CHAPTER 6

GENERALISED LOCAL CONNECTEDNESS

6.1 LOCALLY $\alpha$ CONNECTEDNESS

DEFINITION 6.1.1: Let $X$ be a topological space.

Let $A$ and $B$ be two subsets of $X$. $A$ and $B$ are said to be $\alpha$ separated if

$$A \cap \alpha\text{cl } B = \alpha\text{cl } A \cap B = \emptyset.$$ 

DEFINITION 6.1.2: Let $X$ be a topological space. Let $A$ be a subset of $X$. $A$ is called a $\alpha$ connected set if $A$ can not be expressed as the union of two non empty $\alpha$ separated sets.

The topological space $X$ is said to be $\alpha$ connected if $X$ can not be expressed as the union of two non empty $\alpha$ separated sets.

DEFINITION 6.1.3: Let $X$ be a topological space and let $x \in X$. $X$ is said to be locally $\alpha$ connected at $x$, if for every $\alpha$ open set $U$ containing $x$, $\exists$ a $\alpha$ connected open set $C$ such that $x \in C \subset U$.

DEFINITION 6.1.4: A topological space $X$ is said to be locally $\alpha$ connected if $X$ is locally $\alpha$ connected at each point of $X$. 
THEOREM 6.1.5: If $A$ is $\alpha$ connected and $A \subseteq C \cup D$ where $C$ and $D$ are $\alpha$ separated sets, then either $A \subseteq C$ or $A \subseteq D$.

PROOF: $A$ is $\alpha$ connected and $A \subseteq C \cup D$. $C$ and $D$ are $\alpha$ separated.

Then $A = (A \cap C) \cup (A \cap D)$.

$(A \cap C) \cap \alpha \text{cl} (A \cap D) \subseteq (A \cap C) \cap (\alpha \text{cl} A \cap \alpha \text{cl} D)$.

$\subseteq C \cap \alpha \text{cl} D = \phi$.

Similarly $(A \cap D) \cap \alpha \text{cl} (A \cap C) = \phi$. Hence $A \cap C$ and $A \cap D$ are $\alpha$ separated sets. Since $A$ is $\alpha$ connected and $A = (A \cap C) \cup (A \cap D)$,

$A \cap C = \phi$ or $A \cap D = \phi$. Hence $A = A \cap D$ or $A = A \cap C$.

Therefore $A \subseteq D$ or $A \subseteq C$.

THEOREM 6.1.6: The union of a family of $\alpha$ connected sets having non empty intersection is $\alpha$ connected.

PROOF:

Let $\{A_s / s \in I\}$ be a family of $\alpha$ connected subsets of $X$, where $I$ is an index set and let $\cap A_\alpha \neq \phi$.

Let $A = \cup A_s$. Suppose $A$ is not $\alpha$ connected, then $A = C \cup D$ where $C$ and $D$ are non empty $\alpha$ separated sets. Take $x \in \cap A_s$. Then $x \in A$.

Hence $x \in C \cup D$.

Since $C$ and $D$ are $\alpha$ separated, $x \in C$ or $x \in D$. 
Case 1: Let $x \in C$.

Fix any $s \in I$. Then $A_s \subset A$. Hence $A_s \subset C \cup D$.

$A_s$ is $\alpha$ connected, $C$ and $D$ are $\alpha$ separated. Hence by theorem 6.1.5 $A_s \subset C$ or $A_s \subset D$.

Since $x \in C$ and $x \in A_s$, $x \in A_s \cap C$. Hence $A_s \cap C \neq \emptyset$.

Now $A_s \subset C$ or $A_s \subset D$, $C$ and $D$ are $\alpha$ separated, $A_s \cap C \neq \emptyset$.

Hence $A_s \subset C$. This is true for each $s$ in $I$.

Hence $\bigcup A_s \subset C$. Hence $A \subset C$. Therefore $D = \emptyset$. $\Rightarrow \Leftarrow$

Therefore $A$ is $\alpha$ connected.

Case 2: Let $x \in D$. By a similar proof as in case 1, $A$ is $\alpha$ connected.

**THEOREM 6.1.7:** If $A$ is $\alpha$ connected and $A \subset B \subset \text{acl}(A)$, then $B$ is $\alpha$ connected. In particular $\text{acl}(A)$ is $\alpha$ connected.

**PROOF:** $A$ is $\alpha$ connected and $A \subset B \subset \text{acl}(A)$.

Suppose $B$ is not $\alpha$ connected. Then $B = C \cup D$ where $C$ and $D$ are non empty $\alpha$ separated sets.

$A \subset B \Rightarrow A \subset C \cup D$. By theorem 6.1.5. $A \subset C$ or $A \subset D$.

Case 1: Let $A \subset C$. Then $\text{acl} A \subset \text{acl} C$.

$x \in B \Rightarrow x \in \text{acl} A \Rightarrow x \in \text{acl} C \Rightarrow x \notin D$.

$B = C \cup D$ and $x \in B \Rightarrow x \notin D$. Hence $D = \emptyset$. $\Rightarrow \Leftarrow$

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Hence B is connected.

Case 2: Let \( A \subseteq D \). Similar to case 1, we get B is connected.

**DEFINITION 6.1.8:** Let \( x \in X \). The \( \alpha \) component of \( x \) denoted by \( \alpha c(x) \) is the union of all \( \alpha \) connected subsets of \( X \) containing \( x \).

**THEOREM 6.1.9:** \( \alpha c(x) \) is \( \alpha \) connected.

**PROOF:** Let \( x \in X \). \( \alpha c(x) \) is the union of all \( \alpha \) connected subsets of \( X \) containing \( x \). Hence by theorem 6.1.6, \( \alpha c(x) \) is \( \alpha \) connected.

**DEFINITION 6.1.10:** Let \( E \) be a subset of \( X \) and \( x \in E \). Then the union of all \( \alpha \) connected sets containing \( x \) and contained in \( E \) is called the \( \alpha \) component of \( x \) corresponding to \( E \).

**THEOREM 6.1.11:** Let \( X \) be a topological space which is not \( \alpha \) connected. Then each \( \alpha \) component \( \alpha c(x) \) is a maximal \( \alpha \) connected set in \( X \).

**PROOF:** Let \( \alpha c(x) = A \).

By theorem 6.1.9, \( A \) is \( \alpha \) connected.
If $A$ is not a maximal $\alpha$ connected set, then $\exists$ a $\alpha$ connected set $B$ with

$A \subseteq B$ and $A \neq B$.

Then $B$ is a $\alpha$ connected set containing $x$.

But $A$ is the union of all $\alpha$ connected sets containing $x$.

Hence $B \subseteq A$. Therefore $A = B$. \iff

Hence $A$ is a maximal $\alpha$ connected set.

**THEOREM 6.1.12:** The set of all distinct $\alpha$ components of points of $X$

form a partition of $X$.

**PROOF:**

case 1: Let $X$ be $\alpha$ connected.

For each $x$ in $X$, $\alpha c(x) = X$. Hence $X$ is the only $\alpha$ component.

Case 2: Let $X$ be not $\alpha$ connected. Take $x, y$ in $X$.

Let $\alpha c(x) \cap \alpha c(y) \neq \emptyset$.

Then $\alpha c(x)$ and $\alpha c(y)$ are $\alpha$ connected sets with non empty intersection.

Hence by theorem 6.1.6, $\alpha c(x) \cup \alpha c(y)$ is $\alpha$ connected.

By the property of maximality, $\alpha c(x) = \alpha c(x) \cup \alpha c(y)$.

Hence $\alpha c(y) \subseteq \alpha c(x)$. similarly $\alpha c(x) \subseteq \alpha c(y)$. Hence $\alpha c(x) = \alpha c(y)$.

Therefore $\alpha c(x) \cap \alpha c(y) \neq \emptyset \implies \alpha c(x) = \alpha c(y)$.

Hence distinct $\alpha$ components are disjoint.
Let $U$ be the union of all $\alpha$ components.

Then $U \subset X$. Take $x \in X$. Then $x \in \alpha c(x) \subset U$.

Hence $X \subset U$. Therefore $X = U$.

Hence $X$ is the union of all $\alpha$ components.

Since distinct $\alpha$ components are disjoint, $X$ is the union of all distinct $\alpha$ components. Also each $\alpha$ component is non empty.

Therefore the set of all distinct $\alpha$ components form a partition of $X$.

**THEOREM 6.1.13:** Each $\alpha c(x)$ is $\alpha$ closed.

**PROOF:** Take $A = \alpha c(x)$. Then by theorem 6.1.9, $A$ is $\alpha$ connected.

By theorem 6.1.7, $\alpha cl(A)$ is $\alpha$ connected. Hence $\alpha cl(A)$ is a $\alpha$ connected set containing $x$. Therefore $\alpha cl(A) \subset \alpha cl(x) = A$.

Hence $A$ is $\alpha$ closed. Therefore $\alpha c(x)$ is $\alpha$ closed.

### 6.2 LOCALLY $\beta$ CONNECTEDNESS

**DEFINITION 6.2.1:** Let $X$ be a topological space.

Let $A$ and $B$ be two subsets of $X$. $A$ and $B$ are said to be $\beta$ separated if

$A \cap \beta cl\ B = \beta cl A \cap B = \phi$. 
DEFINITION 6.2.2: Let $X$ be a topological space. Let $A$ be a subset of $X$. $A$ is called a $\beta$ connected set if $A$ cannot be expressed as the union of two non empty $\beta$ separated sets.

The topological space $X$ is said to be $\beta$ connected if $X$ can not be expressed as the union of two non empty $\beta$ separated sets.

DEFINITION 6.2.3: Let $X$ be a topological space and let $x \in X$. $X$ is said to be locally $\beta$ connected at $x$, if for every $\beta$ open set $U$ containing $x$, there exists a $\beta$ connected open set $C$ such that $x \in C \subset U$.

DEFINITION 6.2.4: A topological space $X$ is said to be locally $\beta$ connected if $X$ is locally $\beta$ connected at each point of $X$.

THEOREM 6.2.5: If $A$ is $\beta$ connected and $A \subset C \cup D$ where $C$ and $D$ are $\beta$ separated sets, then either $A \subset C$ or $A \subset D$.

PROOF: $A$ is $\beta$ connected and $A \subset C \cup D$. $C$ and $D$ are $\beta$ separated.

Then $A = (A \cap C) \cup (A \cap D)$.

$$(A \cap C) \cap \beta \text{cl} (A \cap D) \subset (A \cap C) \cap (\beta \text{cl} A \cap \beta \text{cl} D).$$

$$\subset C \cap \beta \text{cl} D = \emptyset.$$
Similarly \((A \cap D) \cap \beta cl(A \cap C) = \emptyset\). Hence \(A \cap C\) and \(A \cap D\) are \(\beta\) separated sets. Since \(A\) is \(\beta\) connected and \(A = (A \cap C) \cup (A \cap D)\), \(A \cap C = \emptyset\) or \(A \cap D = \emptyset\). Hence \(A = A \cap D\) or \(A = A \cap C\).

Therefore \(A \subseteq D\) or \(A \subseteq C\).

**THEOREM 6.2.6:** The union of a family of \(\beta\) connected sets having non-empty intersection is \(\beta\) connected.

**PROOF:**

Let \(\{A_s / s \in I\}\) be a family of \(\beta\) connected subsets of \(X\) where \(I\) is an index set and let \(\cap A_s \neq \emptyset\).

Let \(A = \bigcup A_s\). Suppose \(A\) is not \(\beta\) connected, then \(A = C \cup D\) where \(C\) and \(D\) are non-empty \(\beta\) separated sets. Take \(x \in \cap A_s\). Then \(x \in A_s\).

Hence \(x \in C \cup D\).

Since \(C\) and \(D\) are \(\beta\) separated, \(x \in C\) or \(x \in D\).

Case 1: Let \(x \in C\).

Fix any \(s \in I\). Then \(A_s \subseteq A\). Hence \(A_s \subseteq C \cup D\).

\(A_s\) is \(\alpha\) connected, \(C\) and \(D\) are \(\beta\) separated. Hence by theorem 6.1.5 \(A_s \subseteq C\) or \(A_s \subseteq D\).

Since \(x \in C\) and \(x \in A_s\), \(x \in A_s \cap C\). Hence \(A_s \cap C \neq \emptyset\).

Now \(A_s \subseteq C\) or \(A_s \subseteq D\), \(C\) and \(D\) are \(\beta\) separated, \(A_s \cap C \neq \emptyset\).
Hence $A_s \subseteq C$. This is true for each $s$ in $I$.

Hence $\bigcup A_s \subseteq C$. Hence $A \subseteq C$. Therefore $D = \emptyset$. $\Rightarrow \Leftarrow$

Therefore $A$ is $\beta$ connected.

Case 2: Let $x \in D$. By a similar proof as in case 1, $A$ is $\beta$ connected.

**THEOREM 6.2.7:** If $A$ is $\beta$ connected and $A \subseteq B \subseteq \beta cl(A)$, then $B$ is $\beta$ connected. In particular $\beta cl(A)$ is $\beta$ connected.

**PROOF:** $A$ is $\beta$ connected and $A \subseteq B \subseteq \beta cl(A)$.

Suppose $B$ is not $\beta$ connected. Then $B = C \cup D$ where $C$ and $D$ are non-empty $\beta$ separated sets.

$A \subseteq B \Rightarrow A \subseteq C \cup D$. By theorem 6.2.5. $A \subseteq C$ or $A \subseteq D$.

Case 1: Let $A \subseteq C$. Then $\beta cl A \subseteq \beta cl C$.

$x \in B \Rightarrow x \in \beta cl A \Rightarrow x \in \beta cl C \Rightarrow x \notin D$.

$B = C \cup D$ and $x \in B \Rightarrow x \notin D$. Hence $D = \emptyset$. $\Rightarrow \Leftarrow$

Hence $B$ is $\beta$ connected.

Case 2: Let $A \subseteq D$. Similar to case 1, we get $B$ is $\beta$ connected.

**DEFINITION 6.2.8:** Let $x \in X$. The $\beta$ component of $x$ denoted by $\beta c(x)$ is the union of all $\beta$ connected subsets of $X$ containing $x$. 

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THEOREM 6.2.9: $\beta c(x)$ is $\beta$ connected.

PROOF: Let $x \in X$. $\beta c(x)$ is the union of all $\beta$ connected subsets of $X$ containing $x$. Hence by theorem 6.2.6, $\beta c(x)$ is $\beta$ connected.

DEFINITION 6.2.10: Let $E$ be a subset of $X$ and $x \in E$. Then the union of all $\beta$ connected sets containing $x$ and contained in $E$ is called the $\beta$ component of $x$ corresponding to $E$.

THEOREM 6.2.11: Let $X$ be a topological space which is not $\beta$ connected. Then each $\beta$ component $\beta c(x)$ is a maximal $\beta$ connected set in $X$.

PROOF: Let $\beta c(x) = A$.

By theorem 6.2.9, $A$ is $\beta$ connected.

If $A$ is not a maximal $\beta$ connected set, then $\exists$ a $\beta$ connected set $B$ with $A \subset B$ and $A \neq B$.

Then $B$ is a $\beta$ connected set containing $x$.

But $A$ is the union of all $\beta$ connected sets containing $x$.

Hence $B \subset A$. Therefore $A = B$. $\Rightarrow$ $\Leftarrow$

Hence $A$ is a maximal $\beta$ connected set.
THEOREM 6.2.12: The set of all distinct $\beta$ components of points of $X$ form a partition of $X$.

PROOF:

Case 1: Let $X$ be $\beta$ connected.

For each $x$ in $X$, $\beta c(x) = X$. Hence $X$ is the only $\beta$ component.

Case 2: Let $X$ be not $\beta$ connected. Take $x, y$ in $X$.

Let $\beta c(x) \cap \beta c(y) \neq \phi$.

Then $\beta c(x)$ and $\beta c(y)$ are $\beta$ connected sets with non empty intersection.

Hence by theorem 6.2.6, $\beta c(x) \cup \beta c(y)$ is $\beta$ connected.

By the property of maximality, $\beta c(x) = \beta c(x) \cup \beta c(y)$.

Hence $\beta c(y) \subseteq \beta c(x)$. Similarly $\beta c(x) \subseteq \beta c(y)$. Hence $\beta c(x) = \beta c(y)$.

Therefore $\beta c(x) \cap \beta c(y) \neq \phi \Rightarrow \beta c(x) = \beta c(y)$.

Hence distinct $\beta$ components are disjoint.

Let $U$ be the union of all $\beta$ components.

Then $U \subseteq X$. Take $x \in X$. Then $x \in \beta c(x) \subseteq U$.

Hence $X \subseteq U$. Therefore $X = U$.

Hence $X$ is the union of all $\beta$ components.

Since distinct $\beta$ components are disjoint, $X$ is the union of all distinct $\beta$ components. Also each $\beta$ component is non empty.

Therefore the set of all distinct $\beta$ components form a partition of $X$. 154
THEOREM 6.2.13: Each $\beta c(x)$ is $\beta$ closed.

PROOF: Take $A = \beta c(x)$. Then by theorem 6.2.9, $A$ is $\beta$ connected.

By theorem 6.2.7, $\beta \text{cl}(A)$ is $\beta$ connected. Hence $\beta \text{cl}(A)$ is a $\beta$ connected set containing $x$. Therefore $\beta \text{cl}(A) \subseteq \beta \text{cl}(x) = A$.

Hence $A$ is $\beta$ closed. Therefore $\beta c(x)$ is $\beta$ closed.