The maximal ideals of $C^*(X)$ were described by M. H. Stone [36] in 1937. He did it by using the fact that the rings $C^*(X)$ and $C(\beta X)$ are isomorphic. The maximal ideals of $C(X)$ were completely described by Gelfand and Kolmogoroff [17] in 1939 by using the fact that $\beta X$ can be achieved as the structure space of the $C(X)$.

From the works of Stone, Gelfand and Kolmogoroff it follows that there is an one-to-one correspondence between the set of all maximal ideals of $C(X)$ and those of $C^*(X)$. This correspondence can be achieved via the points of $\beta X$. Since $C^*(X)$ is a subring of $C(X)$, one may think that given a maximal ideal $M$ in $C(X)$, $M \cap C^*(X)$ is a maximal ideal in $C^*(X)$. But Hewitt [21] in 1948 shown that this is not true in general, it is true if and only if $M$ is a real ideal in $C(X)$. In particular for every fixed maximal ideal $M_x$ in $C(X)$, $M_x \cap C^*(X)$ is maximal in $C^*(X)$ which is equal to $M^*_x$. It has also been shown in [19] that $M_x$ is the only maximal ideal in $C(X)$ such that $M^*_x = M_x \cap C^*(X)$ and examples of maximal ideals $M$ in $C(X)$ are given such that $M \cap C^*(X)$ are not maximal in $C^*(X)$.

An analogous study for the maximal congruences on the hemirings...
$C_+(X)$ and $C_+(X)$ will be done in this chapter. The characterization of real congruences on $C(X)$ will also be given here.

Recall that the structure space $W(C_+(X))$ is a model of the Stone-Čech compactification $\beta X$ of $X$. A base for the closed sets of $W(C_+(X))$ is,

$$\{m(f, g): f, g \in C_+(X)\}$$

where for each pair $(f, g)$ of elements of $C_+(X)$, $m(f, g)$ is defined by

$$m(f, g) = \{\rho \in W(C_+(X)): (f, g) \in \rho\}.$$ 

The closure operator $h^*$ on $W(C_+(X))$ is defined by

$$h^*(\mathcal{A}) = \{\rho \in W(C_+(X)): \rho \supseteq \mathcal{A}\} \quad \forall \mathcal{A} \subseteq W(C_+(X)).$$

The embedding map $\eta_X: X \rightarrow W(C_+(X))$ is defined by $\eta_X(x) = \rho_x$ for all $x$ in $X$, where $\rho_x = \{(f, g): f, g \in C_+(X), f(x) = g(x)\}$. For any $f, g$ in $C_+(X)$, the agreement set $E(f, g)$ is defined by $E(f, g) = \{x \in X: f(x) = g(x)\}$. It may be noted that for a Tychonoff space $X$ the family $\{E(f, g): f, g \in C_+(X)\}$ is a base for the closed sets.

We start with the following theorem.

5.1 Theorem. The set of all maximal congruences on $C_+(X)$ is precisely the set

$$\{\rho^*_x: x \in \beta X\},$$

where for any $x$ in $\beta X$,

$$\rho^*_x = \{(f, g) \in C_+(X) \times C_+(X): f^\beta(x) = g^\beta(x)\}.$$
Proof. Note that the hemirings $C^+_+(X)$ and $C^+_+(eta X)$ are isomorphic under the mapping $f \mapsto f^\beta$ for each $f$ in $C^+_+(X)$. Also since $\beta X$ is compact in view of Corollary 2.25, every maximal congruence on $C^+_+(\beta X)$ is fixed. Thus in view of Theorem 2.17 the set of all maximal congruences on $C^+_+(\beta X)$ is

$$\{\rho_x : x \in \beta X\}$$

where for any $x$ in $\beta X$,

$$\rho_x = \{(f, g) : f, g \in C^+_+(\beta X), f(x) = g(x)\}.$$  

Since every element in $C^+_+(\beta X)$ is of the form $f^\beta$ for some $f$ in $C^+_+(X)$, a maximal congruence $\rho_x$ on $C^+_+(\beta X)$ can be written in the form

$$\rho_x = \{(f^\beta, g^\beta) : f, g \in C^+_+(X), f^\beta(x) = g^\beta(x)\}.$$  

Thus each maximal congruence on $C^+_+(X)$ is of the form

$$\rho^*_x = \{(f, g) \in C^+_+(X) \times C^+_+(X) : f^\beta(x) = g^\beta(x)\}, x \in \beta X,$$

and hence the set of all the maximal congruences on $C^+_+(X)$ is precisely the set

$$\{\rho^*_x : x \in \beta X\}$$

We now give a description of the set of all maximal congruences on the hemiring $C^+_+(X)$. To do this we need the following lemma.

S.2 Lemma. For all $f, g$ in $C^+_+(X)$, $h^*(\eta_X(E(f, g))) = m(f, g)$.

Proof. $h^*(\eta_X(E(f, g)))$

$$= \{\rho \in W(C^+_+(X)) : \rho \supseteq \bigcap \{\sigma : \sigma \in \eta_X(E(f, g))\}\}$$

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= \{\rho \in W(C_+(X)) : \rho \supset \bigcap \{\eta_x(x) : x \in E(f, g)\}\}

= \{\rho \in W(C_+(X)) : \rho \supset \bigcap \{\rho_x : x \in E(f, g)\}\}

= \{\rho \in W(C_+(X)) : (f, g) \in \rho\}

= m(f, g).

5.3 Theorem. The set of all the maximal congruences on the hemiring $C_+(X)$ is precisely the set

\[\{\rho^X : x \in X\}\]

where for any $x$ in $X$,

\[\rho^X = \{(f, g) \in C_+(X) \times C_+(X) : x \in \text{cl}_{\beta_X}(E(f, g))\}.\]

Proof. Since $W(C_+(X))$ is equivalent to $\beta X$, there exists a homeomorphism $H$ of $W(C_+(X))$ onto $\beta X$ such that $H \circ \eta_x = \beta$. Thus $W(C_+(X)) = \{H^{-1}(x) : x \in \beta X\}$, i.e., every maximal congruence on $C_+(X)$ is of the form $H^{-1}(x)$ for some $x$ in $\beta X$. Now for all $f, g$ in $C_+(X)$ and $x$ in $\beta X$,

\[(f, g) \in H^{-1}(x) \iff H^{-1}(x) \in m(f, g)\]

\[\iff H^{-1}(x) \in h^*(\eta_x(E(f, g)))\] by Theorem 5.2,

\[\iff x \in H(h^*(\eta_x(E(f, g))))\]

\[\iff x \in \text{cl}_{\beta_X}(H(E(f, g))), \text{ because } H \text{ is a homeomorphism},\]

\[\iff x \in \text{cl}_{\beta_X}(\beta(E(f, g))), \text{ because } H \circ \eta_x = \beta,\]

\[\iff (f, g) \in \rho^x.\]

Thus $H^{-1}(x) = \rho^x$. 
The description of the set of real congruences on \( C_+(X) \) is given by the following.

**5.4 Theorem.** The set of all real congruences on \( C_+(X) \) is precisely the set \( \{ \rho^x_R : x \in vX \} \) where for any \( x \) in \( vX \), \( \rho^x_R \) is defined by,

\[
\rho^x_R = \{ (f, g) \in C_+(X) \times C_+(X): f^v(x) = g^v(x) \}.
\]

**Proof.** Note that the hemirings \( C_+(X) \) and \( C_+(vX) \) are isomorphic under the map \( f \mapsto f^v \) for each \( f \) in \( C_+(X) \). Since \( vX \) is realcompact in view of Corollary 3.30, every real congruence on \( C_+(vX) \) is fixed and hence in view of Theorem 2.17 the set of all real congruence on \( C_+(vX) \) is precisely the set \( \{ \rho^x_x : x \in vX \} \) where for any \( x \) in \( vX \), \( \rho^x_x \) is defined by,

\[
\rho^x_x = \{ (f, g) \in C_+(vX) \times C_+(vX): f(x) = g(x) \}.
\]

But each element of \( C_+(vX) \) is of the form \( f^v \) for some \( f \) in \( C_+(X) \). Since the correspondence is an isomorphism, in view of Corollary 3.29 it follows that the set of all real congruences on \( C_+(X) \) is precisely the set \( \{ \rho^x_R : x \in vX \} \) where for any \( x \) in \( vX \), \( \rho^x_R \) is defined by,

\[
\rho^x_R = \{ (f, g) \in C_+(X) \times C_+(X): f^v(x) = g^v(x) \}.
\]

This completes the proof.

The following gives a relation between the maximal congruences on \( C_+(X) \) and \( C_+^*(X) \).
5.5 Theorem. If \( \rho \) is a maximal congruence on \( C_+(X) \) then \( \rho^* \), defined by,

\[
\rho^* = \rho \cap (C_+^*(X) \times C_+^*(X))
\]
is a prime congruence on \( C_+^*(X) \).

Proof. It can easily be verified that \( \rho^* \) is a congruence on \( C_+^*(X) \). Let \( f_1, f_2, g_1, g_2 \) belong to \( C_+^*(X) \) such that

\[
(f_1.g_1 + f_2.g_2, f_1.g_2 + f_2.g_1) \in \rho^*.
\]

Since \( \rho \) is maximal, it is prime and since \( \rho^* \subseteq \rho \), we have either \( (f_1, f_2) \) belongs to \( \rho \) or \( (g_1, g_2) \) belongs to \( \rho \). But \( f_1, f_2, g_1, g_2 \) are all members of \( C_+^*(X) \) and hence either \( (f_1, f_2) \) belongs to \( \rho^* \) or \( (g_1, g_2) \) belongs to \( \rho^* \). Thus \( \rho^* \) is prime.

In what follows we give an example to show that for an arbitrary congruence \( \rho \) on \( C_+(X) \), the congruence \( \rho \cap (C_+^*(X) \times C_+^*(X)) \) on \( C_+^*(X) \) need not be maximal.

5.6 Example. Let us consider the relation \( \rho_0 \) on \( C_+(\mathbb{N}) \) defined by,

\[
\rho_0 = \{(f, g) \in C_+(\mathbb{N}) \times C_+(\mathbb{N}) : \exists N \in \mathbb{N} \text{ such that } f(n) = g(n) \forall n \geq N\}.
\]

Then \( \rho_0 \) is a congruence on \( C_+(\mathbb{N}) \). Let \( \rho \) be a maximal congruence on \( C_+(\mathbb{N}) \) containing \( \rho_0 \). Let \( \rho^* = \rho \cap (C_+^*(\mathbb{N}) \times C_+^*(\mathbb{N})) \). In view of the above theorem, \( \rho^* \) is a prime congruence on \( C_+^*(\mathbb{N}) \). We claim that \( \rho^* \) is not maximal. If possible suppose that \( \rho^* \) is maximal. Then in view of Theorem 5.1, there exists \( x \) in \( \mathbb{N} \) such that \( \rho^* = \rho^* x \), i.e.,
\[ \rho^* = \{(f, g) \in C^*_+(\mathbb{N}) \times C^*_+(\mathbb{N}) : f^\beta(x) = g^\beta(x) \}. \]

Since \( \rho_* \) is free, so is \( \rho \) and hence \( \rho^* x \) is free. Thus \( x \in \beta\mathbb{N} - \mathbb{N} \). Define a function \( j : \mathbb{N} \to \mathbb{R}_+ \) by, \( j(n) = 1/n \) \( \forall n \in \mathbb{N} \). Since \( \mathbb{N} \) is discrete, \( j \) is continuous. Also \( j^\beta(y) = 0 \) for all \( y \) in \( \beta\mathbb{N} - \mathbb{N} \). Thus \( (j, Q) \in \rho^* \). Clearly \( (j, Q) \not\in \rho \) because \( E(j, Q) = \emptyset \) — a contradiction. Thus \( \rho^* \) is not a maximal congruence on \( C^*_+(\mathbb{N}) \) even though \( \rho \) is a maximal congruence on \( C^*_+(\mathbb{N}) \).

In what follows we shall prove a necessary and sufficient condition on a maximal congruence \( \rho \) on \( C^*_+(X) \) such that \( \rho \cap (C^*_+(X) \times C^*_+(X)) \) is a maximal congruence on \( C^*_+(X) \). To achieve the condition we begin with the following theorem.

5.7 Theorem. For each congruence \( \rho \) on \( C^*_+(X) \), \( C^*_+(X)/\rho \) is isomorphic to the hemiring \( \mathbb{R}_+ \).

Proof. Let \( \rho \) be a maximal congruence on \( C^*_+(X) \). Then by Theorem 5.1, it follows that \( \rho = \rho^* x \) for some \( x \) in \( \beta X \). Define a mapping \( \varphi^* x : C^*_+(X) \to \mathbb{R}_+ \) by,

\[ \varphi^* x(f) = f^\beta(x) \quad \forall f \in C^*_+(X). \]

Then \( \varphi^* x \) is a non-zero hemiring homomorphism of \( C^*_+(X) \) onto \( \mathbb{R}_+ \). Also for each \( x \) in \( X \), \( \varphi^* x = \psi_x \) where \( \psi_x \) is as defined in Remark 4.7.

Let \( f, g \in C^*_+(X) \). Then,

\[ (f, g) \in \text{Core} \varphi^* x \iff \varphi^* x(f) = \varphi^* x(g) \iff f^\beta(x) = g^\beta(x) \]
Thus \( \text{Core } \rho^*_X = \rho^*_X \). Hence in view of Theorem 1.36, \( C^*_+(X)/\rho^*_X \) is isomorphic to \( \mathbb{R}_+ \).

5.8 Remark. From the above theorem it follows that if \( \rho \) is a maximal congruence on \( C^*_+(X) \) then for each \( f \) in \( C^*_+(X) \) there exists an \( r \) in \( \mathbb{R}_+ \) such that \( \rho(f) = \rho(r) \). In this connection it is worth noting that the question of existence of hyper-real congruences on \( C^*_+(X) \) does not arise.

5.9 Theorem. Let \( \rho \) be a maximal congruence on \( C^*_+(X) \). Then \( \rho \) is real if and only if \( \rho^* = \rho \cap (C^*_+(X) \times C^*_+(X)) \) is a maximal congruence on \( C^*_+(X) \).

Proof. Let \( \rho \) be a real congruence on \( C_+(X) \). Then by Theorem 5.4 there exists an \( x \) in \( vX \) such that \( \rho = \rho^X_R \), i.e.,

\[
\rho = \{(f, g) \in C_+(X) \times C_+(X) : f^v(x) = g^v(x)\}.
\]

Thus we have,

\[
\rho^* = \{(f, g) \in C^*_+(X) \times C^*_+(X) : f^v(x) = g^v(x)\}.
\]

Since \( X \) is dense in both \( vX \) and \( \beta X \), it follows that for any \( h \) in \( C^*_+(X) \), \( h^v = h^\beta \mid_{vX} \). Thus,

\[
\rho^* = \{(f, g) \in C^*_+(X) \times C^*_+(X) : f^\beta(x) = g^\beta(x)\}.
\]

In view of Theorem 5.1, it follows that \( \rho^* \) is a maximal congruence on \( C^*_+(X) \).

Conversely suppose that \( \rho \) is hyper-real. Then there exists \( f \) in \( C_+(X) \) such that \( f \geq 1 \) and \( \rho(f) \) is infinitely large. Then \( 1/f \) belongs
to $C^*_+(X)$ and $\rho(1/f)$ is infinitely small and hence $\rho(1/f) \neq \rho(r)$ for each $r$ in $R_+$. This implies that

\[
\left( \frac{1}{f}, r \right) \not\in \rho \quad \forall \ r \in R_+
\]

\[
\Rightarrow \left( \frac{1}{f}, r \right) \not\in \rho^* \quad \forall \ r \in R_+
\]

\[
\Rightarrow \rho^*\left( \frac{1}{f} \right) \neq \rho^*(r) \quad \forall \ r \in R_+.
\]

Thus by the above remark it follows that $\rho^*$ is not a maximal congruence on $C^*_+(X)$.

This completes the proof.

We conclude this chapter with the following remark.

5.10 Remark. If $\rho$ is a fixed congruence on $C_+(X)$ then $\rho$ is obviously real (see 3.14) and hence by the above theorem it follows that $\rho \cap (C^*_+(X) \times C^*_+(X))$ is a maximal congruence which is obviously fixed.