CHAPTER 2

SEMI GENERALISED STAR CLOSED SETS

2.1 SEMI GENERALISED STAR SETS

DEFINITION 2.1.1: A subset A of a topological space X is called a semi generalised star closed set (briefly, sg*closed) if cl(A) ⊆ U whenever A ⊆ U and U is semi open in X.

THEOREM 2.1.2: Every closed set is sg*closed.

PROOF: Let A be a closed subset of X. Let A ⊆ U and U be semi open. cl(A) = A since A is closed. Therefore cl(A) ⊆ U. Hence A is sg* closed.

Converse of this theorem is not true as seen in the following example.

EXAMPLE 2.1.3: Let X = {a, b, c} and T = {∅, {a, b}, X}.

Let A = {a, c}. The only semi open set containing A is X. Hence A ⊆ U and U be semi open implies cl(A) ⊆ U. Therefore A is sg*closed. A^c = {b} is not open. Hence A is not closed. Therefore A is sg*closed but not closed.

THEOREM 2.1.4: Every sg*closed set is g closed.
PROOF: Let $A$ be a sg*closed subset of $X$. Let $A \subseteq U$ and $U$ be open.

$U$ is semi open since every open set is semi open. $A$ is sg*closed and $U$ is semi open. Therefore $\overline{A} \subseteq U$. Hence $A$ is g closed.

We have the following example to show that the converse is not true.

EXAMPLE 2.1.5: Let $X = \{a, b, c\}$. $T = \{\emptyset, \{a\}, X\}$.

Let $A = \{a, b\}$. $\overline{A} = X$.

The only open set containing $A$ is $X$. Therefore $A$ is g closed.

Let $U = \{a, b\}$.

$A \subseteq U$ and $U$ is semi open. But $\overline{A} \not\subseteq U$.

Hence $A$ is not sg*closed. Therefore $A$ is g closed but not sg*closed.

PROPOSITION 2.1.6: The two concepts semi closed and sg*closed are independent.

This is proved by the following two examples.

EXAMPLE 2.1.7: Consider $\mathbb{R}$ with usual Topology. Let $A = (0,1]$.

$A$ is semi closed. Let $U = (0,1]$.

$A \subseteq U$ and $U$ is semi open. $\overline{A} = [0,1] \not\subseteq U$. Hence $A$ is not sg*closed.

Therefore $A$ is semi closed but not sg*closed.
EXAMPLE 2.1.8: Let \( X = \{a,b,c\} \) \( T = \{\emptyset, \{a\}, \{b,c\}, X\} \). In this space every open set is closed and conversely. In this space the list of all open sets, the list of all closed sets, the list of all semi open sets and the list of all semi closed sets are the same.

Let \( A = \{b\} \). \( A \) is not semi closed. \( \text{cl}(A) = \{b,c\} \)

Semi open sets containing \( A \) are \( \{b,c\} \) and \( X \).

\( \text{cl}(A) \) is a subset of each of these sets. Hence \( A \) is sg*closed. Therefore \( A \) is sg*closed but not semi closed.

THEOREM 2.1.9: Union of two sg*closed sets is sg*closed.

PROOF: Let \( A \) and \( B \) be two sg*closed sets.

Let \( A \cup B \subseteq U \) and \( U \) be semi open. Then \( A \subseteq U \) and \( B \subseteq U \). \( \text{cl}(A) \subseteq U \) since \( A \) is sg*closed. \( \text{cl}(B) \subseteq U \) since \( B \) is sg*closed. \( \text{cl}(A) \cup \text{cl}(B) \subseteq U \).

Therefore \( \text{cl}(A \cup B) \subseteq U \). Hence \( A \cup B \) is sg*closed.

PROPOSITION 2.1.10: Finite Union of sg*closed sets is sg*closed.

Proof follows from the above Theorem.

Infinite Union sg*closed sets need not be sg*closed and is proved by the following example.
EXAMPLE 2.1.11: Let \( X=\mathbb{R} \) with usual topology. Let \( A_n=\{\frac{1}{n}\} \) for each \( n \in \mathbb{N} \). Each \( A_n \) is closed and hence sg*closed.

\[
\bigcup A_n = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}. \quad \text{cl}(\bigcup A_n) = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}
\]

Let \( U=(0,1] \). \( U \) is semi open. \( \bigcup A_n \subset U \) and \( \text{cl}(\bigcup A_n) \subset U \). Therefore \( \bigcup A_n \) is not sg*closed.

THEOREM 2.1.12: A sg*closed set which is semi open is closed.

PROOF: Let \( A \) be a sg*closed set which is semi open. \( A \subset A \) and \( A \) is semi open. This implies \( \text{cl}(A) \subset A \). Hence \( A \) is closed.

RESULT 2.1.13: Being semi open is a sufficient condition for a sg*closed set to be closed. However this condition is not necessary. There are sg*closed sets which are closed but not semi open.

EXAMPLE 2.1.14: Let \( X=\{a,b,c\} \) \( T=\{\emptyset, \{a\}, \{b\}, \{a,b\}, X\} \) closed sets are \( \emptyset, \{b,c\}, \{a,c\}, \{c\}, X \). Let \( A=\{c\} \). \( A \) is sg*closed. Also \( A \) is closed but \( A \) is not semi open.

THEOREM 2.1.15: If \( A \) is sg*closed and \( A \subset B \subset \text{cl}(A) \), then \( B \) is sg*closed.

PROOF: Let \( A \) be a sg*closed subset of \( X \).
Let $A \subset B \subset \text{cl}(A)$. Then $\text{cl}(B) \subset \text{cl}(A)$. Let $B \subset U$ and $U$ be semi open. Then $A \subset U$. $\text{cl}(A) \subset U$ since $A$ is $sg^*$-closed. Hence $\text{cl}(B) \subset U$.

Hence $B$ is $sg^*$-closed.

**THEOREM 2.1.16:** Let $A$ be $sg^*$ closed but not closed. Then for every open set $O \subset A$, there exists an open set $V$ such that $A$ intersects $V$ and $\text{cl}(O)$ does not intersect $V$.

**PROOF:** $A$ is $sg^*$ closed but not closed.

Let $O \subset A$ and $O$ be open.

Claim: $A \not\subset \text{cl}(O)$.

If $A \subset \text{cl}(O)$, then $O \subset A \subset \text{cl}(O)$ and $O$ is open.

Hence $A$ is semi open.

A is $sg^*$ closed and semi open $\Rightarrow A$ is closed.

But $A$ is not closed. Hence $A \not\subset \text{cl}(O)$.

Hence there exists $x \in A$ and $x \not\in \text{cl}(O)$. Let $V = \text{cl}(O)^c$.

Then $V$ is open and $\text{cl}(O)$ does not intersect $V$.

$x \not\in \text{cl}(O)$. Therefore $x \in \text{cl}(O)^c$. Hence $x \in V$.

Then $x \in A$ and $x \in V$. Therefore $A \cap V \neq \emptyset$.

Therefore $A$ intersects $V$ and $\text{cl}(O)$ does not intersect $V$. 

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**THEOREM 2.1.17:** Let $A$ be sg* closed and $A \subseteq \Omega$ where $\Omega$ is open. That is,

Then $\text{Fr}(\Omega) \subseteq \text{int } A^c$, that is, every boundary point of $\Omega$ is an interior point of $A^c$.

**PROOF:** For any subset $A$ of $X$, the frontier of $A$, denoted as $\text{Fr}(A)$ is defined to be the set $\text{cl}(A) - \text{int } A$.

Let $A$ be sg* closed and let $A \subseteq \Omega$ where $\Omega$ is open. Then $\text{cl}(A) \subseteq \Omega$.

Take any $x \in \text{Fr}(\Omega)$. Then $x \in \text{cl}(\Omega) - \text{int } \Omega$.

Hence $x \in \text{cl}(\Omega) - \Omega$ since $\Omega$ is open. Hence $x \notin \Omega$.

Therefore $x \in \text{cl}(A)$. Hence $x \in \text{cl}(A)^c$.

Therefore $x \in \text{int } A^c$. Hence $\text{Fr}(\Omega) \subseteq \text{int } A^c$.

**THEOREM 2.1.18:** In a Door space every sg* closed set is closed.

**PROOF:** In a Door space every subset is either open or closed.

Let $X$ be a Door space.

Let $A$ be a sg* closed subset of $X$. Then $A$ is open or closed.

If $A$ is closed then nothing to prove.

If $A$ is open then $A$ is semi open. $A$ is sg* closed and $A$ is semi open.

Therefore $A$ is closed.

The converse of the above Theorem is not true.
EXAMPLE 2.1.19: Let $X=\{a,b,c\}$ and $T=\{\emptyset, X, \{a\}\}$. 

Closed sets are $\emptyset, X, \{b,c\}$. 

Semi open sets are $\emptyset, X, \{a\}, \{a,b\}, \{a,c\}$. 

$\{a\}$ is not sg* closed since $\{a\} \subset \{a\}$ and $\text{cl}(\{a\})=X \nsubseteq \{a\}$. 

$\{b\}$ is not sg* closed since $\{b\} \subset \{a,b\}$ and $\text{cl}(\{b\})=\{b,c\} \nsubseteq \{a,b\}$. 

$\{c\}$ is not sg* closed since $\{c\} \subset \{a,c\}$ and $\text{cl}(\{c\})=\{b,c\} \nsubseteq \{a,c\}$. 

$\{a,b\}$ is not sg* closed since $\{a,b\} \subset \{a,b\}$ and $\text{cl}(\{a,b\})=X \nsubseteq \{a,b\}$. 

$\{a,c\}$ is not sg* closed since $\{a,c\} \subset \{a,c\}$ and $\text{cl}(\{a,c\})=X \nsubseteq \{a,c\}$. 

$\{b,c\}$ is sg* closed since $X$ is the only semi open set containing it. 

Hence sg* closed sets are $\emptyset, X, \{b,c\}$. 

Therefore in $X$ every sg* closed set is closed. 

$X$ is not a Door space since $\{b\}$ is neither open nor closed. 

DEFINITION 2.1.20: A subset $A$ of a topological space $X$ is called semi generalised star open (briefly, sg*open) if $A^c$ is sg* closed. 

THEOREM 2.1.21: Every open set is sg*open. 

PROOF: Let $A$ be a open set of $X$. Then $A^c$ is closed. $A^c$ is sg* closed since every closed set is sg* closed. Hence $A$ is sg*open.
Converse of this Theorem is not true and is proved by the following example.

**EXAMPLE 2.1.22:** Let \( X = \{a, b, c\} \) and \( T = \{\emptyset, \{a, b\}, X\} \). Let \( A = \{b\} \).
\( A^c = \{a, c\} \). \( A^c \) is sg*closed. Hence \( A \) is sg*open. But \( A \) is not open.

**THEOREM 2.1.23:** Every sg*open set is g open.

**PROOF:** Let \( A \) be a sg*open set of \( X \). \( A^c \) is sg*closed. \( A^c \) is g closed since every sg*closed set is g closed. Hence \( A \) is g open.

Converse of this Theorem is not true and is proved by the following example.

**EXAMPLE 2.1.24:** Let \( X = \{a, b, c\} \) \( T = \{\emptyset, \{a\}, X\} \). Let \( A = \{c\} \).
\( A^c = \{a, b\} \). \( A^c \) is g closed. Hence \( A \) is g open. \( A^c \) is not sg*closed. Hence \( A \) is not sg*open.

**THEOREM 2.1.25:** Intersection of two sg*open sets is sg*open.

**PROOF:** Let \( A \) and \( B \) be two sg*open sets in \( (X, T) \). Then \( A^c \) and \( B^c \) are sg*closed sets. \( A^c \cup B^c \) is sg*closed since union of two sg*closed sets is sg*closed. Therefore \( (A \cap B)^c \) is sg*closed. Hence \( A \cap B \) is sg*open.
PROPOSITION 2.1.26: Finite Intersection of sg*open sets is sg*open.

Proof follows from the above Theorem.

Infinite intersection of sg*open sets need not be sg*open and is proved by the following example.

EXAMPLE 2.1.27: Let $X=(-1,1)$ with usual topology.

For each $n \in \mathbb{N}$, define $A_n = (-\frac{1}{n}, \frac{1}{n})$. Each $A_n$ is open. Hence each $A_n$ is sg*open. Let $A = \bigcap A_n$.

$A = \{0\}$. $A^c = (-1,0) \cup (0,1)$. $cl(A^c) = (-1,1)$.

$A^c$ is open. Hence $A^c$ is semi open.

Let $U = (-1,0) \cup (0,1)$.

$A^c \subseteq U$ and $U$ semi open. But $cl(A^c) \not\subseteq U$.

Therefore $A^c$ is not sg*closed.

Hence $A$ is not sg*open.

THEOREM 2.1.28: If $A$ is sg*open and $Int A \subseteq B \subseteq A$. Then $B$ is sg*open.

PROOF: $A$ is sg*open. Hence $A^c$ is sg*closed. $Int A \subseteq B \subseteq A$. Therefore $(Int A)^c \supset B^c \supset A^c$. Therefore $A^c \subseteq B^c \subseteq cl(A^c)$. Hence $B^c$ is sg*closed.

Hence $B$ is sg*open.
THEOREM 2.1.29: If $A$ is $sg^*$ open and $A \supseteq F$, $F$ closed then $Fr(F) \subseteq \text{Int} A$.

PROOF: $A$ is $sg^*$ open and $A \supseteq F$, $F$ is closed.

$A^c$ is $sg^*$closed, $A^c \subseteq F^c$ and $F^c$ is open.

By theorem 2.1.17 $Fr(F^c) \subseteq \text{Int} A$.

$Fr(F) \subseteq \text{Int} A$ since $Fr(F^c) = Fr(F)$.

THEOREM 2.1.30: In a Door space every $sg^*$ open set is open.

PROOF: Let $X$ be a Door space. Let $A$ be a $sg^*$open subset of $X$. Then $A^c$ is $sg^*$closed. Since $X$ is a door space, $A^c$ is closed. Hence $A$ is open.

Converse of the above Theorem is not true.

EXAMPLE 2.1.31: Let $X=\{a,b,c\}$ and $T=\{\emptyset, X, \{a\}\}$.

$sg^*$closed sets are $\emptyset, X, \{b,c\}$. $sg^*$open sets are $\emptyset, X, \{a\}$.

Hence every $sg^*$open set is open.

$X$ is not a door space since $\{b\}$ is neither open nor closed.

THEOREM 2.1.32: Any singleton set is either semi closed or $sg^*$ open.

PROOF: Take $\{x\}$.

If it is semi closed then nothing to prove.

If it is not semi closed, then $\{x\}^c$ is not semi open.
Therefore \( X \) is the only semi open set containing \( \{x\}^c \).

Therefore \( \{x\}^c \) is sg* closed. Hence \( \{x\} \) is sg* open.

Therefore \( \{x\} \) is semi closed or sg* open.

2.2 EQUIVALENT CHARACTERISATIONS

**THEOREM 2.2.1:** A set \( A \) is sg*closed iff for each \((x_a)\) in \( A \) converging to \( x \), \( \exists \ y \in A \) such that every net semiconverging to \( x \), semiconverges to \( y \) also.

**PROOF:** Let \( A \) be sg*closed.

Let \((x_a)\) in \( A \) converge to \( x \). Then \( x \in \text{cl}(A) \).

consider \( \text{scl}(x) \).

claim: \( \text{scl}(x) \) contains a point of \( A \).

If not, then \( \text{scl}(x) \cap A = \emptyset \).

\( A \subset (\text{scl}(x))^c \), \( (\text{scl}(x))^c \) is semi open and \( A \) is sg*closed.

Hence \( \text{cl}(A) \subset (\text{scl}(x))^c \).

\( x \in \text{cl}(A) \Rightarrow x \in (\text{scl}(x))^c \Rightarrow x \notin \text{scl}(x) \Rightarrow \quad \)

Hence \( \text{scl}(x) \) contains a point of \( A \).

Take \( y \in A \ \exists \ y \in \text{scl}(x) \).

Let \((x_a)\) semi converge to \( x \).

Claim: \( (x_a) \) semi converges to \( y \).

Let \( O \) be any semi open set containing \( y \).
Claim: $x \in O$.

If not then $x \not\in O$. Then $x \in O^c$ and $O^c$ is semi closed.

Hence $scl(x) \subseteq O^c$. Since $y \in scl(x)$, $y \in O^c$. $\Rightarrow\Leftarrow$ since $y \in O$.

Hence $x \in O$.

$O$ is a semi open set containing $x$ and $(x_\alpha)$ semi converges to $x$.

Therefore $(x_\alpha)$ is eventually in $O$.

Hence if $O$ is any semi open set containing $y$ then $(x_\alpha)$ is eventually in $O$.

Therefore $(x_\alpha)$ semi converges to $y$.

Therefore $\exists \; y \in A$ such that every net semiconverging to $x$, semiconverges to $y$ also.

Conversely,

If for each $(x_\alpha)$ in $A$ converging to $x$, $\exists \; y \in A \ni$ every net semi converging to $x$ semiconverges to $y$ also.

To prove: $A$ is sg*-closed.

Let $A \subseteq U$, $U$ be semi open.

Take $x \in cl(A)$. Then $\exists \; (x_\alpha)$ in $A$ converging to $x$.

Therefore $\exists \; y \in A \ni$ every net semi converging to $x$, semi converges to $y$ also.

Claim: $y \in scl(x)$.

consider the net $(x_\alpha)$ where $x_\alpha = x$ for each $\alpha$. 

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(xₐ) is a net semi converging to x.

Therefore (xₐ) semi converges to y. Therefore every semi open set containing y contains elements of the net. Therefore every semi open set containing y contains x also.

If y \not\in \text{scl}(x), then y \in (\text{scl}(x))^c and (\text{scl}(x))^c is semi open.

Therefore (\text{scl}(x))^c contains x also. 

Hence y \in \text{scl}(x). Also y \in A. Hence \text{scl}(x) \cap A \neq \phi.

Claim: x \in U.

Suppose not,

Then x \in U^c and U^c is semi closed.

Therefore \text{scl}(x) \subset U^c.

Since \text{scl}(x) \cap A \neq \phi, A \cap U^c \neq \phi. Therefore A \subset U. 

Hence x \in U. Therefore \text{cl}(A) \subset U.

Therefore A is sg*closed.

**THEOREM 2.2.2:** A set A is sg*closed iff for each filter F, such that A belongs to F and F converges to x, \exists y \in A such that every filter semiconverging to x, semiconverges to y also.

**PROOF:** Let A be sg*closed.

Let F be a filter such that A belongs to F and F converges to x.

Then x \in \text{cl}(A). consider \text{scl}(x).
claim: $\text{scl}(x)$ contains a point of $A$.

If not, then $\text{scl}(x) \cap A = \emptyset$.

$A \subset (\text{scl}(x))^c$, $(\text{scl}(x))^c$ is semi open and $A$ is $\text{sg}^*$-closed.

Hence $\text{cl}(A) \subset (\text{scl}(x))^c$. $x \in \text{cl}(A) \Rightarrow x \in (\text{scl}(x))^c \Rightarrow x \notin \text{scl}(x) \Rightarrow \Rightarrow$

Hence $\text{scl}(x)$ contains a point of $A$.

Take $y \in A \ni y \in \text{scl}(x)$.

Let $V$ be a filter semi converging to $x$.

Claim: $V$ semi converges to $y$.

Let $O$ be any semi open set containing $y$.

Claim: $x \in O$.

If not then $x \in O^c$ and $O^c$ is semi closed.

Hence $\text{scl}(x) \subset O^c$.

Since $y \in \text{scl}(x)$, $y \in O^c \Rightarrow \Rightarrow$ since $y \in O$.

Hence $x \in O$.

$O$ is a semi open set containing $x$ and filter $V$ semi converges to $x$.

Therefore $O$ belongs to the filter $V$.

Hence if $O$ is any semi open set containing $y$ then $O$ belongs to the filter $V$. Therefore the filter $V$ semi converges to $y$.

Therefore $\exists y \in A$ such that every filter semiconverging to $x$, semiconverges to $y$ also.

Conversely,
If for each filter $F$ with $A$ belongs to $F$ and $F$ converges to $x$, $\exists y \in A$ such that every filter semiconverging to $x$, semiconverges to $y$ also.

To prove: $A$ is sg*closed.

Let $A \subseteq U$, $U$ be semi open.

Take $x \in \text{cl}(A)$.

Then $\exists$ a filter $F$ with $A$ belongs to $F$ and $F$ converges to $x$.

Therefore $\exists y \in A$ such that every filter semiconverging to $x$, semiconverges to $y$ also.

Claim: $y \in \text{scl}(x)$.

Let $S.O(x)$ be the set of all semi open sets containing $x$.

$S.O(x)$ has finite intersection property. Hence there exist a filter with $S.O(x)$ as a filter subbasis. Let $V$ be a filter containing all the elements of $S.O(x)$.

Then $V$ semi converges to $x$.

Therefore $V$ semi converges to $y$ also. Therefore every semi open set containing $y$ belongs to $V$. Since every element of $V$ contains $x$, every semi open set containing $y$ contains $x$ also.

If $y \notin \text{scl}(x)$, then $y \in (\text{scl}(x))^c$ and $(\text{scl}(x))^c$ is semi open.

Therefore $(\text{scl}(x))^c$ contains $x$ also. $\Rightarrow\Leftarrow$

Hence $y \in \text{scl}(x)$. Also $y \in A$. Hence $\text{scl}(x) \cap A \neq \emptyset$.

Claim: $x \in U$.  

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suppose not, 
Then \( x \in U^c \) and \( U^c \) is semi closed.

Therefore \( \text{scl}(x) \subseteq U^c \). Since \( \text{scl}(x) \cap A \neq \emptyset \), \( A \cap U^c \neq \emptyset \).

Therefore \( A \cap U \). \( \Rightarrow \Leftarrow \)

Hence \( x \in U \).

Therefore \( \text{cl}(A) \subseteq U \).

Therefore \( A \) is \( \text{sg}^* \)closed.

**THEOREM 2.2.3:** A set \( A \) is \( \text{sg}^* \)closed iff whenever \( A \) does not intersect a semi closed set \( F \), \( \text{cl}(A) \) also does not intersect \( F \).

**PROOF:**

**Necessity:** Let \( A \) be a \( \text{sg}^* \)closed subset of \( X \). Let \( F \) be a semi closed set where \( A \cap F = \emptyset \). This implies \( A \subseteq F^c \). \( F^c \) is semi open. \( \text{cl}(A) \subseteq F^c \) since \( A \) is \( \text{sg}^* \)closed. Therefore \( \text{cl}(A) \cap F = \emptyset \). Hence \( \text{cl}(A) \) does not intersect \( F \).

**Sufficiency:** Let \( A \subseteq U \) and \( U \) be semi open. \( A \) does not intersect \( U^c \) and \( U^c \) is semi closed. Hence \( \text{cl}(A) \) does not intersect \( U^c \). Hence \( \text{cl}(A) \subseteq U \).

This implies \( A \) is \( \text{sg}^* \)closed.

**THEOREM 2.2.4:** A set \( A \) is \( \text{sg}^* \)open iff whenever \( A \) contains a semi closed set \( F \), \( \text{Int} A \) also contains \( F \).

**PROOF:**

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Necessity: Let $A$ be a $sg^*$-open subset of $X$. Let $A \supset F$ where $F$ is semi closed. Then $A^c \subset F^c$, $F^c$ is semi open and $A^c$ is $sg^*$ closed. Therefore $\text{cl}(A^c) \subset F^c$. Hence $[\text{cl}(A^c)]^c \supset F$. Hence $\text{Int} A \supset F$.

Sufficiency: Let $A^c \subset U$ and $U$ be semi open. $A \supset U^c$. $U^c$ is semi closed. Hence $\text{Int} A \supset U^c$. Therefore $(\text{Int} A)^c \subset U$. Hence $\text{cl}(A^c) \subset U$. Therefore $A^c$ is $sg^*$-closed. Hence $A$ is $sg^*$-open.

**THEOREM 2.2.5:** A set $A$ is $sg^*$-open iff $G$ is closed and $\text{Int} G \subset A$ implies $A \cap G = \text{Int} A \cap G$.

**PROOF:**

Necessity: Let $A$ be a $sg^*$-open subset of $X$. Let $G$ be a closed set with $\text{Int} G \subset A$. Clearly $\text{Int} A \cap G \subset A \cap G$. Let $x \in A \cap G$. Then $x \in A$ and $x \in G$. Now either $x \in \text{Int} G$ or $x \notin \text{Int} G$.

**Case 1:** Let $x \in \text{Int} G$. $\text{Int} G \subset A$. $\text{Int} G \subset \text{Int} A$ since $\text{Int} A$ is the largest open subset of $A$. Therefore $x \in \text{Int} A$. Hence $x \in \text{Int} A \cap G$.

**Case 2:** Let $x \notin \text{Int} G$. Let $F = \text{Int} G \cup \{x\}$. Then $\text{Int} G \subset F \subset G$. Therefore $F$ is semi closed. $\text{Int} G \subset A$ and $x \in A$. Therefore $F \subset A$. $A$ contains the semi closed set $F$ and $A$ is $sg^*$-open. Therefore $\text{Int} A$ contains $F$. Therefore $x \in \text{Int} A$. Hence $x \in \text{Int} A \cap G$.

Therefore $A \cap G \subset \text{Int} A \cap G$. Hence $A \cap G = \text{Int} A \cap G$. 

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Sufficiency: Let \( F \) be a semi closed set such that \( A \supseteq F \). Since \( F \) is semi closed there exists a closed set \( G \) such that \( \text{Int} \, G \subseteq F \subseteq G \). \( \text{Int} \, G \subseteq F \) and \( F \subseteq A \). Therefore \( \text{Int} \, G \subseteq A \). \( G \) is closed and \( \text{Int} \, G \subseteq A \). Therefore \( A \cap G = \text{Int} \, A \cap G \). \( F \subseteq A \cap G \). Therefore \( F \subseteq \text{Int} \, A \cap G \). Hence \( F \subseteq \text{Int} \, A \). Hence if \( A \) contains a semi closed set \( F \) then \( \text{Int} \, A \) also contains \( F \). Therefore \( A \) is \( \text{sg}^* \)-open.

**THEOREM 2.2.6:** A set \( A \) is \( \text{sg}^* \)-closed iff \( O \) is open and \( A \subseteq \text{cl}(O) \Rightarrow A \cup O = \text{cl}(A) \cup O \).

**PROOF:**

**Necessity:** Let \( A \) be a \( \text{sg}^* \)-closed subset of \( X \). Let \( O \) be open and \( A \subseteq \text{cl}(O) \). Let \( G = O^c \). \( G \) is closed. \( A \subseteq \text{cl}(O) \) implies \( A^c \supseteq [\text{cl}(O)]^c \).

Therefore \( A^c \supseteq \text{Int} \, O^c \). Hence \( A^c \supseteq \text{Int} \, G \). \( A^c \) is \( \text{sg}^* \)-open, \( G \) is closed and \( \text{Int} \, G \subseteq A^c \). Therefore \( A^c \cap G = \text{Int} \, A^c \cap G \).

Therefore \( A^c \cap O^c = [\text{cl}(A)]^c \cap O^c \). Therefore \( (A \cup O)^c = (\text{cl}(A) \cup O)^c \).

Hence \( A \cup O = \text{cl}(A) \cup O \).

**Sufficiency:** Let \( G \) be a closed set and \( \text{Int} \, G \subseteq A^c \). Let \( O = G^c \). \( O \) is open. \( \text{Int} \, G \subseteq A^c \) implies \( \text{Int} \, O^c \subseteq A^c \). Therefore \( [\text{cl}(O)]^c \subseteq A^c \).

Therefore \( A \subseteq \text{cl}(O) \). Therefore \( A \cup O = \text{cl}(A) \cup O \).
Hence \((A \cup O)^c = (cl(A) \cup O)^c\). Therefore \(A^c \cap O^c = [cl(A)]^c \cap O^c\).

Hence \(A^c \cap G = \text{Int} A^c \cap G\). Therefore \(A^c\) is sg*open. Hence \(A\) is sg*closed.

**THEOREM 2.2.7:** A set \(A\) is sg* closed iff \(O\) is open and \(A \subset cl(O)\)

\(\Rightarrow cl(A)-A \subset O\).

**PROOF:** Follows from the above theorem.

**THEOREM 2.2.8:** The following are equivalent.

1. \(A\) is sg* closed.

2. For each \(x \in cl(A)\), \(scl(x) \cap A \neq \emptyset\).

3. \(cl(A)-A\) contains no non empty semiclosed set.

**PROOF:** To prove 1\(\Rightarrow\)2.

\(A\) is sg* closed. Take any \(x \in cl(A)\).

Claim: \(scl(x) \cap A \neq \emptyset\).

Suppose not then \(scl(x) \cap A = \emptyset\).

Then \(A \subset [scl(x)]^c\), \([scl(x)]^c\) is semi open and \(A\) is sg* closed.

Therefore \(cl(A) \subset [scl(x)]^c\).

As \(x \in cl(A)\) we have \(x \in [scl(x)]^c\). \(\Rightarrow\) \(\Rightarrow\)

Therefore \(scl(x) \cap A \neq \emptyset\).

Hence 1\(\Rightarrow\)2 is proved.
To prove 2=>3.

Suppose \( \text{cl}(A) - A \) contains a non empty semi closed set \( F \).

\( F \subseteq \text{cl}(A) - A \), \( F \neq \emptyset \) and \( F \) is semi closed.

Take any \( x \in F \). Then \( x \in \text{cl}(A) \). Therefore \( \text{scl}(x) \cap A \neq \emptyset \).

\( x \in F \) and \( F \) is semi closed.

Therefore \( \text{scl}(x) \subseteq F \). Therefore \( F \cap A \neq \emptyset \).

We have \( F \subseteq \text{cl}(A) - A \) and \( F \cap A \neq \emptyset \). \( \Rightarrow \Leftarrow \)

Hence \( \text{cl}(A) - A \) contains no non empty semiclosed set.

Hence 2=>3 is proved.

To Prove 3=>1.

Suppose \( A \) is not \( sg^* \) closed .

Then there exists semi open set \( U \) with \( A \subseteq U \) and \( \text{cl}(A) \cap U \).

There exists \( x \in \text{cl}(A) \) and \( x \notin U \).

\( x \in \text{cl}(A) - U \). Let \( F = \text{cl}(A) - U \). Then \( F \subseteq \text{cl}(A) - A \).

\( F = \text{cl}(A) \cap U^c \). \( \text{cl}(A) \) is semi closed and \( U^c \) is semi closed .

Therefore \( F \) is semi closed. Since \( x \notin F \), \( F \neq \emptyset \). Also \( F \subseteq \text{cl}(A) - A \).

Therefore \( \text{cl}(A) - A \) contains a non empty semi closed set . \( \Rightarrow \Leftarrow \)

Therefore \( A \) is \( sg^* \) closed. Hence 3=>1 is proved.

**THEOREM 2.2.9:** The following are equivalent.

1. \( A \) is \( sg^* \) open.
2. For each \( x \in A - \text{int} A \), \( \text{scl}(x) \cap A^c \neq \emptyset \).

3. \( A - \text{int} A \) does not contain a non empty semiclosed set.

**PROOF:** To prove \( 1 \Rightarrow 2 \).

Let \( A \) be \( \text{sg}^* \) open. Let \( B = A^c \). Then \( B \) is \( \text{sg}^* \) closed.

\[ x \in A - \text{int} A \Rightarrow x \not\in \text{int} A \Rightarrow x \in (\text{int} A)^c \Rightarrow x \in \text{cl}(A^c) \Rightarrow x \in \text{cl}(B). \]

Since \( B \) is \( \text{sg}^* \) closed, \( \text{scl}(x) \cap B \neq \emptyset \). Hence \( \text{scl}(x) \cap A^c \neq \emptyset \).

Hence \( 1 \Rightarrow 2 \) is proved.

To prove \( 2 \Rightarrow 3 \).

Suppose \( A - \text{int} A \) contains a non empty semiclosed set \( F \).

\( F \subset A - \text{int} A \), \( F \neq \emptyset \) and \( F \) is semi closed.

Take \( x \in F \). Then \( x \in A - \text{int} A \). Then \( \text{scl}(x) \cap A^c \neq \emptyset \).

Since \( x \in F \) and \( F \) is semi closed, \( \text{scl}(x) \subset F \).

\( \text{scl}(x) \subset A \) since \( F \subset A \). Therefore \( \text{scl}(x) \cap A^c = \emptyset \). => <=

\( A - \text{int} A \) does not contain a non empty semiclosed set.

Hence \( 2 \Rightarrow 3 \) is proved.

To prove \( 3 \Rightarrow 1 \).

\( A - \text{int} A \) does not contain a non empty semi closed set.

Let \( B = A^c \). Now \( \text{cl}(B) - B = \text{cl}(A^c) - A^c \).

\[ = \text{cl}(A^c) \cap A. \]

\[ = (\text{int} A)^c \cap A. \]

\[ = A - \text{int} A. \]
Therefore $\text{cl}(B) - B$ does not contain a nonempty closed set.

Therefore $B$ is sg* closed.

Therefore $A$ is sg* open.

**THEOREM 2.2.10:** Let $A$ be sg* closed. Then $A$ is closed iff $\text{cl}(A) - A$ is semi closed.

**PROOF:**

Part 1: $A$ is sg* closed which is also closed.

$\text{cl}(A) - A = \emptyset$. Hence $\text{cl}(A) - A$ is semi closed.

Part 2: $A$ is sg* closed and $\text{cl}(A) - A$ is semi closed.

Since $A$ is sg* closed, $\text{cl}(A) - A$ cannot contain a nonempty semi closed set. Therefore $\text{cl}(A) - A = \emptyset$. Therefore $\text{cl}(A) \subseteq A$. Therefore $A$ is closed.

**THEOREM 2.2.11:** In $X$ every set is sg* closed iff the set of all semi open sets of $X$ is equal to the set of all closed sets of $X$.

**PROOF:**

Part 1: If every set is sg* closed, let $O$ be semi open.

$O$ is semi open, $O \subseteq O$ and $O$ is sg* closed.

Therefore $\text{cl}(O) \subseteq O$. Therefore $O$ is closed.

Therefore every semi open set is closed.
Let $F$ be a closed set. Then $F^c$ is open.

Hence $F^c$ is semi open.

$F^c \subseteq F^c$, $F^c$ is sg* closed.

Hence $\text{cl}(F^c) \subseteq F^c$.

Therefore $F^c$ is closed. Therefore $F$ is open.

Therefore $F$ is semi open.

Hence every closed set is semi open.

Therefore the set of all semi open sets is equal to the set of all closed sets.

Part 2: If the set of all semi open sets is equal to the set of all closed sets.

To prove: every set is sg* closed.

Let $A \subseteq X$. Let $A \subseteq O$ and $O$ be semi open.

Then $\text{cl}(A) \subseteq \text{cl}(O)$.

Since every semi open is closed, $\text{cl}(O) = O$.

Hence $\text{cl}(A) \subseteq O$. Therefore $A$ is sg* closed.

Hence every set is sg* closed.

**DEFINITION 2.2.12:** sg* $T_{1/2}$ space

A topological space $X$ is called sg* $T_{1/2}$ space if every sg* closed set is closed.
THEOREM 2.2.13: A space $X$ is $sg^* T_{1/2}$ iff every $sg^*$open set is open.

PROOF: Let $X$ be $sg^*T_{1/2}$.

$A$ is $sg^*$ open. $\Rightarrow A^c$ is $sg^*$ closed.

$\Rightarrow A^c$ is closed.

$\Rightarrow A$ is open.

conversely,

If every $sg^*$ open set is open, let $A$ be any $sg^*$closed set.

$A$ is $sg^*$ closed. $\Rightarrow A^c$ is $sg^*$open.

$\Rightarrow A^c$ is open.

$\Rightarrow A$ is closed.

Therefore every $sg^*$closed set is closed.

Hence $X$ is $sg^* T_{1/2}$.

THEOREM 2.2.14: A space $X$ is $sg^* T_{1/2}$ iff every singleton set is either semi closed or open.

PROOF: Given $X$ is $sg^* T_{1/2}$.

Take $\{x\}$. If it is semiclosed then nothing to prove.

If $\{x\}$ is not semiclosed then by theorem 2.1.32, $\{x\}$ is $sg^*$open. Since $X$ is $sg^* T_{1/2}$, $\{x\}$ is open.

Hence $\{x\}$ is semi closed or open.
Conversely, if $X$ is such that each singleton subset of $X$ is either semiclosed or open.

To prove: $X$ is $sg^* T_{1/2}$ space.

Let $A$ be any $sg^*$ closed set.

claim: $cl(A) \subseteq A$.

Take $x \in cl(A)$. Then $\{x\}$ is either semi closed or open.

case 1: Let $\{x\}$ be semiclosed.

If $x \not\in A$, then $clA - A$ contains $\{x\}$ a non empty semiclosed set.

Therefore $A$ is not $sg^*$ closed. \(\Rightarrow \Leftarrow\)

Therefore $x \in A$.

case 2: Let $\{x\}$ be open.

$x \in cl(A)$ therefore any open set containing $x$ intersects $A$.

$\{x\}$ is an open set containing $x$. Hence $\{x\} \cap A \neq \emptyset$.

Therefore $x \in A$. Hence $cl(A) \subseteq A$.

Hence $A$ is closed.

Therefore each $sg^*$ closed set is closed. Therefore $X$ is $sg^* T_{1/2}$.

**DEFINITION 2.2.15: $sg^* T_{1/3}$ space**

A topological space $X$ is called $sg^* T_{1/3}$ space if every $g$ closed set in $X$ is $sg^*$closed in $X$. 
THEOREM 2.2.16: A space $X$ is sg*-$T_{\frac{1}{3}}$ iff every g open set is sg* open.

PROOF: Let $X$ be sg*-$T_{\frac{1}{3}}$. A is g open $\Rightarrow A^c$ is g closed $\Rightarrow A^c$ is sg*closed $\Rightarrow A$ is sg*open.

Every g open set is sg*open $\Rightarrow$ every g closed set is sg*closed $\Rightarrow X$ is sg*-$T_{\frac{1}{3}}$.

2.3 SEMI GENERALISED STAR HOMEOMORPHISM

THEOREM 2.3.1: Let $f: X \rightarrow Y$ be a homeomorphism. Then a subset $A$ is sg* closed in $Y$ $\Rightarrow f^{-1}(A)$ is sg* closed in $X$.

PROOF: $f: X \rightarrow Y$ is a homeomorphism. Let $A$ be a sg*closed subset of $Y$. Let $B = f^{-1}(A)$.

To prove: $B$ is sg* closed.

Let $U$ be any semi open set with $B \subseteq U$. Then $f(B) \subseteq f(U)$.

Therefore $f(f^{-1}(A)) \subseteq f(U)$.

Since $f$ is bijective, $f(f^{-1}(A)) = A$. Therefore $A \subseteq f(U)$.

Claim: $f(U)$ is semi open.

$U$ is semi open. Therefore $U \subseteq \text{cl int } U$.

$f(U) \subseteq f(\text{cl int } U)$

$\subseteq \text{cl } f(\text{int } U)$ since $f$ is continuous.
\[ \subset \text{cl int } f(U) \text{ since } f \text{ is open.} \]

Therefore \( f(U) \) is semi open.

\( A \subset f(U), f(U) \) is semi open and \( A \) is sg*closed.

Therefore \( \text{cl}(A) \subset f(U) \). Hence \( f^{-1}[\text{cl}(A)] \subset f^{-1}[f(U)] \).

Since \( f \) is a homeomorphism, \( f^{-1}(\text{cl}(A)) = \text{cl}(f^{-1}(A)) \).

Therefore \( \text{cl}(f^{-1}(A)) \subset f^{-1}[f(U)] \).

Therefore \( \text{cl}(B) \subset U \).

Therefore \( B \) is sg* closed. Therefore \( f^{-1}(A) \) is sg* closed.

**THEOREM 2.3.2:** Let \( f: X \to Y \) be a homeomorphism. A subset \( A \) is sg* open in \( Y \) \( \implies f^{-1}(A) \) is sg* open in \( X \).

**PROOF:** \( A \) is sg* open in \( Y \). \( \implies \text{A}^c \) is sg* closed in \( Y \).

\[ \implies f^{-1}(A^c) \text{ is sg* closed in } X. \]

\[ \implies [f^{-1}(A)]^c \text{ is sg* closed in } X. \]

\[ \implies f^{-1}(A) \text{ is sg* open in } X. \]

**THEOREM 2.3.3:** Let \( f: X \to Y \) be a homeomorphism. A subset \( A \) is sg* closed in \( X \) \( \implies f(A) \) is sg* closed in \( Y \).

**PROOF:** \( f: X \to Y \) is a homeomorphism.

\( A \) is sg* closed in \( X \). Let \( B = f(A) \).

To prove: \( B \) is sg* closed.
Let U be a semi open set with B \subseteq U.

That is f(A) \subseteq U. Hence f^{-1}(f(A)) \subseteq f^{-1}(U).

Since f is bijective, f^{-1}(f(A)) = A. Therefore A \subseteq f^{-1}(U).

Since U is semi open and f is a homeomorphism, f^{-1}(U) is semi open.

A \subseteq f^{-1}(U), f^{-1}(U) is semi open and A is sg* closed.

Therefore \text{cl}(A) \subseteq f^{-1}(U). Hence f(\text{cl}(A)) \subseteq f(f^{-1}(U)).

Since f is a closed map, \text{cl}(f(A)) \subseteq f(\text{cl}(A)).

Therefore \text{cl}(f(A)) \subseteq f[f^{-1}(U)].

Hence \text{cl}(B) \subseteq U. Therefore B is sg* closed.

Therefore image of a sg* closed set is sg* closed.

**THEOREM 2.3.4:** Let f: X \rightarrow Y be a homeomorphism. A is sg* open in X \Rightarrow f(A) is sg* open in Y.

**PROOF:** A is sg* open in X. \Rightarrow A^c is sg* closed in X.

\Rightarrow f(A^c) is sg* closed in Y.

\Rightarrow [f(A)]^c is sg* closed in Y.

\Rightarrow f(A) is sg* open in Y.

**DEFINITION 2.3.5:** Pre sg* closed map

Let X and Y be two topological spaces. A map f: X \rightarrow Y is called a Pre sg* closed map if A is sg* closed in X \Rightarrow f(A) is sg* closed in Y.
THEOREM 2.3.6: Every homeomorphism is a Pre sg* closed map.

PROOF: Follows from theorem 2.3.3.

DEFINITION 2.3.7: Pre sg* open map
A map f: X → Y is called a pre sg* open map if A is sg* open in X => f(A) is sg* open in Y.

THEOREM 2.3.8: Every homeomorphism is a pre sg* open map.

PROOF: Follows from theorem 2.3.4

DEFINITION 2.3.9: sg* irresolute map.
Let X and Y be two topological spaces. A map f: X → Y is called sg* irresolute map if A is sg* closed in Y => f^(-1)(A) is sg* closed in X.

THEOREM 2.3.10: Every homeomorphism is sg* irresolute.

PROOF: Follows from theorem 2.3.1.

THEOREM 2.3.11: f is sg* irresolute iff inverse image of every sg* open set is sg* open.
PROOF: A is sg* open in Y. \( \Rightarrow \) \( A^c \) is sg* closed in Y.

\[ \Rightarrow f^\rightarrow (A^c) \text{ is sg* closed in } X. \]

\[ \Rightarrow [f^\rightarrow (A)]^c \text{ is sg* closed in } X. \]

\[ \Rightarrow f^\rightarrow (A) \text{ is sg* open in } X. \]

Hence inverse image of every sg* open set is sg* open.

**DEFINITION 2.3.12:** sg* homeomorphism

A map \( f : X \rightarrow Y \) is called a sg* homomorphism if \( f \) is bijective, \( f \) is sg* irresolute and \( f^\rightarrow \) is sg* irresolute.

**THEOREM 2.3.13:** \( f : X \rightarrow Y \) is bijective. Following are equivalent.

1. \( f \) is sg* irresolute and \( f \) is pre sg* closed.
2. \( f \) is sg* irresolute and \( f \) is pre sg* open.
3. \( f \) is sg* homeomorphism.

**PROOF:** To prove \( 1 \Rightarrow 2 \).

Now \( f : X \rightarrow Y \) is bijective, \( f \) is sg* irresolute, \( f \) is pre sg* closed.

A is sg* open in \( X \) \( \Rightarrow \) \( A^c \) is sg* closed in \( X \).

\[ \Rightarrow f(A^c) \text{ is sg* closed in } Y. \]

\[ \Rightarrow f(A)^c \text{ is sg* closed in } Y. \]

\[ \Rightarrow f(A) \text{ is sg* open in } Y. \]

Hence \( f \) is a pre sg* open map. Hence \( 1 \Rightarrow 2 \) is proved.
To prove $2 \Rightarrow 3$.

Now $f: X \rightarrow Y$ is bijective, $f$ is sg* irresolute, $f$ is pre sg* open.

A is sg*open in $X$. $\Rightarrow f(A)$ is sg*open in $Y$.

$\Rightarrow (f^{-1})^{-1}$ is sg*open in $Y$.

Hence $f^{-1}$ is sg* irresolute. Hence $f^{-1}$ is a sg* homeomorphism.

Therefore $2 \Rightarrow 3$ is proved.

To prove $3 \Rightarrow 1$.

Now $f: X \rightarrow Y$ is bijective, $f$ is sg* irresolute, $f^{-1}$ is sg* irresolute.

$f^{-1}$ is sg* irresolute $\Rightarrow f$ is pre sg* closed. Hence $3 \Rightarrow 1$ is proved.

**DEFINITION 2.3.14:** sg* closed map

A map $f: X \rightarrow Y$ is called sg* closed map if for each closed set $F$ of $X$, $f(F)$ is sg* closed in $Y$.

**THEOREM 2.3.15:** A map $f: X \rightarrow Y$ is sg* closed iff for each subset $S$ of $Y$ and for each open set $U$ containing $f^{-1}(S)$, $\exists$ a sg* open set $V$ of $Y$ such that $S \subset V$ and $f^{-1}(V) \subset U$.

**PROOF:**

Part 1: Let $f: X \rightarrow Y$ be sg* closed.

Let $S \subset Y$. Let $f^{-1}(S) \subset U$ and $U$ be open.
Since \( U^c \) is closed in \( X \) and \( f: X \to Y \) is sg* closed, \( f(U^c) \) is sg* closed in \( Y \). Take \( V = [f(U^c)]^c \) then \( V \) is sg* open.

claim: \( S \subseteq V \).

\[ x \in U^c \Rightarrow x \not\in f^{-1}(S) \Rightarrow f(x) \not\in S \Rightarrow f(x) \in S^c. \]

Hence \( f(U^c) \subseteq S^c \).

\([f(U^c)]^c \supset S \). Therefore \( V \supset S \).

claim: \( f^{-1}(V) \subseteq U \).

\[ x \in f^{-1}(V) \Rightarrow f(x) \in V \Rightarrow f(x) \in [f(U^c)]^c. \]

\[ \Rightarrow f(x) \not\in f(U^c). \]

\[ \Rightarrow x \not\in U^c. \]

\[ \Rightarrow x \in U. \]

Hence \( f^{-1}(V) \subseteq U \). Therefore \( \exists \) sg* open set \( V \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Part 2: Suppose for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \) there exist sg* open set \( V \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

To prove: \( f \) is sg* closed map.

Let \( F \) be a closed subset of \( X \), \( f(F) \subseteq Y \).

Take \( S = [f(F)]^c \). Then \( f^{-1}(S) \subseteq F^c \) and \( F^c \) is open.

Therefore \( \exists \) sg* open set \( V \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq F^c \).
claim: $S = V$.

Suppose not, then $V \nsubseteq S$.

$\exists y \in V \text{ and } y \notin S$. Then $y \in f(F)$.

Then $\exists x \in F$ such that $f(x) = y$. Then $f(x) \in V$.

Hence $x \in f^{-1}(V) \subseteq F^c$. $\Rightarrow \Leftarrow$ since $x \in F$.

Therefore $S = V$. Hence $S$ is sg* open.

Hence $[f(F)]^c$ is sg* open. Hence $f(F)$ is sg* closed.

Therefore $F$ is closed $\Rightarrow f(F)$ is sg* closed.

Therefore $f$ is sg* closed map.

DEFINITION 2.3.16: sg* open map.

A map $f : X \to Y$ is called sg* open if for each open set $U$ of $X$, $f(U)$ is sg* open in $Y$.

THEOREM 2.3.17: A map $f : X \to Y$ is sg* open iff for each subset $S$ of $Y$ and for each closed set $A$ containing $f^{-1}(S)$, $\exists$ a sg* closed set $B \supset S$ such that $f^{-1}(B) \subseteq A$.

PROOF: Let $f : X \to Y$ be sg* open. Let $S \subseteq Y$, $A \supset f^{-1}(S)$ and $A$ be closed. Then $A^c$ is open in $X$. Since $f : X \to Y$ is sg* open, $f(A^c)$ is sg* open in $Y$. Take $B = [f(A^c)]^c$. Then $B$ is sg* closed set.

claim: $S \subseteq B$. 

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\[ x \in A^{c} \Rightarrow x \not\in f^{1}(S) \Rightarrow f(x) \in S \Rightarrow f(x) \in S^{c}. \]

Hence \( f(A^{c}) \subset S^{c} \). Therefore \( [f(A^{c})]^{c} \supset S \). Hence \( B \supset S \).

claim: \( f^{1}(B) \subset A \).

\[ x \in f^{1}(B) \Rightarrow f(x) \in B \Rightarrow f(x) \in [f(A^{c})]^{c}. \]

\[ \Rightarrow f(x) \notin f(A^{c}). \]

\[ \Rightarrow x \notin A^{c}. \]

\[ \Rightarrow x \in A. \]

Therefore \( f^{1}(B) \subset A \).

converse: For each subset \( S \) of \( Y \) and for each closed set \( A \) containing \( f^{1}(S) \), \( \exists \) sg * closed set \( B \) \( \exists S \subset B \) and \( f^{1}(B) \subset A \).

To prove: \( f \) is sg* open map.

Let \( U \) be a open set of \( X \).

Take \( S = [f(U)]^{c} \). Then \( f^{1}(S) = f^{1}(f(U))^{c} = f^{1}(f(U))^{c} \).

Hence \( S \subset U^{c} \).

\( U^{c} \) is a closed set containing \( S \).

Hence \( \exists \) sg* closed set \( B \) such that \( S \subset B \) and \( f^{1}(B) \subset U^{c} \).

Claim: \( S = B \).

If not then \( S \subset B \). Then \( \exists y \in B \) and \( y \notin S \). Since \( y \in f(U) \), \( \exists x \in U \) \( \exists f(x) = y \). Then \( f(x) \in B \). This implies \( x \in f^{1}(B) \) and hence \( x \in U^{c} \).

\( \Rightarrow \Leftarrow \) since \( x \in U \).
Therefore \( S = \text{B} \) Hence \( S \) is \( \text{sg}^* \) closed.

Therefore \( f(U^*) \) \( \text{sg}^* \) open. Hence \( f \) is \( \text{sg}^* \) open map.

**THEOREM**

The following are equivalent.

- \( \text{sg}^* \) closed map.

2. If \( A \subset X \) is open then \( \{ y \in f^{-1}(y) \subset A \} \) is \( \text{sg}^* \) open in \( Y \).

3. If \( A \subset X \) is closed then \( \{ y \in f^{-1}(y) \cap A \neq \emptyset \} \) is \( \text{sg}^* \) closed in \( Y \).

**PROOF:**

To prove \( 1 \Rightarrow 2 \).

Let \( f : X \to Y \) be a \( \text{sg}^* \) closed map.

Let \( A \) be a open subset of \( X \). Then \( A^c \) is closed.

Since \( f \) is a \( \text{sg}^* \) closed map, \( f(A^c) \) is \( \text{sg}^* \) closed.

**claim:** \( [f(A^c)]^c = \{ y \in f^{-1}(y) \subset A \} \).

\[ y \in [f(A^c)]^c \Rightarrow y \notin f(A^c) \Rightarrow f^{-1}(y) \cap A^c = \emptyset \Rightarrow f^{-1}(y) \subset A. \]

Therefore \( [f(A^c)]^c \subset \{ y \in f^{-1}(y) \subset A \} \).

\[ f^{-1}(y) \subset A \Rightarrow f^{-1}(y) \cap A^c = \emptyset \Rightarrow y \notin f(A^c) \Rightarrow y \in [f(A^c)]^c. \]

Therefore \( [f(A^c)]^c \supset \{ y \in f^{-1}(y) \subset A \} \).

Hence \( [f(A^c)]^c = \{ y \in f^{-1}(y) \subset A \} \).

Since \( [f(A^c)]^c \) is \( \text{sg}^* \) open, \( \{ y \in f^{-1}(y) \subset A \} \) is \( \text{sg}^* \) open.

Hence \( 1 \Rightarrow 2 \) is proved.
To prove $2 \Rightarrow 3$.

Let $A \subseteq X$ be open. $\{y / f^{-1}(y) \cap A \neq \emptyset \}$ is sg* open in $Y$.

Let $F \subseteq X$ be

$F^c$ is open. $\{y / f^{-1}(y) \subseteq F^c \}$ is sg* open.

$\Rightarrow F^c$ is sg* closed.

$F = \emptyset$ is sg* closed.

Hence $2 \Rightarrow 3$ is proved.

To prove $3 \Rightarrow 1$.

Let $A$ be a closed subset of $X$. Then $\{y / f^{-1}(y) \cap A \neq \emptyset \}$ is sg* closed.

$A$ is closed. $\Rightarrow \{y / f^{-1}(y) \cap A \neq \emptyset \}$ is sg* closed.

Claim: $f(A) = \{y / f^{-1}(y) \cap A \neq \emptyset \}$.

$y \in f(A)$ iff $y = f(x)$ for some $x \in A$.

iff $x \in f^{-1}(y)$ and $x \in A$.

iff $f^{-1}(y) \cap A \neq \emptyset$.

Hence $f(A) = \{y / f^{-1}(y) \cap A \neq \emptyset \}$.

Since $\{y / f^{-1}(y) \cap A \neq \emptyset \}$ is sg* closed, $f(A)$ is sg* closed.

Therefore $f$ is sg* closed.

Hence $3 \Rightarrow 1$ is proved.
THEOREM 2.3.19: If \( f: X \rightarrow Y \) is an irresolute closed map, then \( F \) is sg* closed in \( X \Rightarrow f(F) \) is sg* closed in \( Y \).

PROOF: Let \( F \) be a sg* closed subset of \( X \).

To prove : \( f(F) \) is sg* closed.

Let \( f(F) \subset O, O \) be semi open. Then \( F \subset f^{-1}(O) \).

\( O \) is semi open and \( f \) is irresolute.

Therefore \( f^{-1}(O) \) is semi open.

\( F \) is sg* closed, \( F \subset f^{-1}(O) \) and \( f^{-1}(O) \) is semi open.

Therefore \( cl(F) \subset f^{-1}(O) \). Hence \( f(cl(F)) \subset f(f^{-1}(O)) \subset O \).

since \( f \) is a closed map, \( cl(f(F)) \subset f(cl(F)) \).

Hence \( cl(f(F)) \subset O \). Therefore \( f(F) \) is sg* closed.

NOTE 2.3.20: S.R. Malghan [22] proved that if \( f: X \rightarrow Y \) is a continuous g-closed surjection and \( X \) is normal, then \( Y \) is normal.

As a corollary, we have the following theorem.

THEOREM 2.3.21: Let \( X \) and \( Y \) be two topological spaces. If \( f: X \rightarrow Y \) is a continuous, sg* closed surjection. Then \( X \) is normal implies \( Y \) is normal.
PROOF: \( f: X \rightarrow Y \) is continuous, \( sg^* \) closed surjective map. \( X \) is normal.

To prove: \( Y \) is normal.

\( f: X \rightarrow Y \) is \( sg^* \) closed implies \( f: X \rightarrow Y \) is \( g \)-closed.

Hence by Malghan's result stated above, \( Y \) is normal.

2.4 FUZZY SEMI GENERALISED STAR SETS.

In 1965, Zadeh(32) introduced the concept of fuzzy sets. After that K.K.Azad, Chang(8), Ming and Ming and various other mathematicians, developed fuzzy set theory and also fuzzy topology concepts.

A fuzzy set is a map from a non empty set \( X \) to the closed interval \([0,1]\). Let \( X \) be a non empty set and let \( I = [0,1] \). \( I^X \) denote the collection of all mappings from \( X \) to \( I \). Any member \( \lambda \) of \( I^X \) is called a fuzzy set on \( X \). The null fuzzy set 0 is the mapping from \( X \) into \( I \) where \( 0(x) = 0 \) for all \( x \). The fuzzy set \( X \) is denoted by the mapping \( 1 \) from \( X \) into \( I \), where \( 1(x) = x \) for all \( x \). A member \( \lambda \) of \( I^X \) is contained in a member \( \mu \) of \( I^X \), denoted as \( \lambda \subseteq \mu \) iff \( \lambda(x) \leq \mu(x) \), for each \( x \in X \). If \( \lambda \subseteq \mu \) then we say that \( \lambda \) is a subset of \( \mu \) and we write \( \lambda \subset \mu \).

Let \( \{\lambda_k: k \in \Lambda\} \) be a family of fuzzy sets of \( X \) where \( \Lambda \) is the index set. The union \( \bigcup \lambda_k \) is defined to be the mapping \( \text{Sup} \lambda_k \). The
intersection $\cap \lambda_k$ is defined to be the mapping $\text{Inf}\lambda_k$. The complement $\lambda^c$ of a fuzzy set $\lambda$ is $1-\lambda$, where $(1-\lambda)(x) = 1-\lambda(x)$ for each $x \in X$. The support of a fuzzy set $\lambda$ is the set $\{x : \lambda(x) > 0\}$. Let $x \in X$ and $a \in I$. Then the map $\lambda : X \to I$ defined as $\lambda(x) = a$ and $\lambda(y) = 0$ if $y$ is not equal to $x$, is called a fuzzy point and it is denoted by $x_a$. A fuzzy set $A$ is said to be quasi-coincident with $B$, denoted by $A \sim B$, iff there exists $x \in X$, such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$. If $A$ is not quasi-coincident with $B$, then we write $A \not\sim B$.

Let $X$ and $Y$ be two non-empty sets. Let $f : X \to Y$ be a mapping. If $\lambda$ is a fuzzy set of $X$, then $f(\lambda)$ is a fuzzy set of $Y$ defined by

$$f(\lambda) y = \sup \{\lambda(x) \mid x \in f^{-1}(y)\} \text{ if } f^{-1}(y) \neq \emptyset.$$  

$$f(\lambda) y = 0 \text{ if } f^{-1}(y) = \emptyset.$$  

If $\mu$ is a fuzzy set of $Y$, then $f^{-1}(\mu)$ is a fuzzy set of $X$ defined by

$$f^{-1}(\mu)(x) = \mu(f(x)) \text{ for each } x \in X.$$  

A subfamily $T$ of $I^X$ is called a fuzzy topology on $X$ if

(i) $0$ and $1$ belong to $T$,

(ii) any union of members of $T$ is in $T$ and

(iii) a finite intersection of members of $T$ is in $T$.  

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Members of $T$ are called fuzzy open (fo) sets and their complements are called fuzzy closed (fc) sets. For a fuzzy set $\lambda$ of $X$, the T-closure and the T-interior are defined respectively, as $\text{cl} \lambda = \inf \{ v : v \geq \lambda, v \in T \}$ and $\text{int} \lambda = \sup \{ v : v \leq \lambda, v \in T \}$.

A fuzzy set $\lambda$ in a fts $X$ is called a fuzzy semiopen set if there exists a fuzzy open set $U$ such that $U \leq \lambda \leq \text{cl} U$.

A fuzzy set $\lambda$ is called a fuzzy semiclosed set if there exists a fuzzy closed set $V$ such that $\text{int} V \leq \lambda \leq V$. A fuzzy set $\lambda$ is fuzzy semiclosed iff its complement is fuzzy semiopen.

A fuzzy set $\lambda$ in a fts $X$ is called a fuzzy $\alpha$ open set if $\lambda \subseteq \text{int}(\text{cl}(\text{int} \lambda))$. The complement of a fuzzy $\alpha$ open set is defined to be fuzzy $\alpha$ closed.

A fuzzy set $\lambda$ in a fts $X$ is called a fuzzy $\beta$ open set if $\lambda \subseteq \text{cl}(\text{int}(\text{cl} \lambda))$. The complement of a fuzzy $\beta$ open set is defined to be fuzzy $\beta$ closed.

For example, let $X = [0,1]$. $T = \{0, 1, A\}$ where $A : X \rightarrow [0,1]$ is defined as $A(0) = 0.3$ and $A(x) = 0$ for $x$ in $(0,1]$. Then $T$ is a fuzzy topology.

Fuzzy open sets are 0,1 and $A$.

Fuzzy closed sets are 0,1 and $B = A^c$ where $B$ is defined as $B(0) = 0.7$ and $B(x) = 1$ for all $x$ in $(0,1]$.
Take $\lambda = A$. Then $\lambda \subset \text{int}(\text{cl}(\text{int}\lambda))$. Hence $\lambda$ is fuzzy $\alpha$ open.

Take $\lambda = B$. Then $\lambda \subset \text{cl}(\text{int}(\text{cl}\lambda))$. Hence $\lambda$ is fuzzy $\beta$ open.

In a fuzzy topological space, the concepts of fuzzy generalized closed (open) sets, fuzzy semi generalized closed (open) sets, fuzzy generalized $\alpha$ closed (open) sets, fuzzy $\alpha$ generalized closed (open) sets, fuzzy generalized $\beta$ closed (open) sets and fuzzy $\beta$ generalized closed (open) sets are defined similar to the definitions in ordinary topological space.

**DEFINITION 2.4.1**

Let $X$ be a fuzzy topological space. A fuzzy set $A$ of $X$ is called a fuzzy semi generalised star closed (fsg* closed) set if whenever $A \subset U$ and $U$ is a fuzzy semi open set of $X$, $\text{cl}(A) \subset U$.

**DEFINITION 2.4.2**

Let $X$ be a fuzzy topological space. A fuzzy set $A$ is called a fuzzy semi generalised star open (fsg* open) set if $A^c$ is fuzzy semi generalied star closed.
THEOREM 2.4.3

Let $X$ be fuzzy topological space. A fuzzy subset $A$ is $fsg^*$ closed iff whenever $A$ is not $q$ coincident with a semiclosed set $F$, $cl(A)$ is also not $q$ coincident with $F$.

PROOF:

Necessary: Let $A$ be $fsg^*$ closed.

If $A$ is not $q$ coincident with a fuzzy semiclosed set $F$, then

$A(x) + F(x) \leq 1$ for all $x \in X$.

$\Rightarrow A(x) \leq 1 - F^c(x)$ for all $x$.

$\Rightarrow A(x) \leq F^c(x)$ for all $x$.

$A \subseteq F^c$, $F^c$ is Fuzzy semiopen and $A$ is fuzzy sg* closed.

Therefore $cl(A) \subseteq F^c$. Hence $(cl(A)(x)) \leq F^c(x)$ for all $x$.

Hence $(cl(A)(x)) \leq 1 - F(x)$ for all $x$.

$(cl(A)(x)) + F(x) \leq 1$ for all $x$.

Hence $cl(A)$ is not $q$ coincident with $F$.

Conversely,

Let $A \subseteq U$, $U$ be semiopen.

Then $A(x) \leq U(x)$ for all $x$. Hence $A(x) \leq 1 - U^c(x)$ for all $x$.

Therefore $A(x) + U^c(x) \leq 1$ for all $x$.

Hence $A$ is not $q$ coincident with $U^c$ where $U^c$ is fuzzy semiclosed.

Therefore $cl(A)$ is not $q$ coincident with $U^c$.
Hence \((\text{cl}(A)(x)) + U(x) \leq 1\) for all \(x\).

Therefore \( (\text{cl}(A)(x)) + 1 - U(x) \leq 1 \) for all \(x\).

Hence \( (\text{cl}(A)(x)) \leq U(x) \) for all \(x\). Therefore \( \text{cl}(A) \subset U \).

Therefore \(A\) is \(fsg^*\) closed.

2.5 ON SEMI GENERALISED SETS

THEOREM 2.5.1: Every \(sg^*\)closed set is \(sg\) closed.

PROOF: Let \(A\) be a \(sg^*\) closed subset of \(X\). Let \(A \subset U\) and \(U\) be semi open. \(\text{cl}(A) \subset U\) since \(A\) is \(sg^*\)closed. \(\text{scl}(A) \subset U\) since \(\text{scl}(A) \subset \text{cl}(A)\).

Hence \(A\) is \(sg\) closed.

Converse of this Theorem is not true and is proved by the following example.

EXAMPLE 2.5.2: Let \(X = \mathbb{R}\) with usual metric topology. Let \(A = (0, 1]\) \(\text{scl}(A) = A\). Therefore \(A\) is \(sg\) closed. Let \(U = (0, 1]\). \(A \subset U\) and \(U\) is semi open. \(\text{Cl}(A) = [0, 1] \subset U\). Hence \(A\) is not \(sg^*\)closed.

THEOREM 2.5.3: Every \(sg^*\)closed set is \(gs\) closed.

PROOF: Let \(A\) be a \(sg^*\)closed subset of \(X\).
Let $A \subseteq U$ and $V$ open. Every open set is semi open. Therefore $U$ is semi open. $\emptyset$ is $sg^*$closed, $cl(A) \subseteq U$. $scl(A) \subseteq cl(A)$. Therefore $scl(A) \subseteq U$ is $gs$ closed.

Conversely, the theorem is not true and is proved by the following example.

**EXAMPLE 2.5.4:** Let $X=\{a,b,c\}$ $T=\{\emptyset, \{a\}, X\}$. Let $A=\{a,c\}$ The only open set containing $A$ is $X$. Therefore $A$ is $gs$ closed. $cl(A)=X$. Let $U=\{a,c\}$. $A \subseteq U$ and $U$ is semi open. But $cl(A) \not\subseteq U$. Hence $A$ is not $sg^*$closed.

**THEOREM 2.5.5:** Every $sg^*$open set is $sg$ open.

**PROOF:** Let $A$ be a $sg^*$open set in $(X, t)$. The $A^c$ is $sg^*$closed. Therefore $A^c$ is $sg$ closed since every $sg^*$closed set is $sg$ closed. Hence $A$ is $sg$ open.

Converse of this theorem is not true and is proved by the following example.

**EXAMPLE 2.5.6:** Let $X=[0,1]$, $t=$Usual Topology. $A^c=(0,1]$. $A^c$ is $sg$ closed.
A 2.5.7: Every sg*open set is gs open.

PROOF: Let A be a sg*open set in (X, τ).

A^c is sg*closed. A^c is gs closed since every sg*closed set is gs closed. Hence A is gs open.

Converse of this theorem is not true and is proved by the following example.

EXAMPLE 2.5.8: Let X={a,b,c} and τ = {ϕ, {a}, X} Let A={b}. A^c={a,c}. A^c is gs closed but not sg*closed Hence A is gs open but not sg*open.

THEOREM 2.5.9: A set A is sg closed iff for each (xₐ) in A semi converging to x, ∃ y∈A such that every net semiconverging to x, semiconverges to y also.

PROOF: Let A be sg*closed.

Let (xₐ) in A semiconverge to x. Then x∈scl(A).

consider scl(x).
claim: \( \text{scl}(x) \) contains a point of \( A \).

If not, then \( \text{scl}(x) \cap A = \emptyset \).

\( A \subseteq (\text{scl}(x))^c \), \( (\text{scl}(x))^c \) is semi open and \( A \) is sg closed.

Hence \( \text{scl}(A) \subseteq (\text{scl}(x))^c \).

\( x \in \text{scl}(A) \Rightarrow x \in (\text{scl}(x))^c \Rightarrow x \notin \text{scl}(x) \Rightarrow \Rightarrow \)

Hence \( \text{scl}(x) \) contains a point of \( A \).

Take \( y \in A \Rightarrow y \in \text{scl}(x) \).

Let \( (x_n) \) semi converge to \( x \).

Claim: \( (x_n) \) semi converges to \( y \).

Let \( O \) be any semi open set containing \( y \).

Claim: \( x \in O \).

If not then \( x \notin O \). Then \( x \in O^c \) and \( O^c \) is semi closed.

Hence \( \text{scl}(x) \subseteq O^c \). Since \( y \in \text{scl}(x) \), \( y \in O^c \). \( \Rightarrow \Rightarrow \) since \( y \in O \).

Hence \( x \in O \).

\( O \) is a semi open set containing \( x \) and \( (x_n) \) semi converges to \( x \).

Therefore \( (x_n) \) is eventually in \( O \).

Hence if \( O \) is any semi open set containing \( y \) then \( (x_n) \) is eventually in \( O \).

Therefore \( (x_n) \) semi converges to \( y \).

Therefore \( \exists y \in A \) such that every net semiconverging to \( x \), semiconverges to \( y \) also.
Conversely,

If for each \((x_\alpha)\) in \(A\) converging to \(x\), \(\exists y \in A \ni\) every net semi converging to \(x\) semiconverges to \(y\) also.

To prove: \(A\) is \(sg^*\)-closed.

Let \(A \subseteq U\), \(U\) be semi open.

Take \(x \in scl(A)\). Then \(\exists (x_\alpha)\) in \(A\) semiconverging to \(x\).

Therefore \(\exists y \in A \ni\) every net semi converging to \(x\), semi converges to \(y\) also.

Claim: \(y \in scl(x)\).

consider the net \((x_\alpha)\) where \(x_\alpha = x\) for each \(\alpha\).

\((x_\alpha)\) is a net semi converging to \(x\).

Therefore \((x_\alpha)\) semi converges to \(y\). Therefore every semi open set containing \(y\) contains elements of the net. Therefore every semi open set containing \(y\) contains \(x\) also.

If \(y \notin scl(x)\), then \(y \in (scl(x))^c\) and \((scl(x))^c\) is semi open.

Therefore \((scl(x))^c\) contains \(x\) also. \(\Rightarrow\)

Hence \(y \in scl(x)\). Also \(y \in A\). Hence \(scl(x) \cap A \neq \emptyset\).

Claim: \(x \in U\).

suppose not,

Then \(x \in U^c\) and \(U^c\) is semi closed.

Therefore \(scl(x) \subseteq U^c\).
Since \( \text{scl}(x) \cap A \neq \emptyset \), \( A \cap U^c \neq \emptyset \). Therefore \( A \cap U^c \).

Hence \( x \in U \). Therefore \( \text{scl}(A) \subset U \).

\[ \text{Th.} \quad \text{sg closed.} \]

**THEOREM 2.5.10:** A is sg closed iff whenever A does not intersect a closed set F, \( \text{scl}(A) \) also does not intersect F.

**PROOF:** A is sg closed. A does not intersect a semiclosed set F.

Therefore \( A \subset F^c \). Now \( F^c \) is semi open and \( A \subset F^c \).

Therefore \( \text{scl}(A) \subset F^c \). Hence \( \text{scl}(A) \) does not intersect F.

Conversely,

Let \( A \subset U \) and \( U \) be semi open.

A does not intersect the semiclosed set \( U^c \).

Therefore \( \text{scl}(A) \) does not intersect \( U^c \).

Hence \( \text{scl}(A) \cap U^c = \emptyset \).

Therefore \( \text{scl}(A) \subset U \).

Therefore A is sg closed.

**THEOREM 2.5.11:** A is sg open iff G is closed and \( \text{int}G \subset A \Rightarrow A \cap G = \text{sint} A \cap G \).

**PROOF:** A is sg open.

Let \( G \) be a closed set with \( \text{int}G \subset A \).
Let \( x \in A \cap G \). Then \( x \in A \) and \( x \in G \).

Case 1: Suppose \( x \in \text{int } G \).

\( \text{int } G \) is a semi open set contained in \( A \).

\( \text{sint } A \) is the largest semi open set contained in \( A \).

Therefore \( \text{int } G \subseteq \text{sint } A \). Hence \( x \in \text{sint } A \).

Therefore \( x \in \text{sint } A \cap G \). Hence \( A \cap G \subseteq \text{sint } A \cap G \).

Therefore \( A \cap G = \text{sint } A \cap G \).

Case 2: Suppose \( x \notin \text{int } G \).

Let \( F = \text{int } G \cup \{ x \} \). Now \( \text{Int } G \subseteq F \subseteq G \).

Therefore \( F \) is semi closed.

Since \( \text{Int } G \subseteq A \) and \( x \in A \), \( F \subseteq A \).

\( A \) is sg open, \( A \supseteq F \) and \( F \) is semi closed.

Now \( A^c \) is sg closed, \( A^c \subseteq F^c \) and \( F^c \) is semiopen.

Therefore \( \text{scl}(A^c) \subseteq F^c \). Hence \( \text{sint } A \supseteq F \).

Therefore \( x \in \text{sint } A \). Hence \( x \in \text{sint } A \cap G \).

\( A \cap G \subseteq \text{sint } A \cap G \). Hence \( A \cap G = \text{sint } A \cap G \).

Conversely,

Let \( A^c \subseteq U \) where \( U \) is semi open. Let \( U^c = F \).

Let \( A \supseteq F \), and \( F \) is semi closed.

There exists a closed set \( G \) with \( \text{Int } G \subseteq F \subseteq G \).
Since $A \supset F$, $A \supset \text{int} G$. Therefore $A \cap G = \text{sint} A \cap G$.

$F \subseteq A$ and $F \subseteq G$. Therefore $F \subseteq A \cap G$.

Therefore $F \subseteq \text{sint} A \cap G$. Hence $F \subseteq \text{sint} A$.

Therefore $F^c \supset (\text{sint} A)^c$. Hence $\text{cl}(A^c) \subseteq U$.

Therefore $A^c$ is sg closed. Hence $A$ is sg open.

**THEOREM 2.5.12:** $A$ is sg closed iff $O$ is open and $A \subseteq \text{cl}(O) \Rightarrow A \cup O = \text{scl}(A) \cup O$.

**PROOF:** $A$ is sg closed.

Let $O$ be an open set with $A \subseteq \text{cl}(O)$. Let $G = O^c$. Then $G$ is closed.

$A \subseteq \text{cl}(O) \Rightarrow A^c \supset [\text{cl}(O)]^c \Rightarrow A^c \supset \text{int} O^c \Rightarrow A^c \supset \text{int} G$.

Now $A^c$ is sg open, $G$ is closed and $\text{int} G \subseteq A^c$.

Therefore $A^c \cap G = \text{sint} A^c \cap G$. Hence $A^c \cap O^c = \text{sint} A^c \cap O^c$.

Therefore $A^c \cap O^c = (\text{scl}(A))^c \cap O^c$. Hence $(A \cup O)^c = (\text{scl}(A) \cup O)^c$.

Hence $A \cup O = \text{scl}(A) \cup O$.

Conversely,

Let $G$ be closed with $\text{int} G \subseteq A^c$. Let $O = G^c$.

$\text{int} G \subseteq A^c \Rightarrow \text{int} O^c \subseteq A^c \Rightarrow (\text{cl}(O))^c \subseteq A^c \Rightarrow A \subseteq \text{cl}(O)$.

$O$ is open and $A \subseteq \text{cl}(O)$. Therefore $A \cup O = \text{scl}(A) \cup O$.

$A \cup O = \text{scl}(A) \cup O \Rightarrow (A \cup O)^c = [\text{scl}(A) \cup O]^c$. 

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=> A^c \cap O^c= \{scl(A)\}^c \cap O^c.

=> A^c \cap G = \text{int} A^c \cap G.

Therefore $A^c$ is sg open. Hence $A$ is sg closed.

**THEOREM 2.5.13:** The following are equivalent.

1. $A$ is sg closed.
2. For each $x \in \text{scl}(A)$, $\text{scl}(x) \cap A \neq \emptyset$.
3. $\text{scl}(A)$- $A$ contains no non empty semi closed set.

**PROOF:** To prove $1 \Rightarrow 2$.

Let $A$ be a sg closed subset of $X$. Let $x \in \text{scl}(A)$.

Case 1: Let $x \in A$.

Then $\text{scl}(x) \cap A \neq \emptyset$.

Case 2: Let $x \notin A$.

suppose $\text{scl}(x) \cap A = \emptyset$, then $A \subseteq \{\text{scl}(x)\}^c$.

Let $U = \{\text{scl}(x)\}^c$. $A \subseteq U$, $U$ is semi open and $A$ is sg closed.

Therefore $\text{scl}(A) \subseteq U$. Since $x \in \text{scl}(A)$, $x \in U$.

Hence $x \in \{\text{scl}(x)\}^c$. $\Rightarrow \Leftarrow$

Therefore $\text{scl}(x) \cap A \neq \emptyset$.

Hence $1 \Rightarrow 2$ is proved.

To prove $2 \Rightarrow 3$. 

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Suppose \( \text{scl}(A)-A \) contains a non empty semi closed set \( F \).

\[ F \subseteq \text{scl}(A)-A. \]

Take any \( x \in F \). Then \( x \in \text{scl}(A) \). Therefore \( \text{scl}(x) \cap A \neq \emptyset \).

\[ x \in F \text{ and } F \text{ is semi closed. } \Rightarrow \text{scl}(x) \subseteq F. \]

Therefore \( F \neq \emptyset \). \( \Rightarrow \) since \( F \subseteq \text{scl}(A)-A \).

Therefore \( \text{scl}(A)-A \) can not contain any non empty semi closed set.

Hence \( 2 \Rightarrow 3 \) is proved.

To prove \( 3 \Rightarrow 1 \).

Suppose \( A \) is not sg closed.

Then there exists a semi open set \( U \) with \( A \subseteq U \) and \( \text{scl}(A) \subsetneq U \).

Then \( \text{scl}(A)-A \subsetneq U \). Hence there exists \( x \in \text{scl}(A)-A \) and \( x \notin U \).

Let \( F = \text{scl}(A)-U \). Then \( x \in F \). Hence \( F \neq \emptyset \).

Since \( F = \text{scl}(A) \cap U^c \), \( F \) is semi closed.

\[ F = \text{scl}(A)-U \Rightarrow F \subseteq \text{scl}(A)-A. \]

Hence \( \text{scl}(A)-A \) contains a non empty semi closed set.

\[ \Rightarrow \emptyset \text{ Therefore } A \text{ is sg closed. Hence } 3 \Rightarrow 1 \text{ is proved.} \]

**THEOREM 2.5.14:** The following are equivalent.

1. \( A \) is sg open.

2. For each \( x \in A - \text{sint } A \), \( \text{scl}(x) \cap A^c \neq \emptyset \).

3. \( A - \text{sint } A \) does not contain a non empty semi closed set.
prove $1 \Rightarrow 2$.

Let $B = A^c$. Then $B$ is sg closed.

$x \in A \setminus A \Rightarrow x \notin A^c$.

$\Rightarrow x \in (A^c)^c$.

$\Rightarrow x \in \text{scl } A^c$.

$\Rightarrow x \in \text{scl } B$.

$B$ is sg closed and $x \in \text{scl } B$.

$\text{scl}(x) \cap B \neq \emptyset$. Therefore $\text{scl}(x) \cap A^c \neq \emptyset$.

Hence $1 \Rightarrow 2$ is proved.

To Prove $2 \Rightarrow 3$.

Suppose $3$ is not true.

Then $A \setminus A$ contains a non empty semi closed set $F$.

$F \subseteq A \setminus A$. Let $x \in F$. Then $x \in A \setminus A$. Hence $\text{scl}(x) \cap A^c \neq \emptyset$.

$x \in F \Rightarrow \text{scl}(x) \subseteq \text{scl}(F) \Rightarrow \text{scl}(x) \subseteq A \Rightarrow \text{scl}(x) \cap A^c = \emptyset$.

$\Rightarrow \text{scl}(x) \subseteq F$. Therefore $3$ is true.

Hence $2 \Rightarrow 3$ is proved.

To Prove $3 \Rightarrow 1$.

$A \setminus A$ does not contain any non empty semi closed set.

Let $B = A^c$.

$\text{scl}(B) - B = \text{scl}(A^c) \cap B^c$. 


\[= [\text{sint } A]^c \cap A.\]

\[= \text{A-sint } A.\]

Hence \(\text{scl}(B)-B\) does not contain any non empty semi closed set.

Therefore \(B\) is sg closed. Hence \(A\) is sg open. Hence \(3=>1\) is proved.

Relevant definitions in fuzzy topological spaces that we use in the following are from 2.4

**THEOREM 2.5.15:** Let \(X\) be a fuzzy topological space. A fuzzy subset \(A\) is fuzzy sg closed iff whenever \(A\) is not q coincident with a fuzzy semi closed set \(F\), \(\text{scl}(A)\) is also not q coincident with \(F\).

**PROOF:**

If \(A\) is fuzzy sg closed and if \(A\) is not q coincident with fuzzy a semi closed \(F\). Then \(A(x) + F(x) \leq 1\) for all \(x\) in \(X\). Then \(A(x)+1-F^c(x) \leq 1\).

Hence \(A(x) \leq F^c(x)\) for all \(x\).

\(A \subseteq F^c\), \(F^c\) is fuzzy semi open and \(A\) is fuzzy sg closed.

Therefore \(\text{scl}(A) \subseteq F^c\).

\(\text{scl}(A) \subseteq F^c \Rightarrow (\text{scl}(A)(x)) \leq F^c(x)\) for all \(x\).

\[\Rightarrow (\text{scl}(A)(x)) \leq 1-F(x)\] for all \(x\).

\[\Rightarrow (\text{scl}(A)(x))+F(x) \leq 1\] for all \(x\).

Therefore \(\text{scl}(A)\) is not q coincident with \(F\).

Conversely, Let \(A \subseteq U\), \(U\) be fuzzy semi open.
\(A \subseteq U \implies A(x) \leq U(x)\) for all \(x\).

\(\implies A(x) \leq 1 - U^c(x)\) for all \(x\).

\(\implies A(x) + U^c(x) \leq 1\) for all \(x\).

Hence \(A\) is not q coincident with \(U^c\) where \(U^c\) is fuzzy semi closed.

Therefore \(\text{scl}(A)\) is not q coincident with \(U^c\).

Hence \((\text{scl}(A)(x)) + U^c(x) \leq 1\) for all \(x\).

Therefore \((\text{scl}(A)(x)) + 1 - U(x) \leq 1\) for all \(x\).

Hence \((\text{scl}(A)(x)) \leq U(x)\) for all \(x\). Hence \(\text{scl}(A) \subseteq U\).

Therefore \(A\) is fuzzy sg closed.

**THEOREM 2.5.16:** A set \(A\) is gs closed iff for each \((x_\alpha)\) in \(A\) semi converging to \(x\), \(\exists y \in A\) such that every net converging to \(x\), converges to \(y\) also.

**PROOF:** Let \(A\) be gs closed.

Let \((x_\alpha)\) in \(A\) semi converge to \(x\). Then \(x \in \text{scl}(A)\).

Claim: \(\text{cl}(x)\) contains a point of \(A\).

If not, then \(\text{cl}(x) \cap A = \emptyset\).

Then \(A \subseteq (\text{cl}(x))^c\), \((\text{cl}(x))^c\) is open and \(A\) is gs closed.

Hence \(\text{scl}(A) \subseteq (\text{cl}(x))^c\).

\(x \in \text{scl}(A) \implies x \in (\text{cl}(x))^c \implies x \notin \text{cl}(x) \implies \)
Hence \( \text{cl}(x) \) contains a point of \( A \). Take \( y \in A \) such that \( y \in \text{cl}(x) \).

Let \((x_\beta)\) converge to \( x \).

Claim: \((x_\beta)\) converges to \( y \).

Let \( O \) be any open set containing \( y \).

Claim: \( x \in O \).

If not then \( x \not\in O \). Hence \( x \in O^c \) and \( O^c \) is closed.

Therefore \( \text{cl}(x) \subseteq O^c \).

Since \( y \in \text{cl}(x) \), \( y \in O^c \). \( \Rightarrow \Leftrightarrow \) since \( y \in O \).

Hence \( x \in O \).

\( O \) is an open set containing \( x \) and \((x_\beta)\) converges to \( x \).

Therefore \((x_\beta)\) is eventually in \( O \).

Hence if \( O \) is any open set containing \( y \) then \((x_\beta)\) is eventually in \( O \).

Therefore \((x_\beta)\) converges to \( y \).

Therefore \( \exists \ y \in A \) such that every net converging to \( x \), converges to \( y \) also.

Conversely,

If for each \((x_\alpha)\) in \( A \) semi converging to \( x \), \( \exists y \in A \) every net converging to \( x \) converges to \( y \) also.

To prove: \( A \) is gs closed.

Let \( A \subseteq U \) and \( U \) be open.
Take any $x \in \text{cl}(A)$.

Then $\exists (x_\alpha)$ in $A$ semi converging to $x$.

Therefore $\exists y \in A \ni \text{ every net converging to } x$, converges to $y$ also.

Claim: $y \in \text{cl}(x)$.

Consider the net $(x_\alpha)$ where $x_\alpha = x$ for each $\alpha$. $(x_\alpha)$ is a net in $A$ converging to $x$. Therefore $(x_\alpha)$ converges to $y$. Therefore every open set containing $y$ contains elements of the net. Hence every open set containing $y$ contains $x$ also.

If $y \notin \text{cl}(x)$ then $y \in (\text{cl}(x))^c$ and $(\text{cl}(x))^c$ is open.

Therefore $(\text{cl}(x))^c$ contains $x$ also. $\Rightarrow \Leftarrow$

Hence $y \in \text{cl}(x)$.

Since $y \in A$, $\text{cl}(x) \cap A \neq \emptyset$.

Claim: $x \in U$.

Suppose not,

Then $x \in U^c$ and $U^c$ is closed. Then $\text{cl}(x) \subset U^c$.

Since $\text{cl}(x) \cap A \neq \emptyset$, $A \cap U^c \neq \emptyset$. Therefore $A \subset U$. $\Rightarrow \Leftarrow$

Hence $x \in U$. Therefore $\text{cl}(A) \subset U$. Hence $A$ is gs closed.