CHAPTER 4

β GENERALISED STAR CLOSED SETS

4.1 β GENERALISED STAR SETS

DEFINITION 4.1.1: A subset $A$ of a topological space $(X,T)$ is called a β generalised star closed (briefly, βg*closed) set if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is β open in $(X,T)$.

THEOREM 4.1.2: Every closed set is βg*closed.

PROOF: Let $A$ be a closed subset of $X$. Let $A \subseteq U$ and $U$ be β open. $\text{cl}(A) = A$ since $A$ is closed. Therefore $\text{cl}(A) \subseteq U$. Hence $A$ is βg* closed.

THEOREM 4.1.3: Every βg*closed set is g closed.

PROOF: Let $A$ be a βg*closed subset of $X$. Let $A \subseteq U$ and $U$ be open. $U$ is β open since every open set is β open. $A$ is βg*closed and $U$ is β open. Therefore $\text{cl}(A) \subseteq U$. Hence $A$ is g closed.

THEOREM 4.1.4: Union of two βg*closed sets is βg*closed.

PROOF: Let $A$ and $B$ be two βg*closed sets. Let $A \cup B \subseteq U$ and $U$ be
β open. Then $A \subset U$ and $B \subset U$. $\text{cl}(A) \subset U$ since $A$ is $\beta$-closed. $\text{cl}(B) \subset U$ since $B$ is $\beta$-closed. Hence $\text{cl}(A) \cup \text{cl}(B) \subset U$. Therefore $\text{cl}(A \cup B) \subset U$.
Hence $A \cup B$ is $\beta$-closed.

**PROPOSITION 4.1.5:** Finite Union of $\beta$-closed sets is $\beta$-closed.
Proof follows from the above Theorem.

**THEOREM 4.1.6:** A $\beta$-closed set which is $\beta$ open is closed.

**PROOF:** Let $A$ be a $\beta$-closed set which is $\beta$ open. $A \subset A$ and $A$ is $\beta$ open. Hence $\text{cl}(A) \subset A$. Therefore $\text{cl}(A) = A$. Hence $A$ is closed.

**THEOREM 4.1.7:** If $A$ is $\beta$-closed and $A \subset B \subset \text{cl}(A)$, then $B$ is $\beta$-closed.

**PROOF:** Let $A$ be a $\beta$-closed subset of $X$.
Let $A \subset B \subset \text{cl}(A)$. Then $\text{cl}(B) \subset \text{cl}(A)$. Let $B \subset U$ and $U$ be $\beta$ open. Then $A \subset U$. $\text{cl}(A) \subset U$ since $A$ is $\beta$-closed. Hence $\text{cl}(B) \subset U$. Hence $B$ is $\beta$-closed.

**THEOREM 4.1.8:** In a Door space every $\beta$-closed set is closed.

**PROOF:** In a Door space every subset is either open or closed.
Let $X$ be a Door space. Let $A$ be a $\beta g^*$ closed subset of $X$. Then $A$ is open or closed. If $A$ is closed then nothing to prove.

If $A$ is open then $A$ is $\beta$ open. By Theorem 4.1.6 $A$ is closed.

**DEFINITION 4.1.9:** A subset $A$ of a topological space $X$ is called a $\beta$ generalised star open (briefly, $\beta g^*$open) set if $A^c$ is $\beta g^*$ closed.

**THEOREM 4.1.10:** Every open set is $\beta g^*$open.

**PROOF:** Let $A$ be a open subset of $X$. Then $A^c$ is closed. $A^c$ is $\beta g^*$closed since every closed set is $\beta g^*$closed. Hence $A$ is $\beta g^*$open.

**THEOREM 4.1.11:** Every $\beta g^*$open set is $g$ open.

**PROOF:** Let $A$ be a $\beta g^*$open subset of $X$. Then $A^c$ is $\beta g^*$closed. $A^c$ is $g$ closed since every $\beta g^*$closed set is $g$ closed. Hence $A$ is $g$ open.

**THEOREM 4.1.12:** Intersection of two $\beta g^*$open sets is $\beta g^*$open.

**PROOF:** Let $A$ and $B$ be two $\beta g^*$open sets of $X$. Then $A^c$ and $B^c$ are $\beta g^*$closed sets. By Theorem 4.1.4 $A^c \cup B^c$ is $\beta g^*$closed. Therefore $(A \cap B)^c$ is $\beta g^*$closed. Hence $A \cap B$ is $\beta g^*$open.

**PROPOSITION 4.1.13:** Finite Intersection of $\beta g^*$open sets is $\beta g^*$ open.
Proof follows from the above Theorem.

**THEOREM 4.1.14:** If $A$ is $\beta g^\ast$open and $\text{Int}A \subseteq B \subseteq A$. Then $B$ is $\beta g^\ast$open.

**PROOF:** $A$ is $\beta g^\ast$open. Hence $A^c$ is $\beta g^\ast$closed. $\text{Int}A \subseteq B \subseteq A$.

Therefore $(\text{Int}A)^c \supseteq B^c \supseteq A^c$. Therefore $A^c \subseteq B^c \subseteq \text{cl}(A^c)$. By Theorem 4.1.7 $B^c$ is $\beta g^\ast$closed. Hence $B$ is $\beta g^\ast$open.

**THEOREM 4.1.15:** In a Door space every $\beta g^\ast$ open set is open.

Proof follows from theorem 4.1.8.

**4.2 EQUIVALENT CHARACTERISATIONS**

**THEOREM 4.2.1:** $A$ is $\beta g^\ast$closed iff for each net $(x_\alpha)$ in $A$ converging to $x$, $\exists y \in A$ such that every net $\beta$ converging to $x$, $\beta$ converges to $y$ also.

**PROOF:** Let $A$ be $\beta g^\ast$closed.

Let $(x_\alpha)$ in $A$ converge to $x$. Then $x \in \text{cl}(A)$.

Claim: $\beta \text{cl}(x)$ contains a point of $A$.

If not, then $\beta \text{cl}(x) \cap A = \emptyset$. Now $A \subseteq (\beta \text{cl}(x))^c$, $(\beta \text{cl}(x))^c$ is $\beta$ open and $A$ is $\beta g^\ast$closed. Therefore $\text{cl}(A) \subseteq (\beta \text{cl}(x))^c$. Since $x \in \text{cl}(A)$, $x \in (\beta \text{cl}(x))^c$.

$\Rightarrow \Leftarrow$
Hence $\beta cl(x)$ contains a point of $A$.

Take $y \in A \cap \beta cl(x)$. Let $(x_\beta)$ $\beta$ converge to $x$.

Claim: $(x_\beta)$ $\beta$ converges to $y$.

Let $O$ be any $\beta$ open set containing $y$.

Claim: $x \in O$.

If not, then $x \in O^c$ and $O^c$ is $\beta$ closed.

Then $\beta cl(x) \subset O^c$. Since $y \in \beta cl(x)$, $y \in O^c$. $\Rightarrow \Leftarrow$ since $y \in O$.

Therefore $x \in O$.

$O$ is a $\beta$ open set containing $x$ and $(x_\beta)$ $\beta$ converges to $x$.

Therefore $(x_\beta)$ is eventually in $O$.

Hence if $O$ is any $\beta$ open set containing $y$, then $(x_\beta)$ is eventually in $O$.

Hence $(x_\beta)$ $\beta$ converges to $y$.

Conversely,

If for each $(x_\alpha)$ in $A$ converging to $x$, $\exists y \in A$ such that every net $\beta$ converging to $x$, $\beta$ converges to $y$ also.

To prove: $A$ is $\beta g^*$closed.

Let $A \subset U$ and $U$ be $\beta$ closed. Take $x \in cl(A)$.

Then $\exists (x_\alpha)$ in $A$ converging to $x$.

Therefore $\exists y \in A$, $\exists$ every net $\beta$ converging to $x$, $\beta$ converges to $y$ also.

Claim: $y \in \beta cl(x)$.
Consider the net \((x_\alpha)\) where \(x_\alpha = x\) for all \(\alpha\).

\((x_\alpha)\) converges to \(x\) and hence \(x\) converges to \(y\).

Hence every \(\beta\) open set containing \(y\) contains elements of the net.

Therefore every \(\beta\) open set containing \(y\) contains \(x\) also.

If \(y \not\in \beta \text{cl}(x)\), then \(y \in (\beta \text{cl}(x))^c\).

Then \((\beta \text{cl}(x))^c\) is a \(\beta\) open set containing \(y\).

Therefore \((\beta \text{cl}(x))^c\) contains \(x\) also. 

Therefore \(y \in \beta \text{cl}(x)\).

Claim: \(x \in U\).

suppose not, then \(x \in U^c\) and \(U^c\) is \(\beta\) closed.

Therefore \(\beta \text{cl}(x) \subset U^c\). Hence \(y \in U^c\).

Since \(y \in A\), \(y \in U\). Therefore \(y \in U\) and \(y \in U^c\). 

Therefore \(x \in U\). Hence \(\text{cl}(A) \subset U\).

Therefore \(A\) is \(\beta\)-closed.

**THEOREM 4.2.2**: A subset \(A\) is \(\beta\)-closed iff for each filter \(F\), such that \(A\) belongs to \(F\) and \(F\) converges to \(x\), \(\exists \ y \in A\) such that every filter \(\beta\) converging to \(x\), \(\beta\) converges to \(y\) also.

**PROOF**: Let \(A\) be \(\beta\)-closed.

Let \(F\) be a filter such that \(A\) belongs to \(F\) and \(F\) converges to \(x\) .
Then \( x \in \text{cl}(A) \).

Claim: \( \beta \text{cl}(x) \) contains a point of \( A \).

If not, then \( \beta \text{cl}(x) \cap A = \phi \).

\( A \subset (\beta \text{cl}(x))^c \), \( (\beta \text{cl}(x))^c \) is \( \beta \) open and \( A \) is \( \beta \)g*closed.

Hence \( \text{cl}(A) \subset (\beta \text{cl}(x))^c \).

\( x \in \text{cl}(A) \Rightarrow x \in (\beta \text{cl}(x))^c \Rightarrow x \notin \beta \text{cl}(x) \). \( \Rightarrow \leq \)

Hence \( \beta \text{cl}(x) \) contains a point of \( A \).

Take \( y \in \beta \text{cl}(x) \cap A \).

Let \( U \) be a filter \( \beta \) converging to \( x \).

Claim: \( U \) \( \beta \) converges to \( y \).

Let \( O \) be any \( \beta \) open set containing \( y \).

Claim: \( x \in O \).

If not then \( x \in O^c \) and \( O^c \) is \( \beta \) closed.

Hence \( \beta \text{cl}(x) \subset O^c \).

Since \( y \in \beta \text{cl}(x) \), \( y \in O^c \). \( \Rightarrow \leq \) since \( y \in O \).

Hence \( x \in O \).

\( O \) is a \( \beta \) open set containing \( x \) and filter \( U \) \( \beta \) converges to \( x \).

Therefore \( O \) belongs to the filter \( U \).

Hence if \( O \) is any \( \beta \) open set containing \( y \) then \( O \) belongs to the filter \( U \).

Therefore the filter \( U \) \( \beta \) converges to \( y \).
Therefore \( \exists y \in A \) such that every filter \( \beta \) converging to \( x \), \( \beta \) converges to \( y \) also.

Conversely,

If for each filter \( F \) with \( A \) belongs to \( F \) and \( F \) converges to \( x \), \( \exists y \in A \) such that every filter \( \beta \) converging to \( x \), \( \beta \) converges to \( y \) also.

To prove: \( A \) is \( \beta \)g* closed.

Let \( A \subseteq U, U \) be \( \beta \) open. Take \( x \in \text{cl}(A) \).

Then \( \exists \) a filter \( F \) with \( A \) belongs to \( F \) and \( F \) converges to \( x \).

Therefore \( \exists y \in A \) such that every filter \( \beta \) converging to \( x \), \( \beta \) converges to \( y \) also.

Claim: \( y \in \beta \text{cl}(x) \).

Consider the filter \( V \) which has the set of all \( \beta \) open sets containing \( x \) as a filter subbasis.

Then filter \( V \) \( \beta \) converges to \( x \).

Therefore filter \( V \) \( \beta \) converges to \( y \). Therefore every \( \beta \) open set containing \( y \) belongs to \( V \). Every member of \( U \) contains \( x \). Therefore every \( \beta \) open set containing \( y \) contains \( x \) also.

If \( y \not\in \beta \text{cl}(x) \), then \( y \in (\beta \text{cl}(x))^c \) and \( (\beta \text{cl}(x))^c \) is \( \beta \) open.

Therefore \( (\beta \text{cl}(x))^c \) contains \( x \). \( \Rightarrow \Leftarrow \)

Hence \( y \in \beta \text{cl}(x) \). Therefore \( y \in \beta \text{cl}(x) \cap A \).
Claim: $x \in U$.

suppose not,

Then $x \in U^c$ and $U^c$ is $\beta$ closed. Hence $\beta\text{cl}(x) \subset U^c$.

Since $y \in \beta\text{cl}(x)$, $y \in U^c$. But $y \in U$ since $y \in A$ and $A \subset U$.

Therefore $y \in U$ and $y \in U^c$. $\Rightarrow\Leftarrow$

Hence $x \in U$. Therefore $\text{cl}(A) \subset U$.

Therefore $A$ is $\beta g^*$closed.

**THEOREM 4.2.3:** A set $A$ is $\beta g^*$closed iff whenever $A$ does not intersect a $\beta$ closed set $F$, $\text{cl}(A)$ also does not intersect $F$.

**PROOF:**

**Necessity:** Let $A$ be a $\beta g^*$closed subset of $X$. Let $F$ be a $\beta$ closed set where $A \cap F = \emptyset$. Therefore $A \subset F^c$ and $F^c$ is $\beta$ open. Then $\text{cl}(A) \subset F^c$ since $A$ is $\beta g^*$closed. Therefore $\text{cl}(A) \cap F = \emptyset$. Hence $\text{cl}(A)$ does not intersect $F$.

**Sufficiency:** Let $A \subset U$ and $U$ be $\beta$ open. $A$ does not intersect $U^c$ and $U^c$ is $\beta$ closed. Hence $\text{cl}(A)$ does not intersect $U^c$. Hence $\text{cl}(A) \subset U$. This implies $A$ is $\beta g^*$closed.

**THEOREM 4.2.4:** A set $A$ is $\beta g^*$open iff whenever $A$ contains a $\beta$ closed set $F$, $\text{Int} A$ also contains $F$. 

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PROOF:

Necessity: Let $A$ be a $\beta g^*$ open subset of $X$. Let $A \supset F$ where $F$ is $\beta$ closed. $A^c \subset F^c$. $A^c$ is $\beta g^*$ closed and $F^c$ is $\beta$ open. Therefore $\text{cl}(A^c) \subset F^c$. 

$[\text{cl}(A^c)]^c \supset F$. Hence $\text{Int } A \supset F$.

Sufficiency: Let $A^c \subset U$ and $U$ be $\beta$ open. $A \supset U^c$. $U^c$ is $\beta$ closed. Hence $\text{Int } A \supset U^c$. Therefore $(\text{Int } A)^c \subset U$. Hence $\text{cl}(A^c) \subset U$.

Therefore $A^c$ is $\beta g^*$ closed. Hence $A$ is $\beta g^*$ open.

**THEOREM 4.2.5:** The following are equivalent.

1. $A$ is $\beta g^*$ closed.

2. For each $x \in \text{cl}(A)$, $\beta \text{cl}(x) \cap A \neq \emptyset$.

3. $\text{cl}(A) - A$ contains no non empty $\beta$ closed set.

PROOF:

To Prove 1$\Rightarrow$2.

$A$ is $\beta g^*$ closed. Take any $x \in \text{cl}(A)$.

Suppose $\beta \text{cl}(x) \cap A = \emptyset$ then $A \subset [\beta \text{cl}(x)]^c$, $[\beta \text{cl}(x)]^c$ is $\beta$ open and $A$ is $\beta g^*$ closed. Therefore $\text{cl}(A) \subset [\beta \text{cl}(x)]^c$.

As $x \in \text{cl}(A)$ we have $x \in [\beta \text{cl}(x)]^c$. $\Rightarrow \Leftarrow$

Therefore $\beta \text{cl}(x) \cap A \neq \emptyset$.

Hence 1$\Rightarrow$2 is proved.
To prove 2\Rightarrow 3.

Suppose 3 is not true.

Then \( \text{cl}(A) - A \) contains a non empty \( \beta \) closed set \( F \).

\( F \subseteq \text{cl}(A) - A, F \neq \emptyset \) and \( F \) is \( \beta \) closed.

Take any \( x \in F \) then \( x \in \text{cl}(A) \). Therefore \( \beta \text{cl}(x) \cap A \neq \emptyset \).

\( x \in F \) and \( F \) is \( \beta \) closed. Therefore \( \beta \text{cl}(x) \subseteq F \). Hence \( F \cap A \neq \emptyset \).

We have \( F \subseteq \text{cl}(A) - A \) and \( F \cap A \neq \emptyset \). \( \Rightarrow \Leftarrow \).

Therefore 3 is true.

Hence 2\Rightarrow 3 is proved.

To Prove 3\Rightarrow 1.

Suppose \( A \) is not \( \beta g^* \) closed.

Then there exists \( \beta \) open set \( U \) with \( A \subseteq U \) and \( \text{cl}(A) \subset U \).

There exists \( x \in \text{cl}(A) \) and \( x \notin U \).

\( x \in \text{cl}(A) - U \). Let \( F = \text{cl}(A) - U \). Then \( F \neq \emptyset \). Since \( A \subseteq U \), \( F \subseteq \text{cl}(A) - A \).

\( F = \text{cl}(A) \cap U^c \). Now \( \text{cl}(A) \) is \( \beta \) closed and \( U^c \) is \( \beta \) closed.

Therefore \( F \) is \( \beta \) closed.

Therefore \( F \neq \emptyset, F \subseteq \text{cl}(A) - A \) and \( F \) is \( \beta \) closed.

Therefore \( \text{cl}(A) - A \) contains a non empty \( \beta \) closed set. \( \Rightarrow \Leftarrow \).

Therefore \( A \) is \( \beta g^* \) closed.

Hence 3\Rightarrow 1 is proved.
**THEOREM 4.2.6:** The following are equivalent.

1. $A$ is $\beta g^*$ open.

2. For each $x \in A - \text{int}A$, $\beta \text{cl}(x) \cap A^c \neq \emptyset$.

3. $A - \text{int}A$ does not contain a non empty $\beta$ closed set.

**PROOF:**

To prove $1 \Rightarrow 2$.

$A$ is $\beta g^*$ open. Let $B = A^c$. Then $B$ is $\beta g^*$ closed.

Take $x \in A - \text{int}A$.

$x \in A - \text{int}A \Rightarrow x \notin \text{int}A \Rightarrow x \in (\text{int}A)^c \Rightarrow x \in \text{cl} A^c \Rightarrow x \in \text{cl}(B)$.

Now $B$ is $\beta g^*$ closed.

Therefore $\beta \text{cl}(x) \cap B \neq \emptyset$. Hence $\beta \text{cl}(x) \cap A^c \neq \emptyset$.

Hence $1 \Rightarrow 2$ is proved.

To prove $2 \Rightarrow 3$.

Suppose 3 is not true, then $A - \text{int}A$ contains a non empty $\beta$ closed set $F$.

$F \subseteq A - \text{int}A$, $F \neq \emptyset$ and $F$ is $\beta$ closed.

Take $x \in F$. Then $x \in A - \text{int}A$. Therefore $\beta \text{cl}(x) \cap A^c \neq \emptyset$.

Since $x \in F$ and $F$ is $\beta$ closed, $\beta \text{cl}(x) \subseteq F$.

Since $\beta \text{cl}(x) \cap A^c \neq \emptyset$ and $\beta \text{cl}(x) \subseteq F$, $F \cap A^c \neq \emptyset$.

$\Rightarrow \Leftarrow$ since $F \subseteq A$. Hence 3 is true.
Hence 2 => 3 is proved.

To prove 3 => 1.

A-intA does not contain a non empty $\beta$ closed set. Let $B=A^c$.

$\text{cl}(B)-B=\text{cl}(A^c)-A^c=\text{cl}(A^c)\cap A=(\text{int}A)^c\cap A=A-\text{int}A$.

Therefore $\text{cl}(B)-B$ does not contain a nonempty $\beta$ closed set. Therefore $B$ is $\beta g^*$ closed. Therefore $A$ is $\beta g^*$ open.

Hence 3 => 1 is proved.

**THEOREM 4.2.7:** If $A$ is $\beta g^*$ closed, then $A$ is closed iff $\text{cl}(A)-A$ is $\beta$ closed.

**PROOF:** Let $A$ be $\beta g^*$ closed.

If $A$ is closed, then $\text{Cl}(A)-A=\emptyset$. Hence $\text{cl}(A)-A$ is $\beta$ closed.

Conversely, if $\text{cl}(A)-A$ is $\beta$ closed, then we have to prove $A$ is closed.

$A$ is $\beta g^*$ closed $\Rightarrow$ $\text{cl}(A)-A$ cannot contain a non empty $\beta$ closed set.

Now $\text{cl}(A)-A$ is $\beta$ closed. Therefore $\text{cl}(A)-A=\emptyset$.

Therefore $\text{cl}(A)\subseteq A$. Therefore $A$ is closed.

**THEOREM 4.2.8:** In $X$, every set is $\beta g^*$ closed iff the set of all $\beta$ open subsets of $X$ is equal to the set of all closed subsets of $X$. 
PROOF:

Assume every set is $\beta g^*$ closed.

Let $O$ be a $\beta$ open subset of $X$. Then $O$ is $\beta g^*$ closed and $O$ is open.

Therefore $O$ is closed. Hence $O$ is $\beta$ open implies $O$ is closed.

Now, let $F$ be a closed set. Then $F^c$ is open. Hence $F^c$ is $\beta$ open.

$F^c \subseteq F^c$, $F^c$ is $\beta$ open and $F^c$ is $\beta g^*$ closed.

Therefore $\text{cl}(F^c) \subseteq F^c$. Therefore $F^c$ is closed.

Therefore $F$ is open. Hence $F$ is $\beta$ open.

Hence $F$ is closed implies $F$ is $\beta$ open.

Hence set of all $\beta$ open sets is equal to the set of all closed sets.

Conversely,

Let $A$ be any subset of $X$.

Let $A \subseteq O$ and $O$ be $\beta$ open. Then $\text{cl}(A) \subseteq \text{cl}(O)$.

Every $\beta$ open set is closed. Hence $O$ is closed.

Hence $\text{cl}(A) \subseteq O$. Therefore $A$ is $\beta g^*$ closed.

Hence every set is $\beta g^*$ closed.

4.3 $\beta$ GENERALISED STAR HOMEMORPHISM

THEOREM 4.3.1: Let $f: X \rightarrow Y$ be a homeomorphism. Then $A$ is $\beta g^*$ closed in $Y$ $\Rightarrow f^{-1}(A)$ is $\beta g^*$ closed in $X$. 
PROOF: Let $f: X \to Y$ be a homeomorphism.

Let $A$ be a $\beta g^*$ closed subset of $Y$. Let $B = f^{-1}(A)$.

Let $U$ be any $\beta$ open set with $B \subseteq U$. Then $f(B) \subseteq f(U)$.

Therefore $f(f^{-1}(A)) \subseteq f(U)$.

Since $f$ is bijective, $f(f^{-1}(A)) = A$. Therefore $A \subseteq f(U)$.

Claim: $f(U)$ is $\beta$ open.

$U$ is $\beta$ open. Therefore $U \subseteq \text{cl int cl } U$.

$f(U) \subseteq f(\text{cl int cl } U)$.

$\subseteq \text{cl } f(\text{int cl } U)$ since $f$ is continuous.

$\subseteq \text{cl int } f(\text{cl } U)$ since $f$ is open.

$\subseteq \text{cl int cl } f(U)$ since $f$ is continuous.

Therefore $f(U)$ is $\beta$ open.

$A \subseteq f(U)$, $f(U)$ is $\beta$ open and $A$ is $\beta g^*$ closed.

Therefore $\text{cl}(A) \subseteq f(U)$. Hence $f^{-1}[\text{cl}(A)] \subseteq f^{-1}[f(U)]$.

Since $f$ is a homeomorphism, $f^{-1}(\text{cl}(A)) = \text{cl}(f^{-1}(A))$.

Therefore $\text{cl } (f^{-1}(A)) \subseteq f^{-1}[f(U)]$. Therefore $\text{cl}(B) \subseteq U$.

Therefore $B$ is $\beta g^*$ closed. Therefore $f^{-1}(A)$ is $\beta g^*$ closed.

THEOREM 4.3.2: Let $f: X \to Y$ be a homeomorphism. Then a subset $A$ is $\beta g^*$ open in $Y$ if and only if $f^{-1}(A)$ is $\beta g^*$ open in $X$. 
PROOF: A is $\beta g^*$ open in $Y \Rightarrow A^c$ is $\beta g^*$ closed in $Y$.

$\Rightarrow f^1(A^c)$ is $\beta g^*$ closed in $X$.

$\Rightarrow [f^1(A)]^c$ is $\beta g^*$ closed in $X$.

$\Rightarrow f^1(A)$ is $\beta g^*$ open in $X$.

THEOREM 4.3.3: Let $f: X \rightarrow Y$ be a homeomorphism.

Then a subset $A$ is $\beta g^*$ closed in $X \Rightarrow f(A)$ is $\beta g^*$ closed in $Y$.

PROOF: Let $f: X \rightarrow Y$ be a homeomorphism.

Let $A$ be $\beta g^*$ closed in $X$. Let $B=f(A)$.

Claim: $B$ is $\beta g^*$ closed.

Let $U$ be a $\beta$ open set with $B \subseteq U$.

$f(A) \subseteq U$. Hence $f^1(f(A)) \subseteq f^1(U)$.

Since $f$ is bijective, $f^1f(A)=A$. Therefore $A \subseteq f^1(U)$.

Since $U$ is $\beta$ open and $f$ is a homeomorphism, $f^1(U)$ is $\beta$ open.

$A \subseteq f^1(U)$, $f^1(U)$ is $\beta$ open and $A$ is $\beta g^*$ closed.

Therefore $cl(A) \subseteq f^1(U)$. Hence $f(cl(A)) \subseteq f(f^1(U))$.

Since $f$ is a closed map, $cl(f(A)) \subseteq f(cl(A))$.

Therefore $cl(f(A)) \subseteq f(f^1(U))$.

Hence $cl(B) \subseteq U$. Therefore $B$ is $\beta g^*$ closed.

Therefore image of $\beta g^*$ closed set is $\beta g^*$ closed.
THEOREM 4.3.4: Let $f: X \to Y$ be a homeomorphism. Then $A$ is $\beta g^*$ open in $X \Rightarrow f(A)$ is $\beta g^*$ open in $Y$.

PROOF:

$A$ is $\beta g^*$ open in $X$ implies $A^c$ is $\beta g^*$ closed in $X$.

Since $f$ is a homeomorphism, it follows from 4.3.3 that $f(A^c)$ is $\beta g^*$ closed in $Y$. This implies $(f(A))^c$ is $\beta g^*$ closed in $Y$ and hence $f(A)$ is $\beta g^*$ open in $Y$.

Hence $A$ is $\beta g^*$ open in $X \Rightarrow f(A)$ is $\beta g^*$ open in $Y$.

DEFINITION 4.3.5: A map $f: X \to Y$ is called a pre $\beta g^*$ closed map if $A$ is $\beta g^*$ closed in $X \Rightarrow f(A)$ is $\beta g^*$ closed in $Y$.

THEOREM 4.3.6: Every homeomorphism is a pre $\beta g^*$ closed map.

Proof follows from theorem 4.3.3.

DEFINITION 4.3.7: A map $f: X \to Y$ is called a pre $\beta g^*$ open map if $A$ is $\beta g^*$ open in $X \Rightarrow f(A)$ is $\beta g^*$ open in $Y$.

THEOREM 4.3.8: Every homeomorphism is a pre $\beta g^*$ open map.

Proof follows from theorem 4.3.4.
DEFINITION 4.3.9: A function \( f: X \to Y \) is called \( \beta g^* \) irresolute if \( A \) is \( \beta g^* \) closed in \( Y \) \( \Rightarrow f^{-1}(A) \) is \( \beta g^* \) closed in \( X \).

THEOREM 4.3.10: Every homeomorphism is \( \beta g^* \) irresolute.
Proof follows from theorem 4.3.1.

THEOREM 4.3.11: A function \( f: X \to Y \) is \( \beta g^* \) irresolute iff inverse image of every \( \beta g^* \) open set of \( Y \) is \( \beta g^* \) open in \( X \).

PROOF: When \( f \) is \( \beta g^* \) irresolute,

\[ A \text{ is } \beta g^* \text{ open in } Y \Rightarrow A^c \text{ is } \beta g^* \text{ closed in } Y. \]

\[ \Rightarrow f^{-1}(A^c) \text{ is } \beta g^* \text{ closed in } X \text{ since } f \text{ is } \beta g^* \text{ irresolute}. \]

\[ \Rightarrow [f^{-1}(A)]^c \text{ is } \beta g^* \text{ closed in } X. \]

\[ \Rightarrow f^{-1}(A) \text{ is } \beta g^* \text{ open in } X. \]

Conversely,

If inverse image of \( \beta g^* \) open set is \( \beta g^* \) open, then

\[ A \text{ is } \beta g^* \text{ closed in } Y \Rightarrow A^c \text{ is } \beta g^* \text{ open in } Y. \]

\[ \Rightarrow f^{-1}(A^c) \text{ is } \beta g^* \text{ open in } X. \]

\[ \Rightarrow [f^{-1}(A)]^c \text{ is } \beta g^* \text{ open in } X. \]

\[ \Rightarrow f^{-1}(A) \text{ is } \beta g^* \text{ closed in } X. \]
Therefore $f$ is $\beta g^*$ irresolute.

**DEFINITION 4.3.12:** A function $f: X \to Y$ is called $\beta g^*$ homeomorphism if $f$ is bijective, $f$ is $\beta g^*$ irresolute and $f^1$ is $\beta g^*$ irresolute.

**THEOREM 4.3.13:** If $f: X \to Y$ is bijective, then the following are equivalent.

1. $f$ is $\beta g^*$ irresolute and $f$ is pre $\beta g^*$ closed.

2. $f$ is $\beta g^*$ irresolute and $f$ is pre $\beta g^*$ open.

3. $f$ is $\beta g^*$ homeomorphism.

**PROOF:**

To prove $1 \implies 2$.

$f$ is bijective, $f$ is $\beta g^*$ irresolute and $f$ is a pre $\beta g^*$ closed map.

Claim: $f$ is a pre $\beta g^*$ open map.

$A$ is $\beta g^*$ open in $X \implies A^c$ is $\beta g^*$ closed in $X$.

$\implies f(A^c)$ is $\beta g^*$ closed in $Y$ since $f$ is pre $\beta g^*$ closed map.

$\implies [f(A)]^c$ is $\beta g^*$ closed in $Y$ since $f$ is bijective.

$\implies f(A)$ is $\beta g^*$ open in $Y$.

Hence $f$ is a pre $\beta g^*$ open map.
Hence \(1 \Rightarrow 2\) is proved.

To prove \(2 \Rightarrow 3\).

\(f\) is bijective, \(f\) is \(\beta g^*\) irresolute, \(f\) is \(\text{pre } \beta g^*\) open map.

Claim: \(f^1\) is a \(\beta g^*\) irresolute.

\(A\) is \(\beta g^*\) closed in \(X\) \(\Rightarrow\) \(A^c\) is \(\beta g^*\) open in \(X\).

\(\Rightarrow f(A^c)\) is \(\beta g^*\) open in \(Y\).

\(\Rightarrow [f(A)]^c\) is \(\beta g^*\) open in \(Y\).

\(\Rightarrow f(A)\) is \(\beta g^*\) closed in \(Y\).

\(\Rightarrow (f^1)^{-1}(A)\) is \(\beta g^*\) closed in \(Y\).

Therefore \(f^1\) is a \(\beta g^*\) irresolute.

Hence \(f^1\) is a \(\beta g^*\) homeomorphism.

Hence \(2 \Rightarrow 3\) is proved.

To prove \(3 \Rightarrow 1\).

\(f\) and \(f^1\) are \(\beta g^*\) irresolute, \(f\) is bijective.

Claim: \(f\) is \(\text{pre } \beta g^*\) closed map.

\(A\) is \(\beta g^*\) closed in \(X\) \(\Rightarrow\) \((f^1)^{-1}(A)\) is \(\beta g^*\) closed in \(Y\).

\(\Rightarrow f(A)\) is \(\beta g^*\) closed in \(Y\).

Hence \(f\) is a \(\text{pre } \beta g^*\) closed map.

Hence \(3 \Rightarrow 1\) is proved.
DEFINITION 4.3.14: A function \( f: X \to Y \) is called a \( \beta g^* \) closed map if for each closed set \( F \) of \( X \), \( f(F) \) is \( \beta g^* \) closed in \( Y \).

THEOREM 4.3.15: A function \( f: X \to Y \) is \( \beta g^* \) closed map iff for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \), \( \exists \) a \( \beta g^* \) open set \( V \) of \( Y \) such that \( S \subset V \) and \( f^{-1}(V) \subset U \).

PROOF:

Let \( f: X \to Y \) be \( \beta g^* \) closed map. Let \( S \subset Y \). Let \( f^{-1}(S) \subset U \), \( U \) be open.

\( U \) is open in \( X \) \( \Rightarrow \) \( U^c \) is closed in \( X \) \( \Rightarrow f(U^c) \) is \( \beta g^* \) closed in \( Y \).

Let \( V = [f(U^c)]^c \). Then \( V \) is \( \beta g^* \) open.

Claim: \( S \subset V \).

\( x \in U^c \Rightarrow x \notin f^{-1}(S) \Rightarrow f(x) \notin S \Rightarrow f(x) \in S^c \). Therefore \( f(U^c) \subset S^c \).

Hence \( [f(U^c)]^c \supseteq S \). Therefore \( S \subset V \).

Claim: \( f^{-1}(V) \subset U \).

\( x \in f^{-1}(V) \Rightarrow f(x) \in V \Rightarrow f(x) \in [f(U^c)]^c \Rightarrow f(x) \notin f(U^c) \Rightarrow x \notin U^c \).

Hence \( x \in U \). Therefore \( f^{-1}(V) \subset U \).

Therefore \( \exists \beta g^* \) open set \( V \) such that \( S \subset V \) and \( f^{-1}(V) \subset U \).

Conversely,

Suppose for each subset \( S \) of \( Y \) and for each open set \( U \) containing \( f^{-1}(S) \) there exist \( \beta g^* \) open set \( V \) such that \( S \subset V \) and \( f^{-1}(V) \subset U \).
To prove: $f$ is a $\beta g^*$ closed map.

Let $F$ be a closed subset of $X$. Take $S = [f(F)]^c$.

$x \in f^{-1}(S) \Rightarrow f(x) \in S \Rightarrow f(x) \notin f[F] \Rightarrow x \in F^c$.

Hence $f^{-1}(S) \subset F^c$ and $F^c$ is open. Therefore $\exists \; \beta g^*$ open set $V$ such that $S \subset V$ and $f^{-1}(V) \subset F^c$.

Claim: $S = V$.

Suppose not, then $V \not\subset S$.

Hence $\exists \; y \in V$ and $y \notin S$.

Then $y \in f(F)$.

Hence $\exists \; x \in F$ such that $f(x) = y$. Then $f(x) \in V$.

Hence $x \in f^{-1}(V)$.

Since $f^{-1}(V) \subset F^c$, $x \in F^c$. $\Rightarrow \Leftarrow$ since $x \in F$. Therefore $S = V$.

Hence $S$ is $\beta g^*$ open. Therefore $[f(F)]^c$ is $\beta g^*$ open.

Hence $f(F)$ is $\beta g^*$ closed. Therefore $F$ is closed $\Rightarrow f(F)$ is $\beta g^*$ closed.

Hence $f$ is a $\beta g^*$ closed map.

**DEFINITION 4.3.16: $\beta g^*$ open map**

A map $f : X \rightarrow Y$ is called $\beta g^*$ open if for each open set $U$ of $X$, $f(U)$ is $\beta g^*$ open in $Y$.

**THEOREM 4.3.17:** A map $f : X \rightarrow Y$ is $\beta g^*$ open iff for each subset $S$ of $Y$ and for each closed set $A$ containing $f^{-1}(S)$, $\exists \; a \beta g^*$ closed set $B \supset S$ such that $f^{-1}(B) \subset A$. 

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PROOF: \( f : X \to Y \) is a \( \beta \text{g}^* \) open map.

Let \( S \subseteq Y \). Let \( A \supseteq f^1(S) \) and \( A \) be closed.

A is closed \( \Rightarrow A^c \) is open in \( X \) \( \Rightarrow f(A^c) \) is \( \beta \text{g}^* \) open in \( Y \).

Take \( B = [f(A^c)]^c \). \( B \) is \( \beta \text{g}^* \) closed set.

claim: \( S \subseteq B \).

\( x \in A^c \Rightarrow x \not\in f^1(S) \Rightarrow f(x) \not\in S \Rightarrow f(x) \in S^c \).

Hence \( f(A^c) \subseteq S^c \). Therefore \( [f(A^c)]^c \supseteq S \).

Therefore \( B \supseteq S \).

claim: \( f^1(B) \subseteq A \).

\( x \in f^1(B) \Rightarrow f(x) \in B \Rightarrow f(x) \in [f(A^c)]^c \).

\[ \Rightarrow f(x) \not\in f(A^c) \text{.} \]

\[ \Rightarrow x \not\in A^c \text{.} \]

\[ \Rightarrow x \in A \text{.} \]

Hence \( f^1(B) \subseteq A \).

conversely,

Suppose for each subset \( S \) of \( Y \) and for each closed set \( A \) containing \( f^1(S) \), \( \exists \beta \text{g}^* \) closed set \( B \supseteq S \subseteq B \) and \( f^1(B) \subseteq A \).

To prove: \( f \) is \( \beta \text{g}^* \) open map.

Let \( U \) be a open set of \( X \), \( f(U) \subseteq Y \).

Take \( S = [f(U)]^c \). Then \( f^1(S) = f^1([f(U)]^c) \).

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\[ (f^\dagger [f(U)])^c. \]
\[ \subseteq U^c. \]

\( U^c \) is a closed set containing \( f^\dagger (S). \) Hence \( \exists \beta g^* \) closed set \( B \) such that
\( S \subseteq B \) and \( f^{-1}(B) \subseteq U^c. \)

Claim: \( S=B. \)

Suppose not, then \( B \not\subseteq S. \)

\( \exists y \in B \) and \( y \notin S. \) Hence \( y \in f(U). \) Therefore \( \exists x \in U \) s.t. \( f(x)=y. \)

\( y \in B \Rightarrow f(x) \in B \Rightarrow x \in f^\dagger(B) \Rightarrow x \in U^c. \Rightarrow \Leftrightarrow \) since \( x \in U. \)

Therefore \( S=B. \) Therefore \( S \) is \( \beta g^* \) closed.

Therefore \( f(U) \) is \( \beta g^* \) open.

Hence \( f \) is \( \sigma g^* \) open.

**Theorem 4.3.18:** The following are equivalent.

1. \( f: X \to Y \) is \( \beta g^* \) closed map.

2. If \( A \subseteq X \) is open then \( \{ y / f^\dagger(y) \subseteq A \} \) is \( \beta g^* \) open in \( Y. \)

3. If \( A \subseteq X \) is closed then \( \{ y / f^\dagger(y) \cap A \neq \emptyset \} \) is \( \beta g^* \) closed in \( Y. \)

**Proof:**

To prove \( 1 \Rightarrow 2. \)

Let \( f: X \to Y \) be a \( \beta g^* \) closed map.
Let A be a open subset of X. Then $A^c$ is closed.

Since f is a $\beta g^*$ closed map, $f(A^c)$ is $\beta g^*$ closed.

claim: $[f(A^c)]^c = \{ y / f^{-1}(y) \subseteq A \}$.

$y \in [f(A^c)]^c \Rightarrow y \not\in f(A^c) \Rightarrow f^{-1}(y) \cap A^c = \emptyset \Rightarrow f^{-1}(y) \subseteq A$.

Therefore $[f(A^c)]^c \subseteq \{ y / f^{-1}(y) \subseteq A \}$.

$f^{-1}(y) \subseteq A \Rightarrow f^{-1}(y) \cap A^c = \emptyset \Rightarrow y \not\in f(A^c) \Rightarrow y \in [f(A^c)]^c$.

Therefore $[f(A^c)]^c \supseteq \{ y / f^{-1}(y) \subseteq A \}$.

Hence $[f(A^c)]^c = \{ y / f^{-1}(y) \subseteq A \}$.

Since $[f(A^c)]^c$ is $\beta g^*$ open, $\{ y / f^{-1}(y) \subseteq A \}$ is $\beta g^*$ open.

Hence 1$\Rightarrow$ 2 is proved.

To prove 2$\Rightarrow$ 3.

$A \subseteq X$ open $\Rightarrow \{ y / f^{-1}(y) \subseteq A \}$ is $\beta g^*$ open in Y.

Let $F \subseteq X$ be closed.

$F^c$ is open $\Rightarrow \{ y / f^{-1}(y) \subseteq F^c \}$ is $\beta g^*$ open.

$\Rightarrow \{ y / f^{-1}(y) \subseteq F^c \}^c$ is $\beta g^*$ closed.

$\Rightarrow \{ y / f^{-1}(y) \cap F = \emptyset \}$ is $\beta g^*$ closed.

Hence 2$\Rightarrow$ 3 is proved.

To prove 3$\Rightarrow$ 1.

Let A be a closed subset of X. Then $\{ y / f^{-1}(y) \cap A \neq \emptyset \}$ is $\beta g^*$ closed.

claim: $f(A) = \{ y / f^{-1}(y) \cap A \neq \emptyset \}$.
\( j \in f(A) \iff y = f(x) \text{ for some } x \in A. \)
\[ \iff x \in f^{-1}(y) \text{ and } x \in A. \]
\[ \iff f^{-1}(y) \cap A \neq \emptyset. \]

Hence \( f(A) = \{ y/ f^{-1}(y) \cap A \neq \emptyset \}. \)

Since \( \{ y/ f^{-1}(y) \cap A \neq \emptyset \} \) is \( \beta^g \)-closed, \( f(A) \) is \( \beta^g \)-closed.

Therefore \( f \) is a \( \beta^g \)-closed map.

Hence \( 3 \Rightarrow 1 \) is proved.

**THEOREM 4.3.19:** If \( f: X \to Y \) is a \( \beta \)-irresolute closed map, then
\[ F \text{ is } \beta^g \text{-closed in } X \Rightarrow f(F) \text{ is } \beta^g \text{-closed in } Y. \]

**PROOF:** Let \( F \) be a \( \beta^g \)-closed subset of \( X. \)

To prove: \( f(F) \) is \( \beta^g \)-closed.

Let \( f(F) \subset O, O \) be \( \beta \) open. Then \( F \subset f^{-1}(O). \)

\( O \) is \( \beta \) open and \( f \) is \( \beta \)-irresolute. Therefore \( f^{-1}(O) \) is \( \beta \) open.

\( F \) is \( \beta^g \)-closed, \( F \subset f^{-1}(O) \) and \( f^{-1}(O) \) is \( \beta \) open.

Therefore \( \text{cl}(F) \subset f^{-1}(O). \) Hence \( f[\text{cl}(F)] \subset f(f^{-1}(O)) \subset O. \)

since \( f \) is a closed map, \( \text{cl}(f(F)) \subset \text{cl}(f(F)). \)

Hence \( \text{cl}(f(F)) \subset O. \) Therefore \( f(F) \) is \( \beta^g \)-closed.
THEOREM 4.3.20: Let $X$ and $Y$ be two topological spaces. If $f: X \to Y$ is a continuous, $\beta g^*$ closed surjection. Then $X$ is normal implies $Y$ is normal.

PROOF: $f: X \to Y$ is continuous, $\beta g^*$ closed surjective map. $X$ is normal.

$f: X \to Y$ is $\beta g^*$ closed implies $f: X \to Y$ is $g$-closed. Hence by Malghan’s result stated in Note 2.3.20, $Y$ is normal.

4.4 FUZZY $\beta$ GENERALISED STAR CLOSED SETS

We recall the definition and example of fuzzy $\beta$ open set from 2.4

A fuzzy set $\lambda$ in a fts $X$ is called a fuzzy $\beta$ open set if $\lambda \subset \text{cl}(\text{int}(\text{cl}\lambda))$. For example, let $X=[0,1]$. $T=\{0,1,A\}$ where $A: X \to [0,1]$ is defined as $A(0)=0.3$ and $A(x)=0$ for $x$ in $(0,1]$. Then $T$ is a fuzzy topology. Take $\lambda = A^c$. Then $\lambda \subset \text{cl}(\text{int}(\text{cl}\lambda))$. Hence $\lambda$ is fuzzy $\beta$ open.

DEFINITION 4.4.1: Let $X$ be a fuzzy topological space. A fuzzy subset $A$ of $X$ is called fuzzy $\beta$ generalised star closed (briefly, $f\beta g^*$ closed) set if $A \subset U$ and $U$ is fuzzy $\beta$ open $\Rightarrow \text{cl}(A) \subset U$.

THEOREM 4.4.2: Let $X$ be a fuzzy topological space. A fuzzy subset $A$ is $f\beta g^*$ closed iff whenever $A$ is not $q$ coincident with a fuzzy $\beta$ closed set $F$, $\text{cl}(A)$ is also not $q$ coincident with $F$. 
**PROOF:** Necessary: A is fuzzy $\beta g^*$ closed.

A is not $q$ coincident with a fuzzy $\beta$ closed set $F$. Then $A(x)+F(x) \leq 1$ for all $x$.

This implies $A(x)+1- F^c(x) \leq 1 \Rightarrow A(x) \leq F^c(x)$ for all $x$.

Hence $A \subseteq F^c$. $F^c$ is fuzzy $\beta$ open and $A$ is $f\beta g^*$ closed.

Therefore $\text{cl}(A) \subseteq F^c$. Hence $(\text{cl}(A)(x)) \leq F^c(x)$ for all $x$.

Therefore $(\text{cl}(A)(x)) \leq 1-F(x)$ for all $x$.

Hence $(\text{cl}(A)(x))+F(x) \leq 1$ for all $x$.

Therefore $\text{cl}(A)$ is not $q$ coincident with $F$.

Sufficient: Let $A \subseteq U$, $U$ be fuzzy $\beta$ open.

$A(x) \leq U(x)$ for all $x \Rightarrow A(x) \leq 1-U^c(x)$ for all $x$.

This implies $A(x)+ U^c(x) \leq 1$ for all $x$.

Hence $A$ is not $q$ coincident with $U^c$ where $U^c$ is fuzzy $\beta$ closed.

Therefore $\text{cl}(A)$ is not $q$ coincident with $U^c$.

$(\text{cl}(A)(x))+ U^c(x) \leq 1$ for all $x$. Hence $(\text{cl}(A)(x))+1-U(x) \leq 1$ for all $x$.

Therefore $(\text{cl}(A)(x)) \leq U(x)$ for all $x$. Hence $\text{cl}(A) \subseteq U$.

Therefore $A$ is $f\beta g^*$ closed.