = \xi \quad \text{where}$$

$$G(n) = \alpha \frac{1}{\xi^n} \quad \text{where } \alpha \text{ is an arbitrary constant.}$$

Also we have

$$r_1 \frac{F_1(k_1+1)}{F_1(k_1)} . \frac{F_2(k_2-1)}{F_2(k_2)} + r_2 \frac{F_2(k_2+1)}{F_2(k_2)} = (1 + r_1 + r_2) - \xi \tag{5}$$

Since k_1 and k_2 are independent, we shall assume that

$$\frac{F_1(k_1+1)}{F_1(k_1)} = \text{Constant} = \eta_1 \tag{6(a)}$$

$$\text{and } \frac{F_2(k_2+1)}{F_2(k_2)} = \text{Constant} = \eta_2 \text{ say} \tag{6(b)}$$

setting equations 6(a) and 6(b) in equation (5) we obtain

$$r_1 \frac{\eta_1}{\eta_2} + r_2 \eta_2 = (1 + r_1 + r_2) - \xi \tag{7}$$

From equations 6(a) and 6(b) we obtain

$$F_1 (k_1) = \beta \, \eta_1^{k_1} \qquad 8(a)$$

$$F_2 (k_2) = \gamma \, \eta_2^{k_2} \qquad 8(b)$$

respectively, where β and γ are arbitary constants. Hence finally we obtain

$$P_{k_1, k_2, n} = \alpha \, \beta \, \gamma \, (\eta_1)^{k_1} (\eta_2)^{k_2} \frac{1}{\xi^n}$$

$$= C (\eta_1)^{k_1} (\eta_2)^{k_2} \frac{1}{\xi^n}$$

$$k_1 \geq 1 \, , \; k_2 \geq 1 \, , \; 1 \leq n \leq Q - 1 \tag{9}$$

Proposition 1 — The equation (9) is also valid for $k_1 \geq 1$, $k_2 \geq 0$, $0 \leq n \leq Q - 1$. Setting $k_2 = 1$ in equation 2(a)

$$P_{k_1, 1, n-1} + r_1 P_{k_1+1, 0, n} + r_2 P_{k_1, 2, n} - (1 + r_1 + r_2) P_{k_1, 1, n} = 0$$

or $$r_1 P_{k_1+1, 0, n} = C \left\{ (1 + r_1 + r_2) \eta_1^{k_1} \eta_2 \frac{1}{\xi^n} - \eta_1^{k_1} \eta_2 \frac{1}{\xi^{n-1}} - r_2 \eta_1^{k_1} \eta_2^2 \frac{1}{\xi^n} \right\}$$

$$= C \frac{\eta_1^{k_1} \eta_2}{\xi^n} \left[1 + r_1 + r_2 - \xi - r_2 \eta_2 \right] \tag{10}$$

Using equation (7) in equation (10) we obtain

$$r_1 P_{k_1+1,0,n} = C\frac{\eta_1^{k_1}\eta_2}{\xi^n}\left[r_1\frac{\eta_1}{\eta_2} + r_2\eta_2 - r_2\eta_2\right]$$

$$= C\frac{r_1\eta_1^{k_1+1}}{\xi^n}$$

$$\therefore \quad P_{k_1,0,n} = \frac{C\eta_1^{k_1}}{\xi^n} \tag{11}$$

This proves the first part of the proposition. To prove the second part, set $n = 1$ in equations 2(a)

$$P_{k_1 k_2 0} + r_1 P_{k_1+1,k_2-1,1} + r_2 P_{k_1,k_2+1,1}$$

$$-(1 + r_1 + r_2) P_{k_1,k_2,1} = 0$$

we obtain

$$P_{k_1 k_2 0} = (1 + r_1 + r_2) P_{k_1,k_2,1} - r_1 P_{k_1+1,k_2-1,1}$$

$$- r_2 P_{k_1,k_2+1,1}$$

$$= C\left\{(1 + r_1 + r_2)\eta_1^{k_1}\eta_2^{k_2}\frac{1}{\xi} - r_1\eta_1^{k_1+1}\eta_2^{k_2-1}\frac{1}{\xi} - r_2\eta_1^{k_1}\eta_2^{k_2+1}\frac{1}{\xi}\right\}$$

$$= \frac{C\eta_1^{k_1}\eta_2^{k_2-1}}{\xi}\left[(1 + r_1 + r_2)\eta_2 - r_1\eta_1 - r_2\eta_2^2\right] \tag{12}$$

Using equation (7) in equation (12), we obtain

$$P_{k_1,k_2,0} = C\left[\left(r_1\frac{\eta_1}{\eta_2} + r_2\eta_2 + \xi\right)\eta_2 - r_1\eta_1 - r_2\eta_2^2\right]\frac{\eta_1^{k_1}\eta_2^{k_2-1}}{\xi}$$

$$= C\left[r_1\eta_1 + r_2\eta_2^2 + \xi\eta_2 - r_1\eta_1 - r_2\eta_2^2\right]\frac{\eta_1^{k_1}\eta_2^{k_2-1}}{\xi}$$

$$= C\,\eta_1^{k_1}\eta_2^{k_2}$$

which proves the second part. Thus we can finally state that

$$P_{k_1,k_2,n} = C\ \eta_1^{\ k_1}\ \eta_2^{k_2}\left(\frac{1}{\xi}\right)^n \tag{13}$$

for $k_1 \geq 1$, $k_2 \geq 0$, $0 \leq n \leq Q-1$. We shall describe the above property of $P_{k_1,k_2,n}$ as the shielding effect of k_1.

Again from equation 2(d) and using equation (11), we obtain

$$\frac{\eta_1^{k_1}}{\xi^{n-1}} + r_2\,\eta_1^{k_1}\,\eta_2\,\frac{1}{\xi^n} - \left(1 + r_1\right)\eta_1^{k_1}\,\frac{1}{\xi^n}$$

or $\quad \xi + r_2\,\eta_2 = 1 + r_1$

$$\therefore \quad \eta_2 = \frac{1 + r_1 - \xi}{r_2} \qquad 14(a)$$

Setting this value of η_2 in equation (7), we obtain

$$\eta_1 = \frac{1 + r_1 - \xi}{r_1} \qquad 14(b)$$

Now we shall proceed to determine $P_{0,k_2,0}$:– To determine $P_{0,k_2,0}$ we write equation 2(g) in the form of a difference equation in $P_{0,k_2,0}$ as follows:–

$$\left[\,r_2\,E_2 - (\,1 + r_2\,)\,\right] P_{0,k_2,0} = -\,r\,P_{1,k_2-1,0} \tag{15}$$

where E_2 is the shifting operator with respect to k_2. The complementary solution of equation (15) is given by

$$A\left(\frac{1 + r_2}{r_2}\right)^{k_2} \tag{16}$$

where A is an arbitrary constant.

The particular solution is given by

$$-\frac{r_1 P_{1,k_2-1,0}}{r_2 E_2-(1+r_2)}=\frac{C r_1 \eta_1 \eta_2^{k_2-1}}{(1+r_2)-r_2\eta_2}; \quad k_2 \geq 1 \tag{17}$$

$$\therefore \quad P_{0,k_2,0} = A\left(\frac{1+r_2}{r_2}\right)^{k_2}+\frac{C r_1 \eta_1 \eta_2^{k_2-1}}{(1+r_2)-r_2\eta_2}; \; k_2 \geq 1 \tag{18}$$

For convergence consideration, we take $A = 0$

Thus finally we obtain

$$P_{0,k_2,0} = \frac{C r_1 \eta_1 \eta_2^{k_2-1}}{(1+r_2)-r_2\eta_2}=\frac{C(1+r_1-\xi)\eta_2^{k_2-1}}{r_2-r_1+\xi} \quad k_2 \geq 1 \tag{19}$$

Setting $k_2 = 1$ in equation (19) we obtain

$$P_{0,1,0}=\frac{C(1+r_1-\xi)}{r_2-r_1+\xi} \qquad 20(a)$$

by virtue of equation 14(a) and 14(b). Using the boundary condition 2(h), we get

$$P_{0,0,0}=\frac{C(1+r_1-\xi)r_2}{r_2-r_1+\xi} \qquad 20(b)$$

Now we shall proceed to determine $P_{0,k_2,n}$:-

To determine P_{0k_2n} we first set n = 1 in equation 2(c) and write the resulting equation in the form of a difference equation in P_{0k_2n} as follows:-

$$\left[r_2 E_2-(1+r_2)\right] P_{0,k_2,1}=-P_{0,k_2,0}-r_1\eta_1\eta_2^{k_2-1}\cdot\frac{1}{\xi} \tag{21}$$

Using equations (19) and (13) in equation (21) we obtain

$$\left[r_2 E_2-(1+r_2)\right] P_{0,k_2,1}=-\frac{C r_1 \eta_1 \eta_2^{k_2-1}}{r_2-r_1+\xi}-r_1\eta_1\eta_2^{k_2-1}\cdot\frac{1}{\xi}$$

The complementary solution is given by $A\left(\frac{1+r_2}{r_2}\right)^{k_2}$ which we discard by putting A = 0 as before, because of convergence consideration. The particular solution is given by

$$P_{0,k_2,1} = C\, r_1\, \eta_1\, \eta_2^{k_2-1}\left[\frac{1}{(r_2-r_1+\xi)^2}+\frac{1}{(r_2-r_1+\xi)}\frac{1}{\xi}\right]$$

It can be easily shown by induction that

$$P_{0,k_2,n} = C\, r_1\, \eta_1\, \eta_2^{k_2-1}\left[\frac{1}{(r_2-r_1+\xi)^{n+1}}+\frac{1}{(r_2-r_1+\xi)^n\,\xi}\right.$$

$$\left.+\ldots\ldots+\frac{1}{(r_2-r_1+\xi)\,\xi^n}\right]$$

$$=\frac{C\, r_1\, \eta_1\, \eta_2^{k_2-1}}{(r_2-r_1+\xi)^{n+1}\,\xi^n}\left[\frac{(r_2-r_1+\xi)^{n+1}-\xi^{n+1}}{r_2-r_1}\right]$$

$$k_2 \geq 1,\ 0 \leq n \leq Q-1,\ r_1 \neq r_2, \qquad 22(a)$$

If $r_1 = r_2$, we obtain from above

$$P_{0,k_2,n} = \frac{C\, r_1\, \eta_1\, \eta_2^{k_2-1}(n+1)}{\xi^{n+1}}, \qquad k_2 \geq 1:\ 0 \leq n \leq Q-1 \qquad 22(b)$$

We now proceed to determine $P_{0,0,n}$:- To determine $P_{0,0,n}$ we shall write equation 2(f) in the form of a difference equation in $P_{0,0,n-1}$ as follows

$$\left(E_3 - 1\right) P_{0,0,n-1} = r_2\, P_{0,1,n} \qquad 23$$

where E_3 is the shifting operator with respect to n.

Using equation 22(a) and performing the operation, we obtain

$$P_{0,0,n-1} = A + \frac{C r_2 (1 + r_1 - \xi)}{r_2 - r_1} \left[\frac{\xi}{(r_2 - r_1 + \xi - 1)(r_2 - r_1 + \xi)^n} - \frac{1}{(\xi - 1)\xi^{n-1}} \right]$$

Setting n = 1 above and using the value of $P_{0,0,0}$ from equation 20(b), we obtain

$$A = \frac{C(1 - r_1 - \xi) r_2 \xi}{(\xi - 1)(r_2 - r_1 + \xi - 1)}$$

Hence finally, we obtain

$$P_{0,0,n-1} = \frac{C(1 + r_1 - \xi) r_2 \xi}{(r_2 - r_1 + \xi - 1)(\xi - 1)} + \frac{C r_2 (1 + r_1 - \xi)}{r_2 - r_1} \times$$

$$\left[\frac{\xi}{(r_2 - r_1 + \xi - 1)(r_2 - r_1 + \xi)^n} - \frac{1}{(\xi - 1)\xi^{n-1}} \right] \quad r_1 \neq r_2$$

whence

$$P_{0,0,n} = C \left[\frac{(1 + r_1 - \xi) r_2 \xi}{(r_2 - r_1 + \xi - 1)(\xi - 1)} + \frac{r_2 (1 + r_1 - \xi)}{r_2 - r_1} \times \right.$$

$$\left. \left[\left\{ \frac{\xi}{(r_2 - r_1 + \xi - 1)(r_2 - r_1 + \xi)^{n+1}} - \frac{1}{(\xi - 1)\xi^n} \right\} \right] \right] \qquad 24(a)$$

$$0 \leq n \leq Q - 1, \; r_1 \neq r_2$$

Again if $r_1 = r_2$, we obtain from above

$$P_{0,0,n} = C \left[\frac{(1 + r_1 - \xi) r_2 \xi}{(\xi - 1)^2} + \frac{\xi + (n + 1)(\xi - 1)}{(\xi - 1)^2 \xi^{n+1}} \right] \qquad 24(b)$$

Now the inventory position will have the Q-values, s + 1, . . s + Q = S. Also the probability of each of the inventory positions = $\frac{1}{Q}$ [36].

Let p^j be the probability that the inventory position is $s + j = S - (k_1 + k_2)Q - j,\ 0 \le j \le Q-1,\ k_1 \ge 0,\ k_2 \ge 0$

$$P^j = \sum_{k_2=0}^{\infty}\sum_{k_1=0}^{\infty} P_{k_1,k_2,j} = \frac{1}{Q} \qquad (25)$$

$$\text{or,}\quad P_{0,0,j} + \sum_{k_2=1}^{\infty} P_{0,k_2,j} + \sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} P_{k_1,k_2,j} = \frac{1}{Q} \qquad (26)$$

Using equations, 24(a), 22(a), and (13), we obtain,

$$\frac{C(1 + r_1 - \xi)\, r_2 \xi}{(r_2 - r_1 + \xi - 1)(\xi - 1)} = \frac{1}{Q}$$

whence

$$C = \frac{(r_2 - r_1 + \xi - 1)(\xi - 1)}{Q(1 + r_1 - \xi)\, r_2 \xi} \qquad r_1 \ne r_2 \qquad 26(a)$$

and

$$C = \frac{(\xi - 1)^2}{Q(1 + r_1 - \xi)\, r_1 \xi} \qquad \text{when } r_1 = r_2 \qquad 26(b)$$

Now setting equation (13) in equation 2(b) which is a mixed boundary condition at n = 0, and n = Q - 1, and dividing throughout by $C\eta_1^{k_1}\eta_2^{k_2}$, we obtain

$$1 - \eta_1 \quad = 0$$

setting the value of η_1 from 14(b) and simplyfying, we obtain the following equation in ξ.

$$\xi^{Q+1} - (1 + r_1)\xi^Q + r_1 = 0 \tag{27}$$

We shall call equation (27), the characteristic equation of the system. It can be easily verified that equation 2(e) yields the same equation for ξ.

Now equa on (13) can be written as

$$P_{k_1,k_2,n} = C\left(\frac{1 + r_1 - \xi}{r_1}\right)^{k_1}\left(\frac{1 + r_1 - \xi}{r_2}\right)^{k_2}\frac{1}{\xi^n} \qquad 13(b)$$

$$k_1 \geq 1,\ k_2 \geq 0,\ 0 \leq n \leq Q - 1$$

and $P_{0,k_2,n}$ is given by equation 22(a) for $k_2 \geq 1$, $0 \leq n \leq Q - 1$.

It can be easily seen that the equation (27) has $\xi = 1$ as a root. However we can not use this roof of ξ in equation 13(b), since then $P_{k_1,k_2,n}$ will become independent of k_1, which is absurd.

Now equation (27) can be written as

$$(\xi - 1)(\xi^Q - r_1\xi^{Q-1} - r_1\xi^{Q-2} \ldots r_1\xi - r_1) = 0 \tag{28}$$

Since we have discarded the root $\xi = 1$, we shall have to find a positive root of the derived equation:–

$$\xi^Q - r_1\xi^{Q-1} - r_1\xi^{Q-2} \ldots\ldots - r_1\xi - r_1 = 0 \tag{29}$$

Proposition 2 – Equation (29) has only one positive root ξ_1 where $1 < \xi_1 < 1 + r_1$ irrespective of whether Q is odd or even.

Proof – for convergence we must have

$$0 < \frac{1 + r_1 - \xi_1}{r_1} < 1 \qquad 30(a)$$

$$0 < \frac{1 + r_1 - \xi_1}{r_2} < 1 \qquad 30(b)$$

from 30(a) $\quad 1 < \xi_1 < 1 + r_1 \qquad$ 30(c)

from 30 (b) $\quad 0 < 1 + r_1 - \xi_1 < r_2$

or $\quad r_2 > 1 + r_1 - \xi \qquad$ 30(d)

Thus the root ξ_1 satisfies the condition stated. in equation 30(c). Equation 30(d) simply lays down a condition on r_2. Now let us consider the two cases (i) Q is even, (ii) Q is odd.

Case (i) →Q is even — It can be seen by change of sign that equation (27) has at most 2 real positive roots and one real negative root. Thus the derived equation (29) will possess at most one positive and one negative root. From DeGua's rule [31], equation (27) and hence the equation (29) by inheretance, will have at least Q-2 imaginary roots. Since equation (29) is of even degree and the last term is negative, it will possess at least one positive and one negative root. Thus equation (29) will have one positive root, one negative root and (Q - 2) imaginary roots. For our purpose we must take this single positive root of equation (29).

Case (ii) → Q is odd — Since (Q – 1) term are missing from equation (27) and Q – 1 is even, hence by De Gua's rule, the equation (27) has at least Q – 1 imaginary roots. By inheritance, equation (29) will have at least the some number of imaginary roots. Thus the derived equation will possess only one real root.

Again since equation (29) is of odd degree and the sign of the last term is negative, it must have at least one positive root.

Thus equation (29) will possess only one positive root, irrespective of whether Q is even or odd, proving the proposition. We shall denote this root by ξ_1.

Proposition 3 — Let $\xi_1(Q)$ be the only positive root of equation (29), then $\xi_1(Q)$ is an increasing function of Q.

Proof – we have $\xi_1(1)$ is the root of $\xi_1 - r_1 = 0$

$$\xi_1 = r_1 \text{ and } r_1 > 1 \text{ \{by equation (36)\}} \tag{31}$$

Now $\xi_1(2)$ will be the positive root of

$$\xi^2 - r_1\xi - r_1 = 0 \quad (32)$$

Setting $\xi = r_1$ in equation (32), we have

$$r_1^2 - r_1^2 - r_1 = - r_1 < 0$$

Hence $\xi_1(2) > \xi_1(1)$

Now the equation (29) can be written as

$$\xi^Q - r_1 \frac{\xi^Q - 1}{\xi - 1} = 0 \qquad (33)$$

So if $\xi_{1(Q)}$ is the positive root of this equation then

$$\xi_1^Q(Q) - r_1 \frac{\xi_1^Q(Q)}{\xi_1(Q) - 1} = - \frac{r_1}{\xi_1(Q) - 1} \tag{34}$$

Now

$$\xi^{Q+1} - r_1 \frac{\xi^{Q+1} - 1}{\xi - 1}$$

$$= \quad \xi\left(\xi^Q - \frac{r_1\xi^Q}{\xi - 1}\right) + \frac{r_1}{\xi - 1}$$

If we set $\xi_1(Q)$ for ξ in the above expresion then

$$-\xi_1(Q)\frac{r_1}{\xi_1(Q)-1}+\frac{r_1}{\xi_1(Q)-1}=\frac{r_1}{\xi_1(Q)-1}\left\{1-\xi_1(Q)\right\}<0$$

for all Q.

Hence $\xi_1(Q+1)>\xi_1(Q)$. Thus by induction,

$\xi_1(Q)$ is an increasing function of Q as was stated

Now, $\underset{Q\to\alpha}{Lt}\ \xi_1(Q)=1+r_1$

Thus $r_1<\xi_1(1)\ \ldots\ldots\ldots <\xi_1(Q)<1+r_1$.

$\xi_1(Q)$ is tending asymptotically to $1+r_1$.

Now from equation 30(b)

$$r_2>1+r_1-\xi \qquad (35)$$

Thus, as Q increases, the lower bound of r_2 decreases.

If Q = 1, the characteristic equation becomes,

$$\xi^2-(1+r_1)\xi+r_1=0$$

and the derived equation becomes

$$(\xi-r_1)=0$$

Thus $\xi_1=r_1$ and then C becomes

$$C=\frac{(r_2-1)(r_1-1)}{r_1 r_2}$$

$$=(1-\rho_1)(1-\rho_2)$$

and $P_{k_1, k_2} = (1 - \rho_1)(1 - \rho_2)\left(\frac{1}{r_1}\right)^{k_1}\left(\frac{1}{r_2}\right)^{k_2}$

$$= (1 - \rho_1)(1 - \rho_2)\, \rho_1^{k_1} \rho_2^{k_2}$$

which coincides with the result in standard queueing theory.

In the above $r_1 = \frac{1}{\rho_1} = \frac{\mu_1}{\lambda}$

and $r_2 = \frac{1}{\rho_2} = \frac{\mu_2}{\lambda}$

and $r_1 > 1$, $r_2 > 1$ see (35) and (36)

Now in the general case –

we must have

$$0 < \frac{1 + r_1 - \xi_1}{r_1} < 1 \tag{37}$$

and $$0 < \frac{1 + r_1 - \xi_1}{r_2} < 1 \tag{38}$$

from 38(a) $1 < \xi_1 < 1 + r_1$ and from 38(b)

in any case $1 < 1 + r_1 - r_2 < \xi_1 < 1 + r_1$ if $r_2 < r_1$. And when $r_2 \geq r_1$, then $1 < \xi_1 < 1 + r_1$.

Proposition 4 :— $r_1 > 2^{\frac{1}{Q}} - 1$ (36)

Proof :– we have from equation (29) and proposition (3)

$$r_1 = \frac{\xi_1^Q}{\xi_1^{Q-1} + \xi_1^{Q-2} + \ldots + 1}$$

$$> \frac{r_1}{(1 + r_1)^Q - 1}$$

or $\quad (1 + r_1)^Q > 2$

$$r_1 > (2^{\frac{1}{Q}} - 1)$$

whn Q = 1, $r_1 > 1$ as it should be according to the standard queneing theory. This proves the proposition.

Again the probability of k-orders outstanding is given by

$$\Phi_k = \sum_{k_1=0}^{k} \sum_{n=0}^{Q-1} P_{k_1, k-k_1, n}$$

$$= \sum_{n=0}^{Q-1} P_{0,k,n} + \sum_{k_1-1}^{k} \sum_{n=0}^{Q-1} P_{k_1, k-k_1, n}$$

$$= \frac{C r_1 \eta_1 \eta_2^{k-1}}{r_2 - r_1} \left[\frac{\xi_1^Q - 1}{(\xi_1 - 1)\xi_1^{Q-1}} - \frac{\xi_1 (\Theta_1^Q - 1)}{\Theta_1^Q (\Theta_1 - 1)} \right]$$

$$+ \frac{C \eta_1 (\eta_2^k - \eta_1^k)}{\eta_2 - \eta_1} \frac{(\xi_1^Q - 1)}{(\xi_1 - 1)\xi_1^{Q-1}} \tag{39}$$

where $\Theta_1 = r_2 - r_1 + \xi$

The probability of no-orders outstanding will be given by

$$\Phi_0 = \sum_{n=0}^{Q-1} P_{0,0,n}$$

$$= C r_1 r_2 \eta_1 \left[\frac{\xi_1 Q}{(\Theta_1 - 1)(\xi_1 - 1)} + \frac{1}{r_2 - r_1} \left\{ \frac{\xi_1}{(\Theta_1 - 1)^2} \frac{(\Theta_1^Q - 1)}{\Theta_1^Q} \right. \right.$$

$$\left[\left\{-\frac{1}{(\xi_1-1)^2}\frac{\xi_1^Q-1}{\xi_1^{Q-1}}\right\}\right] \tag{40}$$

The expected number of orders outstanding will be given by

$$L=\sum_{k=1}^{\infty} k\,\Phi_k$$

$$=C\,r_1\,\eta_1\,r_2\left[\frac{r_2}{(\Theta_1-1)^2\,\Theta_1}+\frac{r_2}{(\Theta_1-1)^2(r_2-r_1)}\right.$$

$$\left\{\frac{\xi_1^{Q-1}-1}{(\xi-1)(\xi_1^Q-1)}-\frac{\xi_1(\Theta_1^{Q-1}-1)}{\Theta_1^Q(\Theta_1-1)}\right\}$$

$$\left.+\frac{(\xi_1^Q-1)}{(r_2-r_1)\,\xi_1^{Q-1}(\xi_1-1)}\left\{\frac{r_2}{(\Theta_1-1)^2}-\frac{r_1}{(\xi_1-1)^2}\right\}\right] \tag{41}$$

The probability of the first server being idle will be given by

$$\Phi_0^1=\sum_{k_2=0}^{\infty}\sum_{n=0}^{Q-1}P_{0,k_2,n}=\frac{C\,r_1\,\eta_1}{(1-\eta_2)(r_2-r_1)}$$

$$\left[\frac{\xi_1^Q-1}{(\xi_1-1)\,\xi_1^{Q-1}}-\frac{\xi_1(\Theta_1^Q-1)}{\Theta_1^Q(\Theta_1-1)}\right] \tag{42}$$

and the probability that the second server will be idle will be given by

$$\Phi_0^2 = \sum_{k_1=0}^{\infty} \sum_{n=0}^{Q-1} P_{k_1,0,n}$$

$$= C r_1 r_2 \eta_1 \xi_1 \left[\frac{Q}{(\Theta_1 - 1)(\xi_1 - 1)} + \frac{1}{(r_2 - r_1)} \cdot \left\{ \frac{(\Theta_1^Q - 1)}{(\Theta_1 - 1)^2 \Theta_1^Q} - \frac{\xi_1^Q - 1}{\xi_1^Q (\xi_1 - 1)^2} \right\} \right]$$

$$+ \frac{C}{1 - \eta_1} \left[\frac{\xi_1^Q - 1}{(\xi_1 - 1)\xi_1^{Q-1}} \right] \qquad (43)$$

The optimal values of S,s and Q=(S–s) can be obtained either by setting the probability of stockout, P_{out} equal to a specified small value, or by minimising the sum of ordering, inventory carrying and back-order costs.

we have for $0 \leq S < Q$

$$P_{out} = 1 - \sum_{j=0}^{S-1} P_{0,0,j}$$

$$= 1 - C r_1 r_2 \xi_1 \left[\frac{S}{(Q-1)(\xi_1 - 1)} + \frac{1}{r_2 - r_1} \left\{ \frac{\Theta_1^S - 1}{(\Theta_1 - 1)^2 \Theta_1^S} - \frac{(\xi_1^S - 1)}{(\xi_1 - 1)^2 \xi_1^S} \right\} \right] \qquad (44)$$

$$r_1 \neq r_2$$

and for the range $l\,Q < S < (l+1)\,Q$

where l is a positive integer and $l \geq 1$

$$P_{out} = 1 - \left[\sum_{j=0}^{l-1} \sum_{i=0}^{j} \sum_{n=0}^{Q-1} P_{i,j-i,n} + \sum_{i=0}^{l} \sum_{n=0}^{S-lQ-1} P_{i,l-i,n} \right] \qquad 45(a)$$

and for $l\,Q = S$

$$P_{out} = 1 - \left[\sum_{j=0}^{l-1} \sum_{i=0}^{j} \sum_{n=0}^{Q-1} P_{i,j-i,n} \right] \qquad 45(b)$$

Again the expected onhand inventory I is given by

(i) for $0 \le S < Q$

$$I = \sum_{n=0}^{S-1} (S - n) P_{0,0,j} \qquad (46)$$

(ii) in the range $l\,Q < S < (l + 1) Q$

$$I = \sum_{m=0}^{l-1} \sum_{k_1=0}^{m} \sum_{n=0}^{Q-1} \left[S - \{ k_1 Q + (m - k_1) Q \} - n \right] \times P_{k_1, m-k_1, n}$$

$$+ \sum_{k_1=0}^{l} \sum_{n=0}^{S-lQ-1} \left[S - \{ k_1 Q + (l - k_1) Q \} - n \right] P_{k_1, l-k_1, n} \qquad 47(a)$$

(iii) for $l\,Q = S$,

$$I = \sum_{m=0}^{l-1} \sum_{k_1=0}^{m} \sum_{n=0}^{Q-1} \left[S - \{ k_1 Q + (m - k_1) Q \} - n \right] P_{k_1, m-k_1, n} \qquad 47(b)$$

The expected net inventory N is given by

$$N = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{n=0}^{Q-1} \left\{ S - \left(k_1 + k_2 \right) Q - n \right\} P_{k_1, k_2, n} \tag{48}$$

The expected number of back-order B at any time t on the books is given by

$$B = I - N \tag{49}$$

and the expected number of back-orders per unit time E is given by

$$E = \lambda P_{out} \tag{50}$$

Optimal policy of replenishment can either be based on (i) the specified value of P_{out} (probabily of stock-out) or (ii) the minimisation of the expected total cost per unit time

$$K(S, Q) = \frac{A\lambda}{Q} + IC_1 + \Pi E + \hat{\Pi} B + \lambda C \tag{51}$$

where A = the ordering cost

C_1 = the inventory carrying cost per unit per unit time

Π = the cost of back-order per unit

$\hat{\Pi}$ = the cost of back-order per unit per unit time

and C = the price per unit of the item.

Example – Consider the following example : Let λ=5 units/unit time; $r_1 = 3$; $r_2 = 4$, $\mu_1 = 15$, $\mu_2 = 20$; A = \$60; C = \$100, C_1 = \$10, $\hat{H}$ = \$50, H = 0. The following table shows the value for different combinations of S and Q. The table shows that the minimum cost is attained at Q = and S = , the value being equal to \$

Table : The value of K (S,Q) in Dollers for different values of S and Q.

		S = 1	S = 2	S = 3	S = 4	S = 5	S = 6	S = 7	S = 8	S = 9	S = 10
Q = 1	$\xi_1(1)$ = 3.000000	821.66	809.16	798.54	788.38	778.34	768.33	758.33	748.33	738.33	728.33
Q = 2	$\xi_1(2)$ = 3.791282	672.38	665.59	661.76	662.82	662.57	662.77	662.85	662.97	663.07	663.18
Q = 3	$\xi_1(3)$ = 3.951380	639.81	619.07	617.55	618.07	621.539	625.08	628.38	631.735	635.09	638.44
Q = 4	$\xi_1(4)$ = 3.988140	636.62	608.65	595.09	596.37	599.21	604.24	609.27	614.31	619.30	624.31
Q = 5	$\xi_1(5)$ = 3.997060	644.96	612.61	591.78	582.83	585.83	590.09	596.09	602.10	608.11	614.12
Q = 6	$\xi_1(6)$ = 3.999267	658.96	623.66	597.98	582.18	576.36	580.53	585.74	592.40	599.07	605.74
Q = 7	$\xi_1(7)$ = 3.999817	676.12	638.72	609.57	588.89	576.75	573.18	578.18	584.08	591.22	598.36
Q = 8	$\xi_1(8)$ = 3.999954	695.25	656.28	624.51	600.17	583.30	573.93	572.05	577.68	584.08	591.58
Q = 9	$\xi_1(9)$ = 3.999989	715.69	675.49	641.70	614.50	593.95	580.06	572.84	572.29	578.40	585.20

The optional value of K(S,Q) = K (7,8) = 572.0579(Approx)

For detail please turn over page 83 (Appendix)