CHAPTER - II

(S, s) Inventory Policy for Single Replenishment Channel

INTRODUCTION

An (S,s) inventory model has been developed, on the assumption that there are an infinite number of replenishment channels [10]. In practical situations however there are at most a finite number of replenishment channels. In this paper an attempt has been made to analyse an (S, s) policy where there is a single replenishment channel. The procedure for replenishment of such a system is, of course, as follows :– "as soon as the inventory position touches the level s, an order of size Q = S – s, is placed so that the inventory position is instantly raised to the level S." In this paper we shall make the assumption that demand for the item takes place one at a time and that the inter-demand time is exponentially distributed with mean $1/\lambda$. In accordance with the above replenishment rule, orders of size Q (Q = S – s) is placed with the "supplier" every time Q units are consumed. Consequently the inter arrival time of replenishment orders of size Q will be governed by the Erlang distribution.

$$\frac{\lambda(\lambda t)^{Q-1-\lambda t}e}{(Q-1)!}$$

Our primary aim is to determine the stationary distributions of net inventory. The knowledge of the above will enable us to determine the following parameters –

(i) Probability of Stock out — P_{out}

(ii) Expected on-hand inventory — I

(iii) Expected net-inventory — N

(iv) Expected size of the back-orders — B

The knowledge of these parameters will enable us to compute the best values of S, s and Q also either on the basis of probability of stockout or minimising the sum of ordering, Inventory carrying and back-order costs.

Analysis of the Model

Let P_i denotes the Stationary probability of net-inventory being equal to i. We shall write i in the form $i = S - kQ - n,\ k \geq 0,\ 0 \leq n \leq Q-1$, where k denotes the number of orders outstanding and n denotes the number of units demanded in the interval between two successive placements of orders. We observe that $P_i = P_{s-kQ-n} = P_{k,n}$.

It will be advantageous to determine the steady state probabilities $P_{k,n}$. Let $P_{k,n}(t)$ be the probability that there are k – orders outstanding and n demands $(0 \leq n \leq Q-1)$ have occured in time t. The dynamic equations satisfied by $P_{k,n}(t)$, will be as follows:

$$P_{k,n}(t + \Delta t) = P_{k,n}(t)\{1 - (\lambda + \mu)\Delta t\} + P_{k,n-1}(t) . \lambda \Delta t$$

$$+ P_{k+1,n}(t) . \mu \Delta t . k > 0, \quad 0 < n \leq Q-1 \qquad \text{... 1 (a)}$$

$$P_{k,0}(t + \Delta t) = P_{k,0}(t)\{1 - (\lambda + \mu)\Delta t\} + P_{k-1,Q-1}(t) . \lambda \Delta t$$

$$+ P_{k+1,0}(t) . \mu \Delta t . \quad k > 0 \qquad \text{... 1 (b)}$$

$$P_{0,n}(t+\Delta t) = P_{0,n}(t)(1-\lambda \Delta t) + P_{0,n-1}(t) \cdot \lambda \Delta t$$

$$+ P_{1,n}(t) \cdot \mu \Delta t \qquad \ldots 1\,(c)$$

$$P_{0,0}(t+\Delta t) = P_{0,0}(t)(1-\lambda \Delta t) + P_{1,0}(t) \cdot \lambda \Delta t \qquad \ldots 1\,(d)$$

Hence the steady state probabilities $P_{k,n}$ will satisfy the following equations

$$P_{k,n-1} + r_1 P_{k+1,n} - (1+r_1) P_{k,n} = 0, \; k > 0, \; 0 < n \leq Q-1 \qquad \ldots 2(a)$$

$$P_{k-1,Q-1} + r_1 P_{k+1,0} - (1+r_1) P_{k,0} = 0, \; k > 1 \qquad \ldots 2(b)$$

$$P_{0,n-1} + r_1 P_{1,n} - P_{0,n} = 0, \; 0 < n \leq Q-1 \qquad \ldots 2(c)$$

$$r_1 P_{1,0} = P_{0,0} \qquad \ldots 2(d)$$

where $r_1 = \frac{\mu}{\lambda}$

Equations 2 (b) are mixed boundary conditions with respect to n (at $n = Q-1$ and $n = 0$) of equations of state 2(a) and 2(d) is the boundary condition at $n = 0$ of equation 2(c). It may be observed that equations 2(c) are mixed boundary conditions with respect to k (at $k = 0$, and $k = 1$).

We shall assume that the solution of the system of equations 2(a) is of the form:

$$P_{k,n} = C\,F(k)\,G(n), \; k > 0, \; 0 < n \leq Q-1 \qquad \ldots 3$$

where C is an arbitrary constant.

Substituting equation (3) in 2(a) and dividing by $F(k)\,G(n)$, we obtain

$$\frac{G(n-1)}{G(n)} + r_1 \frac{F(k+1)}{F(k)} - (1 + r_1) = 0$$

or $$\frac{G(n-1)}{G(n)} = (1 + r_1) - r_1 \frac{F(k+1)}{F(k)} \qquad \dots 4$$

Since L.H.S is a function of n only and R.H.S. is a function of k only, each one of them should be equal to some constant ξ say.

Then

$$\frac{G(n-1)}{G(n)} = \xi, \text{ where } G(n) = \frac{a}{\xi^n} \qquad \dots 5$$

where a is an arbitrary constant

Put $\frac{F(k+1)}{F(k)} = \eta_1$, (constant)

From equation (4) and (5), we obtain

$$r_1 \eta = 1 + r_1 - \xi$$

$$\therefore \; \eta_1 = \frac{1 + r_1 - \xi}{r_1} \qquad \dots 6$$

for convergence $\eta_1 < 1$ and $\xi < 1$

Hence $F(k) = b\,\eta_1^k$ $\qquad \dots 7$

where b is an arbitrary constant. Setting equations (5) and (7) in (3)

We obtain

$$P_{k,n} = ab\ \eta_1^k \frac{1}{\xi^n}$$

$$= C\ \eta_1^k \frac{1}{\xi^n},\ k > 0,\ 0 < n \leq Q - 1 \qquad \ldots 8$$

where $\quad C = ab$

Proposition (1) :- Equation (8) is also valid for

$$n = 0,\ k \neq 0$$

Proof : - Setting n = 1 in equation 2(a) we

obtain $\quad P_{k,o} + r_1 P_{k+1,1} - (1 + r_1) P_{k,1} = 0$

or $\quad P_{k,o} = (1 + r_1) P_{k,1} - r_1\ P_{k+1,1}$

$$= (1 + r_1)\, C \eta_1^k \frac{1}{\xi} - C r_1\, \eta_1^{k+1} \frac{1}{\xi}$$

$$= \frac{C \eta_1^k}{\xi} (1 + r_1 - r_1 \eta_1)$$

Using equation (6), we obtain

$$P_{k,o} = C \eta_1^k \qquad \text{Proved}$$

Now setting equation (8) in equation 2(b) and making use of equation (6), we obtain the condition

$$\xi^{Q+1} - (1 + r_1)\, \xi^Q + r_1 = 0 \qquad \ldots 9(a)$$

we shall call the above as the characteristic equation for ξ.

It can be easily seen that the equation (9a) has $\xi = 1$ as a root. However we can not use the root of ξ in expression (6), since then $P_{k,n}$ will become independent of k, which is absurd.

Now equation (9a) can be written as

$$(\xi - 1)(\xi^{Q} - r_1 \xi^{Q-1} - r_1 \xi^{Q-2} - \ldots\ldots - r_1 \xi - r_1) = 0$$

Since we have discarded the root $\xi = 1$, we shall have to find a positive root of the derived equation.

$$\xi^{Q} - r_1 \xi^{Q-1} - r_1 \xi^{Q-2} - \ldots\ldots - r_1 \xi - r_1 = 0 \qquad \ldots 9(b)$$

Proposition (2):- Equation (9b) has only one positive root ξ_1, where $1 < \xi_1 < 1 + r_1$, irrespective of whether Q is odd or even.

Prof :- For convergence we must have

$$0 < \frac{1 + r_1 - \xi_1}{r_1} < 1 \quad \text{from equation (6)}$$

which immediately follows $1 < \xi_1 < 1 + r_1$.

Now let us consider the two cases, (i) Q is even and (ii) Q is odd.

Case (i) — Q is even :– It can be seen by change of sign that the equation (9a) has at most two real positive roots and one real negative root. Thus the derived equation (9b) will possess at most one positive and one negative root. From DeGua's rule [3] equation (9a) and hence the equation (9b) by inheritance, will have at least Q – 2 imaginary roots. Since equation (9b) is of even degree and the last term is negative, it will possess at least one positive and one negative root. Thus equation (9b) will have one positive root, one negative root and (Q – 2) imaginary roots. For our purpose, we must take this single positive root of equation (9b).

Case (ii) — Q is odd :– Since (Q – 1) terms are missing from equation (9a) and Q – 1 is even, hence by De Gha's rule, the equation (9a) has at bast Q – 1 imaginary roots. By inharitance, equation (9b) will have at least the same number of imaginary roots. Thus the derived equation (9b) will possess only one real root. Again since equation (9b) is of odd degree and the sign of the last term is negative, it must have at least one positive root.

Thus equation (9b) will possess only one positive root, irrespective of whether Q is even or odd, prouing the proposition. We shall denote this root by ξ_1.

Proposition :- 3 $\qquad r_1 > 2^{\frac{1}{Q}} - 1$

Proof : - We have from equation (9b)

$$r_1 = \frac{\xi_1^Q}{\xi_1^{Q-1} + \xi_1^{Q-2} + \ldots + 1} > \frac{r_1}{(1+r_1)^Q - 1}$$

or $\quad (1 + r_1)^Q > 2$

or $\quad r_1 > 2^{\frac{1}{Q}} - 1$

Thus $\qquad P_{k,n} = C.\dfrac{\eta_1^k}{\xi_1^n},\ k \neq 0 \quad \ldots\ 10$

The value of ξ_1 of ξ would be used in all the expressions involving ξ.

Now we shall determine $P_{o,n}$ from the set of equations 2(c) and 2(d), separately as follows. Writing equations 2(c) in the form of a difference in n, we have

$$(E_2 - 1)\ P_{o,n-1} = r_1 P_{1,n} = \frac{c\, r_1 \eta_1}{\xi_1^n} \qquad \ldots 11$$

where E_2 is the shifting operator with respect to n. The solution of equation (11) is

$$P_{o,n-1} = A - \frac{C r_1 \eta_1}{(\xi_1 - 1)\,\xi_1^{n-1}}$$

$$P_{o,n} = A - \frac{C r_1 \eta_1}{(\xi_1 - 1)\,\xi_1^{n}} \qquad \ldots 12$$

where A is an arbitrary constant. Setting n = 0, in the equation (12) and making use of equation 2(d), we obtain

$$P_{o,n} = \frac{C r_1 \eta_1}{(\xi_1 - 1)} \left[\xi_1 - \frac{1}{\xi_1^n} \right], \quad 0 \le n \le Q - 1 \qquad \ldots 13$$

Now the inventory position (inventory on order + net inventory) varies from s+1 to s+Q = S, each with common probability of $\frac{1}{Q}$ [6]

Now $\sum_{k=0}^{\infty} P_{k,j} = \sum_{k=0}^{\infty} P_{s-kQ-j}$

is the probability of the inventory position $S - j = s + Q - j$, $o \le j \le Q - 1$.

Hence –

$$\sum_{k=o}^{\infty} P_{k,j} = \frac{1}{Q}, \qquad 0 \le j \le Q - 1$$

or $P_{o,j} + \sum_{k=1}^{\infty} P_{k,j} = \frac{1}{Q} \quad 0 \le j \le Q - 1$

Using the appropriate expression for P_{oj}, and P_{kj}, are ultimately obtain

$$C = \frac{\xi_1 - 1}{(1 + r_1 - \xi_1)\,\xi_1\, Q} \qquad \ldots 14$$

where Q = 1, the characteristic equation

becomes $\xi^2 - (1 + r_1)\,\xi + r_1 = 0$

and the derived equation becomes

$$(\xi - r_1) = 0$$

Therefore for Q = 1, $\xi_1 = r_1$ and n = 0, setting

$Q = 1$, $\xi_1 = r_1$ in equation (14)

We obtain

$$C = (1 - \rho)$$

Where $\rho = \frac{1}{r_1}$, so that $P_k = P_{k,o} = (1 - \rho)\,\rho^k$

which is the same as the standard result in queueing theory.

The probability of k-orders outstanding Φ_k, will be given by

$$\Phi_k = \sum_{j=0}^{Q-1} P_{k,j} = \frac{C\,\eta_1^k\,(\xi_1^Q - 1)}{(\xi_1 - 1)\,\xi_1^{Q-1}} = \frac{\eta_1^{k-1}\,(\xi_1^Q - 1)}{r_1\,\xi_1^Q\, Q} \qquad \ldots 15$$

The probability of no-order outstanding will be given by

$$\Phi_o = \sum_{j=0}^{Q-1} P_{o,j} = \frac{C\,r_1\,\eta_1\,\xi_1}{\xi_1 - 1}\,Q - \frac{C\,r_1\,\eta_1\,(\xi_1^Q - 1)}{(\xi_1 - 1)^2\,\xi_1^{Q-1}}$$

$$\Phi_o = 1 - \frac{(\xi_1^Q - 1)}{(\xi_1 - 1)\,\xi_1^Q\,Q} \qquad \ldots 16$$

The average number of orders outstanding is given by

$$L = \sum_{k=1}^{\infty} k\,\Phi_k = \frac{C(\xi_1^Q - 1)\,\eta_1 r_1^2}{(\xi_1 - 1)\,\xi_1^{Q-1}(\xi_1 - 1)^2}$$

$$= \frac{(\xi_1^Q - 1)\,r_1}{\xi_1^Q(\xi_1 - 1)^2\,a} \qquad \ldots 17$$

Incidentally equations (15) and (16) represent the steady state probabilities for the $E_Q \mid M \mid 1 \mid \alpha$ problem in queueing theory.

The optimum values of S, s, and Q can be obtained either by putting the probability of stock-out equal to a specified small value or by minimising the sum of ordering, inventory carrying and back order costs.

We have

$$P_{out} = 1 - \sum_{j=0}^{S-1} P_{oj}, \quad 0 \le S < Q$$

$$= 1 - \frac{1}{Q}\left[S - \frac{\xi_1^s - 1}{(\xi_1 - 1)\,\xi_1^s}\right] \qquad \ldots 18(a)$$

and in the range $kQ < S < (k+1)Q$

$$P_{out} = 1 - \left\{\sum_{i=0}^{k-1}\sum_{j=0}^{Q-1} P_{i,j} + \sum_{j=0}^{S-kQ-1} P_{k,j}\right\}$$

$$= \frac{\eta_1^{k-1}}{Q}\left[\frac{\xi_1^Q - 1}{\xi_1^Q(\xi_1 - 1)} - \frac{\xi_1^{S-kQ} - 1}{\xi_1^{S-kQ}\, r_1}\right] \qquad \text{... 18(b)}$$

and when $kQ = S$

$$P_{out} = 1 - \sum_{i=0}^{k-1} \sum_{j=0}^{Q-1} P_{i,j}$$

$$= 1 - \frac{\eta_1^{k-1}}{Q}\left[\frac{\xi_1^Q - 1}{\xi_1^Q(\xi_1 - 1)}\right] \text{... 18(c)}$$

The expected net inventory N is given by

$$N = \sum_{k=o}^{\infty} \sum_{j=o}^{Q-1} (S - kQ - j)\; P_{k,j} \qquad \text{... 19}$$

The expected on hand inventory I becomes equal to

(a) for the range $O \leq S < Q$

$$I = \sum_{j=o}^{S-1} (S - j)\, P_{oj}$$

$$= \frac{S(S+1)}{2Q} - \frac{S(\xi^s - 1)}{(\xi - 1)\,\xi^s\, Q} + \frac{\xi_i^s - S\,\xi_1 + S - 1}{(\xi_1 - 1)^2\; \xi_1^s\; Q} \qquad \text{... 20(a)}$$

(b) for the range $kQ < S < (k+1)\,G$

$$I = \sum_{l=o}^{k-1} \sum_{j=o}^{Q-1} (S - lQ - j)\, P_{l,j} + \sum_{j=o}^{S-kQ-1} (S - kQ - j)\, P_{k,j} \qquad \text{... 20(b)}$$

(c) for $kQ = S$

$$I = \sum_{l=o}^{k-1} \sum_{j=o}^{Q-1} (S - lQ - j)\, P_{l,j} \qquad \text{... 20(c)}$$

Expected number of back-order B at any time t on the books is given by

$$B = I - N \qquad \text{... 21}$$

and the expected number of back orders on the books per unit time E is given by

$$E = \lambda\, P_{out} \qquad \text{... 22}$$

Optimal policy of replenishment can either be based (1) on specified small value of P_{out} or (2) minimisation of the expected total cost.

$$K(S, Q) = \frac{A\lambda}{Q} + I\,C_1 + \Pi\, E + \hat{\Pi}\, B + \lambda\, C$$

per unit time

where A = ordering cost

C_1 = inventory carrying cost per unit per unit time

Π = cost of back-order per unit

$\hat{\Pi}$ = cost of back-order per unit per unit time

C = price per unit of the item.

Example – Consider the following example : Let $\lambda=5$ units/unit time; $r_1 = \frac{5}{3}$, A = \$60; C = \$100; C_1 = \$10, $\hat{\Pi}$ = \$50, Π = 0. The following table shows the value for different combinations of S and Q. The table shows that the minimum cost is attained at Q = 5 and S = 4, the value being equal to \$ 585.49. Thus the optimum values of S, Q and s are 4, 5 and –1 respectively.

Table : The value of K (S,Q) in Dollers for different values of S and Q.

		S = 1	S = 2	S = 3	S = 4	S = 5
Q = 1	ξ_1 (1) = 1.66	849	855.4			
Q = 2	ξ_1 (2) = 2.37	678.9	670.84	673.22		
Q = 3	ξ_1 (3) = 2.57	644.09	623.29	621.28	626.63	
Q = 4	ξ_1 (4) = 2.63	639.96	612.09	598.40	599.4	
Q = 5	ξ_1 (5) = 2.654	647.706	615.48	594.60	585.49*	588.28
Q = 6	ξ_1 (6) = 2.66	681.09	645.92	620.21	604.29	607.90