CHAPTER - I

1.1 Introduction :- From early stage of civilisation, we are concious about the control and maintenance of stock of physical goods. It is a common problem to every facet of a given economy. Control and maintenance of inventories is one of the vital tasks of management. Almost all the business situations face the following variable factors :-

(a) Demand for products
(b) Production lead times.
(c) Procurement lead times and raw materials.
(d) Production rates.
(e) Block of capital in raw materials.
(f) Block of capital in finished product.
(g) Variation in labour forces.
(h) Capacity of Godown.
(i) Some costs associated with shortage or delayed deleveries.

Deferent situations face the above mentioned factors in some or other form. Problems of productivity and profitability are intimately tied up with proper inventory management.

1.2 Systematic recording and scientific management of inventory have only been started from the early decades of the twentieth century. First analyitical work is "Simple lot size formula" by Ford Harris in 1915 [ ] . This work was also done by R. H. Wilson (1929) which is known as Wilson formula [3]. The Wilson model is based on the assumptions.

(i) Demad for a product is deterministic and uniform over time, and (ii) the only costs considered are ordering and holding costs.

Application of queueing theory in management of inventories has been started from 1957. Consideration of different models are different. One of the cosideration is that in which the units in inventory are regarded as forming a queue, and the arrival of a demand is likened to service completion as because an arrival results in depletion of the queue. The "units" may arrive in inventory singly or in a lot of fixed or variable size. The time between the arrivals of demand may be regarded as the service time (when inventory is not empty). The time of fulfilling the demad may be considered negligible or included in the service time. Another point of view is to considered the arrival of demands as the arrival of customers in the queueing system, the replenishment lead time then can be considered as the service time of the queueing system.

It was in 1957, L.G.Mitten for the first time applied queueing theory to a number of inventory problems [22]. The goods arrived at the selling points at a rate λ with Poisson distributions. Customers are assured to arrive at the selling point with a rate μ also characterised by the Poisson's distribution. In this model the goods are waiting in the inventory and are serviced by customers. The individual items were assumed to arrive randomly being characterised by a Poisson distribution. Hence ordering costs were not considetred in this model. In this model the goods are waiting in the Inventory and are serviced by customers.

In another model he considered a replacement unit being ordered as soon as a unit is sold. The some of the units on hand plus the unfilled replenishment orders is a constant M, which is maximum liability. In his paper orders for replenishment were of size one. He treated each order as a channel of a multi-server queue system.

In 1956, M. Beckman and R. Muth [2] used dynamic Programming technique to determine the optimal (S,s) inventory policy in the presence of lead time under assumptions that the demand is gonerned by Poissons distribution and takes place one unit at a time. In their model they minimised the discounted expected cost over an infinite horizon.

In 1959-H.P. Galliher, Philip M. Morse and N Simond, [10] developed an optimal (S,s) - policy where replenishment doctrine was - as soon as the inventory position (on hand+on order-backorder) touches a level S, an order of size Q=S-s is immedeately placed with the "supplier"so that inventory position is immedeately raised to S. They called the system as (M,R) system, so that M=S, s=R. They assumed that the demand on the primary stock - point by customers takes place one at a time, and follows a Poissons distribution. Regarding the process of replenishment of order of size Q they have assumed that the numbers servers available with the supplier at any time = number of order outstanding, which virtually means that as it will have an infinite number of servers they have developed the models for the following two situations (a) the service time for replenishment for each server is equal to a common finite value.

(b) The sevice times of each of the servers for replenishment is exponentialy ditributed with common parameter μ.

In our model, we first consider single replenishment channel. In next two chapters we consider single replenishment channel but the total service is completed through two/three stations in series (tandem).

In 1961 Martin Beckman made an inventory model [ ] for arbitrary interval and quantity distribution of demand. His assumptions were (1) an abitrary distribution of the length of intervals between successive demands; (2) a distribution of the quantity demanded which is independent of the last quantity demanded and any previous events but may delayed on the time elapsed since the last demand ; (3) unfilled orders are back logged. The delivery time is fixed. The approach of the model yield convenient expressions for loss function and a set of two equations for the reorder points and the minimum order quantity.

1.3 In this thesis application of queueing theory to a certain class of (S,s) Inventory systems has been attempted. Possible examples of these models are inventories of spare parts for maintenance, of military supplies ; of costly items etc. Here we assumed that the demand takes place one at a time and follows poisson distribution which implies that the probability of more than one unit being demanded at a time is zero, for a small period, the probability of a demand for one unit is proportional to the length of time with a probability factor independent of time. It is well known that under these conditions, the time intervals between successive demands obey a negative exponential distribution and that the number of units demanded during any time interval of fixed length is subject to a poisson distribution and mean of interdemand time is $1/\lambda$. The state of the system at each point in time (occurance of demand, placement of order, receipt of order quantity) is recorded and reported as it occurs. In this proces, it is possible to make decesions concerning the operation of the system.

Here we consider the procurement lead time is also a random variable. The procedure for replenishment of such a system is, of course, as follows :-
"as soon as the inventory touches the level s, an order of size Q = (S-s), is placed with the "supplier" every time Q units are consumed. Consequently the inter arrival time of replenishment orders of size Q will be governed by the Erlang distribution :-

$$\frac{\lambda(\lambda t)^{Q-1} e^{-\lambda t}}{(Q-1)\,!}$$

The value of s can be either positive, nagative or zero. Here we consider the orders are served by a single replenishment channel. Our aim is to determine the optimal value of the order quantity Q and the reorder point s.
The optimal value of Q, S and s will be found by minimizing the average annual variable cost K (S,Q). We assume the unit cost of the item is C and it is independent of size Q. The backorder cost per unit backorder is II, and the backorder cost per unit time is II. The cost of operating the information system is independent of Q and s and is ignored. The inventory position varies between s and s+Q = S during each cycle and the length of the cycle is a random variable obviously. Our first aim is to determine the stationary probability distribution of the different inventory positions.
The first system described in chapter II is given in the following diagram.

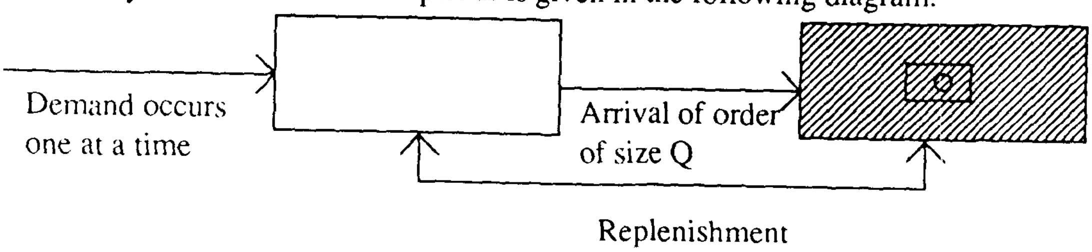

Fig. 1 : 1

In actual practice the replenishment activity may not be performed by a single operator. A number of operation to be performed in series before the replenishment order can be fulfilled.

In chapter III and IV, the method has been extended to a situation where the inventory is replenished after completion of two and three operations in series (tandom).

To define tandom, we consider m service stations arranged in series. The fig of the illustration is given below.

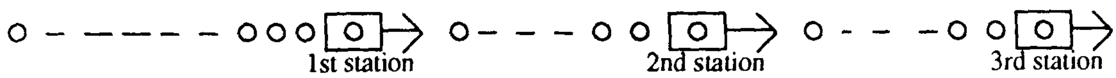

Fig 1 : 2

Lots of size Q of replenishment order arrive one at a time at Ist station. After completion of service at Ist service station the order join the next queue for service in 2nd station and so on. After the service of m thstation the total service is completed. We consider two such stations in chapter III three and such stations in chapter IV. We also consider the service times at stations 1 and 2 be both exponentially distributed with mean service rates μ_1 and μ_2 respectively. The total system thus consists of two distinct queue in chapter III and three distinct queue in chapter IV. We consider the same demand pattern with single replenishment channel but the total service is completed through two stations in series and through three stations in series. The second system is given in the following diagram.

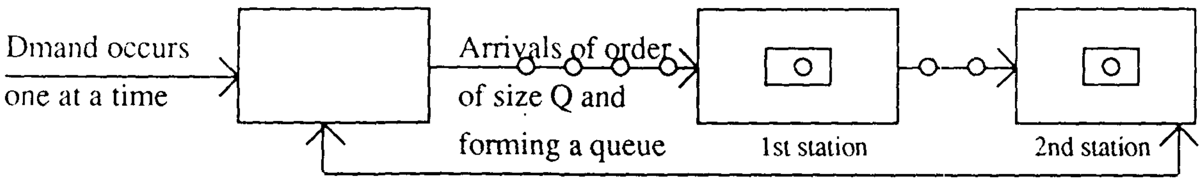

Fig 1 : 3

In chapter III and IV we shall make the following assumptions :-

(i) Demand for the item takes place one at a time and the interdemand time is exponentially distributed with mean $1/\lambda$.

(ii) In accordance with the replenishment rule in the (S,s) inventory policy, replenishment are triggered every time the inventory position touches the level s, and the inventory position is instantly raised to the level S by placing an order of size Q (Q=S-s) with the "Supplier" Consequently replenishment order is placed every time that Q units are consumed. Thus the orders of size Q to the supplier will governed by the Erlang distribution.

$$EQ = \frac{\lambda(\lambda t)^{Q-1} e^{-\lambda t}}{(Q-1)\ !}$$

We shall assume that the time required for processing the replenishment order in stage 1 and 2 in series in exponentially with means $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ respectively.

As in the 2nd chapter, the aim of this model is first to determine the stationary probabilities of net-inventory. The knowledge of this will be ensured to determine the following parameters of the inventory system.

(i) Probability of stock-out — P_{out}.

(ii) Expected on hand inventory I.

(iii) Expected net inventory — N.

(iv) Expected size of backorder years B.

(v) Expected size of the backorder on the books E.

The optimum values of S and s and therefore Q = S-s can then be determine on this basis of any one of the following criteria :-

(i) Given small probability of stockout.

(ii) Minimising the sum of ordering, inventory carrying and backorder cost.

The third system is given in the following diagram :-

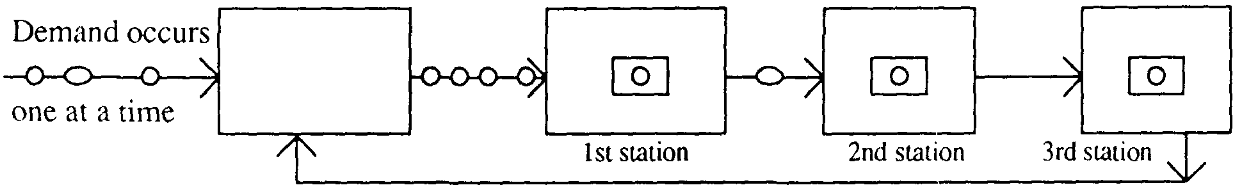

Fig 1 : 4

About cost functions of our models :-

K(S,Q) denotes the total which incorporates the total coost λC per unit time, ordering cost $\frac{A\lambda}{Q}$ where A is the ordering cost for each order, the storage cost for the time IC_1, the total cost of back order per unit $\hat{\Pi}$ E, (where expected number of back orders per unit time is denoted by $E = \lambda P_{out}$) and the cost of back order per unit per unit time $\hat{\Pi}B$.

The expected total cost per unit time would thus be $K(S,Q) = \frac{A\lambda}{Q} + IC_1 \hat{\Pi} E + \hat{\Pi} B + \lambda C$.

The dynamic equations are written in the following way :-

Let Pi denotes the stationary probability of net-inventory being equal to i. We shall write i in the form $i = S-kQ-n$; $k \geq 0, 0 \leq n \leq Q-1$. Where k denotes the number of orders outstanding and n denotes the number of units demanded in the inerval between two successive placements of orders. We observe that $P_i = P_{S-kQ-n} = P_{k,n}$.

Let $P_{k,n}(t)$ be the probability that there are k-orders outstanding and n demands $(0 \leq n \leq Q-1)$ have occured in time t.

$P_{k,n}(t+\Delta t) = P_{k,n}(t) P$ { no arrival of demand and no replenishment of order in $(t, t+\Delta t)$} $+ P_{k,n-1}(t) P$ { one arrival of demand in $(t, t+\Delta t)$} $+ P_{k+1,n}(t) P$ { one replenishment of order in $(t, t+\Delta t)$}

$$= P_{k,n}(t) \left\{1 - (\lambda + \mu)\Delta t\right\} + P_{k,n-1}(t) . \lambda \Delta t + P_{k+1,n}(t) . \mu \Delta t$$

$$k > 0, \quad 0 < n \leq Q-1.$$

In this way we derive the other equations and boundary conditions.

The method of partial difference equations used to solve the following equations of the three models described in Chapter II, III and IV can easily solve the classical model of queueing theory of two stations intrandem [ ].

Let us consider two service facilities arranged in series. Let customers arrive in a Poisson fashion with mean λ for service at service station 1. After completing service at service station 1 the units joi the other queue for service at service station 2. The service time of the two stations follows exponential distribution with mean service rates μ_1 and μ_2 respectively. We donot discuss about the formation of the difference equations. We begin with the steady state equations.

$$P_{k_1-1,k_2} + r_1 P_{k_1+1,k_2-1} + r_2 P_{k_1,k_2+1} - (1 + r_1 + r_2) P_{k_1,k_2} = 0$$

$$k_1, k_2 \geq 1 \qquad \text{... (1)}$$

$$P_{k_1-1,0} + r_2 P_{k_1,1} - (1 + r_1) P_{k_1,0} = 0$$

$$k_1 \geq 1 \qquad \text{... (2)}$$

$$r_1 P_{1,k_2-1} + r_2 P_{0,k_2+1} - (1 + r_2) P_{0,k_2} = 0$$

$$k_2 \geq 1 \qquad \text{... (3)}$$

$$P_{0,0} = r_2 P_{0,1} \qquad \text{... (4)}$$

We shall solve the equation (1); the other equations may be considered as boundary conditions. We shall assume that the solution of equation (1) will be of the form.

$$P_{k_1,k_2} = C F_1(k_1) F_2(k_2) \qquad \text{... (5)}$$

$k_1 \geq 1$, $k_2 \geq 1$, and C be a arbitrary constant to be determined later. Setting equation (5) in equation (1) and deviding throughout by $F_1(k_1)\ F_2(k_2)$, we obtain the following equation.

$$\frac{F_1(k_1-1)}{F_1(k_1)} + r_1 \frac{F_1(k_1+1)}{F_1(k_1)} \cdot \frac{F_2(k_2-1)}{F_2(k_2)} + r_2 \frac{F_2(k_2+1)}{F_2(k_2)}$$

$$-(1 + r_1 + r_2) = 0 \qquad \text{... (6)}$$

Since k_1 and k_2 are independent, we shall assume that

$$\frac{F_1(k_1+1)}{F_1(k_1)} = \eta_1 \text{ (constant) and } \frac{F_2(k_2+1)}{F_2(k_2)} = \eta_2 \text{ (constant)} \qquad \text{... 7(a)}$$

Setting equation 7(a) and 7(b) in equation (6) we obtain

$$\frac{1}{\eta_1} + \frac{r_1 \eta_1}{\eta_2} + r_2 \eta_2 = 1 + r_1 + r_2 \qquad \text{...(8)}$$

From equation 7(a) and 7(b) we obtain

$$F_1(k_1) = \alpha \eta_1^{k_1} \qquad \text{... 9(a)}$$

$$F_2(k_2) = \beta \eta_2^{k_2} \qquad \text{... 9(b)}$$

where α and β are arbitrary constants.

Hence finally we obtain

$$P_{k_1,k_2} = \alpha\,\beta\,(\eta_1)^{k_1}(\eta_2)^{k_2}$$

$$= C\,\eta_1^{k_1}\eta_2^{k_2} \qquad k_1 \geq 1\,,\; k_2 \geq 1 \qquad \text{... (10)}$$

Proposition 1 — The equation (10) is also valid for $k_1 \geq 1$, $k_2 \geq 0$; setting $k_2 = 1$ in equation (1)

$$P_{k_1-1,1} + r_1\,P_{k_1+1,0} + r_2\,P_{k_1,2} - (1 + r_1 + r_2)\,P_{k_1,1} = 0$$

or $$r_1\,P_{k_1+1,0} = C\left\{(1 + r_1 + r_2)\,\eta_1^{k_1}\eta_2 - \eta_1^{k_1-1}\,\eta_2 - r_2\,\eta_1^{k_1}\eta_2^2\right\}$$

$$= C\,\eta_1^{k_1}\eta_2\left[1 + r_1 + r_2 - \frac{1}{\eta_1} - r_2\,\eta_2\right] \qquad \text{... (11)}$$

Substituting the result of equation (8) in equation (11) we obtain.

$$r_1\,P_{k_1+1,0} = C\,\eta_1^{k_1}\eta_2\left[\frac{1}{\eta_1} + \frac{r_1\,\eta_1}{\eta_2} - \frac{1}{\eta_1}\right]$$

$$= C\,r_1\,\eta_1^{k_1+1}$$

$$\therefore\; P_{k_1+1,0} = = C\,\eta_1^{k_1+1} \qquad \text{... (12)}$$

This proves the proposition

we state that $P_{k_1,k_2} = C\,\eta_1^{k_1}\eta_2^{k_2}$ ... (13)

for $k_1 \geq 1$, $k_2 \geq 0$. We shall describte the above property of P_{k_1,k_2} as the shielding effect of k_1.

In equation (2) using equation (13) we get

$$\frac{1}{\eta_1} + r_2\,\eta_2 = 1 + r_1$$

i,e $\quad r_2 \eta_2 = 1 + r_1 - \dfrac{1}{\eta_1}$ ... (14)

Setting equation (14) in equation (8) we get

$$r_1 \eta_1 = r_2 \eta_2 \qquad \text{... (15)}$$

To determine P_{0,k_2} we use equation (3) in the form of a difference equation in P_{0,k_2} as follows

$$r_1 P_1, k_2 - 1 = [(1 + r_1) - r_2 E_2] P_0, k_2 \qquad \text{... (16)}$$

where E_2 is the shifting operator with respect to k_2. The complementary solution of equation (16) is given by.

$$P_{0,k_2} = A\left(\frac{1 + r_2}{r_2}\right)^{k_2} \qquad \text{... (17)}$$

where A is an arbitrary constant.

The particular solution is given by

$$P_{0,k_2} = \frac{r_1 P_{1,k_2-1}}{(1 + r_2) - r_2 E_2} = \frac{C r_1 \eta_1 \eta_2^{k_2-1}}{(1 + r_2) - r_2 E_2}$$

$$= \frac{C r_2 \eta_2^{k_2}}{1 + r_2 - r_2 \eta_2} \qquad \text{... (18)}$$

For convergence consideration, we take $A = 0$

Thus finally we obtain

$$P_{0,k_2} = \frac{C r_2 \eta_2^{k_2}}{1 + r_2 - r_2 \eta_2} \qquad \text{... (19)}$$

For $k_2 = 1$ $\qquad P_{0,1} = \dfrac{C r_2 \eta_2}{1 + r_2 - r_2 \eta_2}$ ... (20)

i,e $\quad r_2 \eta_2 = 1 + r_1 - \dfrac{1}{\eta_1}$ ... (14)

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where A is an arbitrary constant.

The particular solution is given by

$$P_{0,k_2} = \frac{r_1 P_{1,k_2-1}}{(1 + r_2) - r_2 E_2} = \frac{C r_1 \eta_1 \eta_2^{k_2-1}}{(1 + r_2) - r_2 E_2}$$

$$= \frac{C r_2 \eta_2^{k_2}}{1 + r_2 - r_2 \eta_2} \qquad \text{... (18)}$$

For convergence consideration, we take $A = 0$

Thus finally we obtain

$$P_{0,k_2} = \frac{C r_2 \eta_2^{k_2}}{1 + r_2 - r_2 \eta_2} \qquad \text{... (19)}$$

For $k_2 = 1$ $\qquad P_{0,1} = \dfrac{C r_2 \eta_2}{1 + r_2 - r_2 \eta_2}$ ... (20)

From equation (4) $$P_{0,0} = \frac{C r_2^2 \eta_2}{1 + r_2 - r_2 \eta_2} \qquad \text{... (21)}$$

To determine the value of C

we know $$\sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} P_{k_1,k_2} = 1$$

or $$P_{0,0} + \sum_{k_2=1}^{\infty} P_{0,k_2} + \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} P_{k_1,k_2} = 1$$

Using equation (20), (21) we get

$$\frac{C r_2^2 \eta_2}{1 + r_2 - r_2 \eta_2} + \frac{C r_2 \eta_2}{(1 + r_2 - r_2 \eta_2)(1 - \eta_2)} + \frac{C \eta_1}{(1 - \eta_1)(1 - \eta_2)} = 1$$

or $$C \eta_1 \left[\frac{r_1 - r_1 \eta_1 + 1}{(1 - \eta_2)(1 - \eta_1)} \right] = 1$$

or $$C \left[\frac{(1 - \eta_1)(1 - \eta_2)}{\eta_1 (r_1 - r_1 \eta_1 + 1)} \right]$$

using equation (8) we get

$$C = \frac{(1 - \eta_1)(1 - \eta_2)}{\eta_1 \times \frac{1}{\eta_1}} = (1 - \eta_1)(1 - \eta_2) \qquad \text{... (22)}$$

From equation (8) and (15) we get $\eta_1 = \frac{1}{r_1}$ and 1 where $\eta_1 = 1$

is impossible as it will then be undependent of k_1 and $\eta_2 = \frac{1}{r_2}$

Hence get the solution

$$P_{k_1,k_2} = (1 - \eta_1)(1 - \eta_2) \eta_1^{k_1} \eta_2^{k_2}$$

which is the standard result of queueing theory