CHAPTER 7

GENERALISED SETS OF FUZZY TOPOLOGICAL SPACE

7.1 INTRODUCTION:

In chapter 2, some concepts of fuzzy topological spaces have been seen. In this chapter a crisp topology is derived from a fuzzy topology.

In 2000 P.V. Ramakrishnan and V. Lakshmana Gomathi Nayagam,(29) studied nearly fuzzy spaces. They have introduced the following.

Two fuzzy sets $\mu$ and $\nu$ of $X$ are said to intersect at $x$ if $\mu(x)+\nu(x)>1$.

Two fuzzy sets $\mu_1$ and $\mu_2$ of $X$ are said to be disjoint if they do not intersect at any point of $X$ i.e. $\mu_1(x)+\mu_2(x) \leq 1$, for all $x \in X$.

Let $(X,\delta)$ be a fuzzy topological space. A singleton subset $\{x\}$ of $X$ is said to be fuzzy closed if there exists a fuzzy closed set $\mu$ with support $\{x\}$ and $\mu(x) > 1/2$.

A sequence of points $x_n$ of $X$ is said to converge fuzzily to $x \in X$ in $(X,\delta)$, if, for all $\mu \in \delta$ such that $\mu(x)>1/2$ there exists $N$ such that $\mu(x_n)>1/2$ for all $n \geq N$. 
A fuzzy topological space \((X, \delta)\) is said to be nearly fuzzy \(T_1\) (n.f. \(T_1\)) if for every pair \(x, y \in X\) such that \(x \neq y\) there exist \(\mu, \nu \in \delta\) such that \(\mu(x) > \frac{1}{2}\) and \(\mu(y) \leq \frac{1}{2}\) or \(\nu(y) > \frac{1}{2}\) and \(\nu(x) \leq \frac{1}{2}\).

### 7.2 INDUCED CRISP TOPOLOGY

**DEFINITION 7.2.1:** Let \(X\) be a fuzzy topological space.

Let \(\mu\) be a fuzzy subset of \(X\). Then crisp of \(\mu\) is defined to be the set \(\{x \mid \mu(x) > 1/2\}\) and the crisp of \(\mu\) is denoted be \(A_\mu\).

**REMARK 7.2.2:** \(A_\mu = \{x \mid \mu(x) > 1/2\}\) has significance in certain situations.

For example, let us assume that \(\mu(x)\) stands for the mark obtained by a student in any talent examination. For any \(x\), if \(\mu(x) > 1/2\) then we can say that \(x\) is talented. \(\mu^c\) measures non-talent. When \(\mu(x) > 1/2\), we have \(\mu(x) > \mu^c(x)\). This justifies our statement that \(x\) is talented if \(\mu(x) > 1/2\).

It follows that \(x\) is not talented if \(\mu^c(x) > 1/2\) or \(\mu(x) < 1/2\). The case \(\mu(x) \neq 1/2\) gives rise to an ambiguity. Henceforth discard this ambiguous case. That is, we assume \(\mu(x) \neq 1/2\) for every \(x\) and for every \(\mu\) considered. Clearly, \(\alpha \subset \beta\) implies \(A_\alpha \subset A_\beta\). Further here we assume a consistency relation namely \(A_\alpha \subset A_\beta\) implies \(\alpha \subset \beta\).
**THEOREM 7.2.3:** Let \((X, \delta)\) be Fuzzy topological space. Then 
\[ T = \{ A_\mu / \mu \in \delta \} \] is a topology on \(X\).

**PROOF:**  
\((X, \delta)\) is a Fuzzy topology. 
\[ T = \{ A_\mu / \mu \in \delta \} \] Since Zero map \( e \in \delta \), we have \( \varnothing \in T \). The map \( 1 \in \delta \).  
\( 1: X \rightarrow [0,1] \) defined as \( 1(x) = 1 \) for all \( x \). Now \( A_1 = \{ x / 1(x) > 1/2 \} \) \( =X \). Therefore \( X \in T \).

Claim: arbitrary union of elements of \( T \) belongs to \( T \).

Let \( \{ A_{\mu \alpha} / \alpha \in I \} \) be a family of elements of \( T \).

For each \( \alpha \), \( A_{\mu \alpha} \in T \). Therefore \( \mu \alpha \in \delta \). Since \( \delta \) is a topology, \( \cup \mu \alpha \in \delta \).

Let \( \mu = \cup \mu \alpha \).

Claim: \( \cup A_{\mu \alpha} = A_{\cup \mu \alpha} = A_\mu \).

\[ x \in \cup A_{\mu \alpha} \Rightarrow x \in A_{\mu \alpha} \text{ for some } \alpha \Rightarrow \mu \alpha (x) > 1/2 \text{ for some } \alpha. \]

\[ \Rightarrow \cup \mu \alpha (x) > 1/2 \Rightarrow x \in A_\mu. \]

Hence \( \cup A_{\mu \alpha} \subset A_\mu \).

\[ x \in A_\mu \Rightarrow \mu (x) > 1/2 \Rightarrow \cup \mu \alpha (x) > 1/2 \Rightarrow \mu \alpha (x) > 1/2 \text{ for some } \alpha. \]

\[ \Rightarrow x \in A_{\mu \alpha} \text{ for some } \alpha \Rightarrow x \in \cup A_{\mu \alpha}. \]

Hence \( A_\mu \subset \cup A_{\mu \alpha} \). Therefore \( \cup A_{\mu \alpha} = A_\mu \).

Therefore arbitrary union of elements of \( t \) belongs to \( T \).

Claim: Finite intersection of elements \( t \) belongs to \( t \).

Claim \( \cap A_{\mu i} = A_{\cap \mu i} \).

\[ x \in \cap A_{\mu i} \Rightarrow x \in A_{\mu i} \text{ for each } i \Rightarrow \mu i (x) > 1/2 \text{ for each } i. \]
\[\Rightarrow \min \{\mu_i(x)/i=1,2..n\}>1/2 \Rightarrow \cap \mu_i(x)>1/2.\]

\[\Rightarrow x \in \cap A_{\mu_i}.\]

Hence \(\cap A_{\mu_i} \subset A \cap \mu_i.\)

\[x \in \cap A_{\mu_i} \Rightarrow \cap \mu_i(x)>1/2 \Rightarrow \min \{\mu_i(x)/i=1,2..n\}>1/2.\]

\[\Rightarrow \mu_i(x)>1/2 \text{ for each } i \Rightarrow x \in A_{\mu_i} \text{ for each } i.\]

\[\Rightarrow x \in \cap A_{\mu_i}.\]

Hence \(\cap A_{\mu_i} = A \cap \mu_i.\)

\(\mu_i \in \delta \) for each \(i = 1,2...n\) and \(\delta\) is a topology.

Hence \(\cap \mu_i \in \delta.\) Therefore \(A \cap \mu_i \in T.\)

Therefore \(\cap A_{\mu_i} \in T.\)

Therefore finite intersection of elements of \(i\) is in \(i.\)

Hence \(T\) is a topology.

**DEFINITION 7.2.4:** Induced crisp topology

Let \((X,\delta)\) be a fuzzy topology. Let \(T = \{A_\mu/ \mu \in \delta\}.\) Then the topology \(T\) is called the induced crisp topology of the fuzzy topology \(\delta.\)

**RESULT 7.2.5:** \(x \in A_{\mu_1} \cap A_{\mu_2} \Rightarrow \mu_1 \text{ and } \mu_2 \text{ intersect at } x.\)

\(x \in A_{\mu_1} \cap A_{\mu_2} \Rightarrow x \in A_{\mu_1} \text{ and } x \in A_{\mu_2}.\)

\[\Rightarrow \mu_1(x) > \frac{1}{2} \text{ and } \mu_2(x) > \frac{1}{2}.\]
\[ \Rightarrow \mu_2(x) + \mu_2(x) > 1. \]

\[ \Rightarrow \mu_1 \text{ and } \mu_2 \text{ intersect at } x. \]

**RESULT 7.2.6:** \( x \in A_{\mu_1 \cap \mu_2} \Rightarrow \mu_1 \text{ and } \mu_2 \text{ intersect at } x. \)

True since \( A_{\mu_1 \cap \mu_2} = A_{\mu_1} \cap A_{\mu_2} \).

**RESULT 7.2.7:** Converse is not true.

Let \( \mu_1(x) = 0.8 \) and \( \mu_2(x) = 0.3 \). \( x \in A_{\mu_1} \) and \( x \notin A_{\mu_2} \).

\( x \notin A_{\mu_1 \cap \mu_2} \) but \( \mu_1(x) + \mu_2(x) > 1 \). Therefore \( \mu_1 \) and \( \mu_2 \) intersect at \( x \).

**THEOREM 7.2.8:** Let \((X, \delta)\) be a Fuzzy topological space and let \((X, T)\) be the induced crisp topological space. Then \((X, \delta)\) is n.f..T1 iff \((X, T)\) is T1.

**PROOF:** Follows from the definition.

**THEOREM 7.2.9:** Let \((X, \delta)\) be a fuzzy topological space and Let \((X, T)\) be the induced crisp topology. Then \( \{x\} \) is fuzzy closed in \((X, \delta)\) implies \( \{x\} \) is closed in \((X, T)\).

**PROOF:** Let \( \{x\} \) be fuzzy closed in \((X, \delta)\). Then there exist a fuzzy closed set \( \mu \) with support \( \{x\} \) and \( \mu(x) > 1/2 \).

Hence \( \mu(y) = 0 \) if \( y \neq x \) and \( \mu(x) > 1/2 \).
Then $\mu^c(y) = 1$ if $y \neq x$ and $\mu^c(x) < \frac{1}{2}$.

Now $\mu$ is closed $\Rightarrow \mu^c$ is open $\Rightarrow \{y/\mu^c(y) > 1/2\}$ is open $(X,T)$.

$\Rightarrow\{y/y \neq x\}$ is open in $(X,T)$.

$\Rightarrow\{x\}^c$ is open in $(X,T)$.

$\Rightarrow\{x\}$ is closed in $(X,T)$.

Therefore $\{x\}$ is fuzzy closed $\Rightarrow \{x\}$ is closed in $(X,T)$.

Converse of the above theorem is not true.

**EXAMPLE 7.2.10:** $X=\{a,b,c\}$.

Let $\delta = \{0,1,\mu\}$ where $\mu(a) = 0.1$

$\mu(b) = 0.7$

$\mu(c) = 0.8$

$(x,\delta)$ is a Fuzzy topological space.

$A_{\mu} = \{b,c\}$.

The induced crisp topology is $(X,\tau)$ where $\tau = \{\phi, X, \{b,c\}\}$.

$\{b,c\}$ is open. Therefore $\{a\}$ is closed in $(X,\tau)$.

$\{a\}$ is not Fuzzy closed in $(X,\delta)$.

Then is no $\mu$ with support $\{a\}$ and $\mu(a) > \frac{1}{2}$.

**RESULT 7.2.11:** $A_{\mu^c} = (A_{\mu})^c$. 

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\[ x \in (A\mu)^c \iff x \notin A\mu \iff \mu(x) \leq \frac{1}{2} . \]

iff \( \mu(x) < \frac{1}{2} \) iff \( -\mu(x) > -\frac{1}{2} . \)

iff \( 1 - \mu(x) > 1 - \frac{1}{2} . \)

iff \( \mu^c(x) > \frac{1}{2} . \)

\[ x \in A\mu^c . \]

\[ x \in (A\mu)^c \iff x \in A\mu^c . \] Therefore \( A\mu^c = (A\mu)^c . \)

RESULT 7.2.12: \( \mu \) is closed iff \( A\mu \) is closed.

\( \mu \) is closed in \( \tau \) iff \( \mu^c \) is open in \( \tau \).

iff \( A\mu^c \) is open in \( \delta \).

iff \( (A\mu)^c \) is open in \( \delta \).

iff \( A\mu \) is closed in \( \delta \).

RESULT 7.2.13: \( \cap A\mu^c = A \cap_{\mu^c} \mu i \in T \).

We recall from 7.1 that a sequence of points \( x_n \) of \( X \) is said to converge fuzzily to \( x \in X \) in \((X,\delta)\), if, for all \( \mu \in \delta \) such that \( \mu(x) > 1/2 \) there exists \( N \) such that \( \mu(x_n) > 1/2 \) for all \( n \geq N \).

THEOREM 7.2.14: Let \((X,\delta)\) be a fuzzy topological space and let \((X,T)\) be the induced crisp topological space. Let \( (x_n) \) be a sequence of points in \( X \). Then \( (x_n) \) converges fuzzily to \( x \) iff \( (x_n) \) converges to \( x \) in the induced crisp topology.
PROOF:

Part 1

Let \((x_n)\) converge to \(x\) fuzzily.

To prove \((x_n)\) converges to \(x\) in the induced topology.

Let \(x \in A_\mu\) Then \(\mu(x) > 1/2\) Hence \(\exists N \ni \mu(x) > 1/2 \ \forall n \geq N\).

Hence \(x_n \in A_\mu \ \forall n \geq N\). Therefore \((x_n)\) converges to \(x\).

Conversely,

Let \((x_n)\) converge to \(x\) in the induced topology.

To prove: \((x_n)\) converges to \(x\) fuzzily.

Take \(\mu \in \delta\) with \(\mu(x) > \frac{1}{2}\).

\(\mu \in \delta \Rightarrow A_\mu \in \mathfrak{t}\) and also \(x \in A_\mu\).

\((x_n)\) converges to \(x\) and \(A_\mu\) is an open set containing \(x\).

Hence \(\exists N \ni x_n \in A_\mu \ \forall n \geq N\). Hence \(\mu(x) > 1/2 \ \forall n \geq N\).

Therefore \((x_n)\) converges to \(x\) fuzzily.

7.3 BEHAVIOUR OF FUNCTIONS

THEOREM 7.3.1: X and Y be two Fuzzy topological spaces.

\(f : X \rightarrow Y\) be a surjective map. \(\mu\) is a Fuzzy set on \(X\). Then \(f(A_{f^{-1}(\mu)}) = A_\mu\)

PROOF: X and Y are Fuzzy topological spaces.

\(f : X \rightarrow Y\) be a surjective map.

\(\mu\) is a fuzzy set on \(X\). Let \(\alpha = f^{-1}(\mu)\).
Then $\alpha$ is a Fuzzy set on $X$ defined as $\alpha(x) = \mu(f(x))$.

Claim $f(A_{\alpha}) = A_{\mu}$.

Let $y \in f(A_{\alpha})$.

Then $\exists x \in A_{\alpha} \ni f(x) = y$.

$x \in A_{\alpha} \Rightarrow \alpha(x) > 1/2$.

$\Rightarrow \mu(f(x)) > 1/2$.

$\Rightarrow \mu(y) > 1/2$.

$\Rightarrow y \in A_{\mu}$.

Hence $f(A_{\alpha}) \subset A_{\mu}$.

Let $y \in A_{\mu}$ Since $f$ is onto, $\exists x \in X \ni f(x) = y$.

$y \in A_{\mu} \Rightarrow \mu(y) > 1/2$.

$\Rightarrow \mu(f(x)) > 1/2$.

$\Rightarrow \alpha(x) > 1/2$.

$\Rightarrow x \in A_{\alpha}$.

$\Rightarrow f(x) \in f(A_{\alpha})$.

$\Rightarrow y \in f(A_{\alpha})$.

Hence $A_{\mu} \subset f(A_{\alpha})$.

Therefore $f(A_{\alpha}) = A_{\mu}$.

Hence $f(A_{\alpha \cap \mu}) = A_{\mu}$. 

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**THEOREM 7.3.2:** X and Y be two Fuzzy topological spaces.

f : X -> Y be a any map. μ be a Fuzzy set on X. Then \( A_{f^{-1}(μ)} = f^{-1}(A_μ) \)

**PROOF:** Similar to previous theorem.

**THEOREM 7.3.3:** Let \((X, δ_1)\) and \((Y, δ_2)\) be two Fuzzy topological spaces and let \((X, τ_1)\) and \((Y, τ_2)\) be the corresponding induced crisp topological spaces. Let \(f : X -> Y\) be a any onto map. If \(f : (X, δ_1) -> (Y, δ_2)\) is Fuzzy continuous, then \(f : (X, τ_1) -> (Y, τ_2)\) is continuous [that is if \(f\) is a fuzzy continuous function then it is continuous in the induced crisp topologies]

**PROOF:** Let \((X, δ_1)\) and \((Y, δ_2)\) be two Fuzzy topological spaces.

let \((X, τ_1)\) and \((Y, τ_2)\) be the corresponding induced crisp topological spaces.

Let \(f : X -> Y\) be onto Fuzzy continuous.

Claim: \(f : (X, τ_1) -> (Y, τ_2)\) is continuous.

Let B be a open subset of \((Y, τ_2)\).

Then \(∃μ ∈ δ_2 \ ∃ B = A_μ\).

\(μ\) is open in \((Y, δ_2)\) and \(f : X -> Y\) is Fuzzy continuous.

Hence \(f^{-1}(μ)\) is open in \((X, δ_1)\).

Hence \(A_{f^{-1}(μ)}\) is open in \((X, τ_1)\).
Now \( f^{-1}(\mu) = f'(A_{\mu}) \).

Hence \( f'(A_{\mu}) \) is open in \((X, t_1)\).

Hence \( f'(B) \) is open in \((X, t_1)\).

Therefore \( f : (X, t_1) \to (Y, t_2) \) is continuous.

**THEOREM 7.3.4:** Let \((X, \delta_1)\) and \((Y, \delta_2)\) be two Fuzzy topological spaces and let \((X, t_1)\) and \((Y, t_2)\) be the corresponding induced crisp topological spaces. If \( f : (X, \delta_1) \to (Y, \delta_2) \) is Fuzzy Homeomorphic, then \( f : (X, t_1) \to (Y, t_2) \) is Homeomorphic.

**PROOF:** Follows from the previous theorem.

**THEOREM 7.3.5:**

Let \((X, \delta)\) be a fuzzy topological space and \((X, t)\) be the induced crisp topological space. Let \( \mu \) be a fuzzy set of \(X\). Then

1. \( \text{cl}(A_{\mu}) = A_{\text{cl}(\mu)} \).
2. \( \text{int}(A_{\mu}) = A_{\text{int}(\mu)} \).
3. \( \mu \) is semi open implies \( A_{\mu} \) is semi open.
4. \( \mu \) is \( \text{sg}^* \) closed \( \Rightarrow A_{\mu} \) is \( \text{sg}^* \) closed.
5. \( \mu \) is \( \alpha \text{g}^* \) closed \( \Rightarrow A_{\mu} \) is \( \alpha \text{g}^* \) closed.
6. \( \mu \) is \( \beta \text{g}^* \) closed \( \Rightarrow A_{\mu} \) is \( \beta \text{g}^* \) closed.
7. $\mu$ is $pg^*$ closed $\Rightarrow A_\mu$ is $pg^*$ closed.

Proof follows from the definition of induced crisp topology with the assumptions mentioned in the remark 7.2.2.