CHAPTER 4

ODD GRACEFUL GRAPHS

The concept of odd graceful graphs was introduced by Gnanajothi [11]. In this chapter we prove that several classes of graphs such as \( rC_{4k} (k > 1) \), \( C_n \cup K_{1,m} \) where \( n = 4k, \ k > 1 \) and \( m \) is any positive integer, and \( C_n \cup P_m \) where \( n=2k \) are odd graceful. We also prove that the graph obtained by identifying a vertex of an even cycle with the centre or with a pendant vertex of a star is odd graceful and the graph obtained by joining a vertex of an even cycle to a pendant vertex of a star is odd graceful. We also construct several families of odd graceful graphs.

**Theorem 4.1** Let \( G \) be the graph obtained by identifying a pendant vertex of \( P_m \) with a vertex of \( C_n \) where \( n=4k \). Then \( G \) is odd graceful.

**Proof.** Let \( V(G) = \{ u_1, u_2, \ldots, u_{4k}, v_1, v_2, \ldots, v_m \} \),
\( C_{4k} = (u_1, u_2, \ldots, u_{4k}, u_1) \) and \( P_m = (u_{4k}, v_1, v_2, \ldots, v_m) \).

Define \( f : V(G) \to \{0,1,2,\ldots,2q-1\} \) as follows.

\[
f(u_i) = \begin{cases} 
  i - 1 & \text{if } i \text{ is odd and } i \leq 2k - 1 \\
  i + 1 & \text{if } i \text{ is odd and } 2k + 1 \leq i \leq 4k - 1 \\
  2q - i + 1 & \text{if } i \text{ is even}
\end{cases}
\]

and \( f(v_i) = \begin{cases} 
  i + n + 1 & \text{if } i \text{ is odd} \\
  2q - i - n + 1 & \text{if } i \text{ is even}
\end{cases}\).
The induced edge labeling $f^*$ is given by

$$f^*(u_i u_{i+1}) = \begin{cases} 2q - (2i - 1) & \text{if } 1 \leq i \leq 2k - 1 \\ 2q - (2i + 1) & \text{if } 2k \leq i \leq 4k - 1 \end{cases}$$

$$f^*(u_n u_1) = 2q - (n - 1),$$

$$f^*(v_i v_{i+1}) = 2q - (2n + 2i + 1) \text{ if } 1 \leq i \leq n - 1$$

and

$$f^*(u_n v_1) = 2q - (2n + 1).$$

The set of edge labels of the cycle $C_n$

$$= \{ 2q - 1, 2q - 3, \ldots, 2q - (n - 3), 2q - (n + 1), \ldots, 2q - (2n - 1), 2q - (n - 1) \}.$$ 

The set of edge labels of the path $P_m$

$$= \{ 2q - (2n + 1), 2q - (2n + 3), \ldots, 5, 3, 1 \}.$$ 

Hence $G$ is odd graceful. □

**Example 4.2**  The odd graceful labeling of the graph $G$ obtained by attaching a pendant vertex of $P_{19}$ to a vertex of $C_{12}$ is given in Figure 4.1.

Fig 4.1
Theorem 4.3 The graph $G$ obtained by attaching an even cycle $C_n$ to each vertex of $K_2$ is odd graceful.

Proof. Let $V(G) = \{ u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \}$

and $E(G) = \{ u_1 u_2, \ldots, u_n u_1, u_1 v_1, v_1 v_2, \ldots, v_n v_1 \}$.

Let $n=2k$.

Define $f: V \rightarrow \{ 0, 1, 2, \ldots, 2k-1 \}$ as follows.

If $n=4$, $f(u_1) = 0$, $f(u_2) = 17$, $f(u_3) = 4$, $f(u_4) = 15$, $f(v_1) = 1$, $f(v_2) = 8$, $f(v_3) = 5$ and $f(v_4) = 10$.

If $n=6$, $f(u_1) = 0$, $f(u_2) = 25$, $f(u_3) = 8$, $f(u_4) = 21$, $f(u_5) = 14$, $f(u_6) = 23$, $f(v_1) = 1$, $f(v_2) = 20$, $f(v_3) = 15$, $f(v_4) = 18$, $f(v_5) = 7$ and $f(v_6) = 22$.

If $n = 8$, $f(u_1) = 0$, $f(u_2) = 33$, $f(u_3) = 8$, $f(u_4) = 29$, $f(u_5) = 20$, $f(u_6) = 27$, $f(u_7) = 14$, $f(u_8) = 31$, $f(v_1) = 1$, $f(v_2) = 28$, $f(v_3) = 13$, $f(v_4) = 24$, $f(v_5) = 21$, $f(v_6) = 26$, $f(v_7) = 7$ and $f(v_8) = 30$.

If $n = 10$, $f(u_1) = 0$, $f(u_2) = 41$, $f(u_3) = 8$, $f(u_4) = 37$, $f(u_5) = 20$, $f(u_6) = 33$, $f(u_7) = 26$, $f(u_8) = 35$, $f(u_9) = 14$, $f(u_{10}) = 39$, $f(v_1) = 1$, $f(v_2) = 36$, $f(v_3) = 13$, $f(v_4) = 32$, $f(v_5) = 21$, $f(v_6) = 24$, $f(v_7) = 19$, $f(v_8) = 34$, $f(v_9) = 7$ and $f(v_{10}) = 38$. 
If \( n = 12 \), \( f(u_1) = 0 \), \( f(u_2) = 49 \), \( f(u_3) = 8 \), \( f(u_4) = 45 \),
\( f(u_5) = 20 \), \( f(u_6) = 41 \), \( f(u_7) = 32 \), \( f(u_8) = 39 \), \( f(u_9) = 26 \),
\( f(u_{10}) = 43 \), \( f(u_{11}) = 14 \), \( f(u_{12}) = 47 \),
\( f(v_1) = 1 \), \( f(v_2) = 44 \), \( f(v_3) = 13 \), \( f(v_4) = 40 \), \( f(v_5) = 21 \),
\( f(v_6) = 18 \), \( f(v_7) = 23 \), \( f(v_8) = 34 \), \( f(v_9) = 19 \), \( f(v_{10}) = 42 \),
\( f(v_{11}) = 7 \) and \( f(v_{12}) = 46 \).

If \( n = 14, 18, 22, \ldots \),
\[
f(v_i) = \begin{cases} 
1 & \text{if } i = 1 \\
4i + 1 & \text{if } i \text{ is odd and } 3 \leq i \leq k - 2 \\
2q - 4i + 5 & \text{if } i \text{ is odd and } k \leq i \leq n - 3 \\
7 & \text{if } i = n - 1 \\
4n - 4 & \text{if } i = 2 \\
2q - 4i + 6 & \text{if } i \text{ is even and } 4 \leq i \leq k - 1 \\
2n + 10 & \text{if } i = k + 1 \\
4i + 2 & \text{if } i \text{ is even and } k + 3 \leq i \leq n - 2 \\
4n - 2 & \text{if } i = n 
\end{cases}
\]

and
\[
f(u_i) = \begin{cases} 
0 & \text{if } i = 1 \\
8, 20 & \text{if } i = 3, 5 \\
4i + 4 & \text{if } i \text{ is odd and } 7 \leq i \leq k \\
2q - 4i + 12 & \text{if } i \text{ is odd and } k + 2 \leq i \leq n - 3 \\
14 & \text{if } i = n - 1 \\
2q - 1 & \text{if } i = 2 \\
2q - 5 & \text{if } i = 4 \\
2q - 4i + 15 & \text{if } i \text{ is even and } 6 \leq i \leq k + 1 \\
4i + 7 & \text{if } i \text{ is even and } k + 3 \leq i \leq n - 4 \\
2q - 7 & \text{if } i = n - 2 \\
2q - 3 & \text{if } i = n
\end{cases}
\]
If \( n = 16, 20, 24, \ldots \),

\[
f(v_i) = \begin{cases} 
1 & \text{if } i = 1 \\
4i + 1 & \text{if } i \text{ is odd and } 3 \leq i \leq k - 1 \\
4i - 5 & \text{if } i = k + 1 \\
2q - 4i + 5 & \text{if } i \text{ is odd and } k + 3 \leq i \leq n - 3 \\
7 & \text{if } i = n - 1 \\
4n - 4 & \text{if } i = 2 \\
2q - 4i + 6 & \text{if } i \text{ is even and } 4 \leq i \leq k - 2 \\
2n + 6 & \text{if } i = k \\
4i + 2 & \text{if } i \text{ is even and } k + 2 \leq i \leq n - 2 \\
4n - 2 & \text{if } i = n 
\end{cases}
\]

and,

\[
f(u_i) = \begin{cases} 
0 & \text{if } i = 1 \\
8, 20 & \text{if } i = 3, 5 \\
4i + 4 & \text{if } i \text{ is odd and } 7 \leq i \leq k - 1 \\
2n + 12 & \text{if } i = k + 1 \\
2q - 4i + 12 & \text{if } i \text{ is odd and } k + 3 \leq i \leq n - 3 \\
14 & \text{if } i = n - 1 \\
2q - 1 & \text{if } i = 2 \\
2q - 7 & \text{if } i = 4 \\
2q - 4i + 15 & \text{if } i \text{ is even and } 6 \leq i \leq k \\
4i + 7 & \text{if } i \text{ is even and } k + 2 \leq i \leq n - 4 \\
2q - 7 & \text{if } i = n - 2 \\
2q - 3 & \text{if } i = n 
\end{cases}
\]

All the odd numbers between 1 and \( 2q - 1 \) appear as edge labels in the above labeling.

Hence \( G \) is odd graceful. ■

**Example 4.4** An odd graceful labeling of the graph obtained by attaching a copy of \( C_{16} \) to each vertex of \( K_2 \) is given in Figure 4.2.
Example 4.5 An odd graceful labeling of the graph obtained by attaching a copy of $C_{18}$ to each vertex of $K_2$ is given in Figure 4.3.

Theorem 4.6 Let $G = C_n \odot K_{1,m}$ be the graph obtained by identifying a vertex of $C_n = (u_1, u_2, ..., u_n, u_1)$ to the centre of the star $K_{1,m}$. Then $G$ is odd graceful if $m \geq \frac{(n-6)}{2}$ and $n$ is even.

Proof. Let $n = 2k$ and $m \geq k - 3$. 
Let $V(G) = \{ u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_m \}$ and
\[ E(G) = \{ u_1 u_2, u_2 u_3, \ldots, u_{2k-1} u_{2k}, u_{2k} u_1 \} \cup \{ u_i v_i / 1 \leq i \leq m \}. \]

Define $f: V \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ as follows.

\[
f(u_i) = \begin{cases} 
  i - 2 & \text{if } i \text{ is even and } 2 \leq i \leq 2k - 2 \\
  i + 2k - 4 & \text{if } i = 2k \\
  2q - i & \text{if } i \text{ is odd}
\end{cases}
\]

If $m = k - 3$, $f(v_i) = 4k + 2i - 4$, $1 \leq i \leq m$.

If $m > k - 3$,

\[
f(v_i) = \begin{cases} 
  4k + 2i - 4 & \text{if } 1 \leq i \leq k - 2 \\
  4k + 2i - 2 & \text{if } i > k - 2
\end{cases}
\]

Let $f^*$ denote the induced edge labeling.

Then $f^*(u_i u_{i+1}) = 2q - 2i + 1$ if $1 \leq i \leq n - 2$,

\[ f^*(u_{n-1} u_n) = 2m - 2k + 5 \quad \text{and} \]

\[ f^*(u_n u_1) = 2q - 4k + 3. \]

If $m = k - 3$,

\[ f^*(u_1 v_i) = 2q - 2i - 4k + 3, 1 \leq i \leq m, \]

and if $m > k - 3$,

\[
f^*(u_1 v_i) = \begin{cases} 
  2q - 2i - 4k + 3 & \text{if } 1 \leq i \leq k - 2 \\
  2q - 2i - 4k + 1 & \text{if } i > k - 2
\end{cases}
\]

Thus $f^*(E(G)) = \{1, 3, \ldots, 2q - 1\}$ and hence $G$ is odd graceful.
Example 4.7 An odd graceful labeling of $C_{12} \odot K_{1,7}$ is given in Figure 4.4

![Fig 4.4](image)

Theorem 4.8. Let $G = C_n \odot K_{1,m}$ be the graph obtained by identifying a vertex of an even cycle $C_n$ with a pendant vertex of a star $K_{1,m}$. Then $G$ is odd graceful.

Proof. Let $n = 2k$. Let $V(G) = \{ u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_m \}

and $E(G) = \{ u_1 u_2, \ldots, u_{2k} u_1 \} \cup \{ v_i v_{i+1} \mid 1 \leq i \leq m \} \cup \{ u_1 v_1 \}$.

Define $f : V \to \{ 0, 1, 2, \ldots, 2q - 1 \}$ as follows.

If $n = 4$,

$$f(u_i) = \begin{cases} i - 1 & \text{if } i = 1, 3 \\ 2q - 2i + 1 & \text{if } i = 2, 4 \end{cases}$$

and

$$f(v_i) = \begin{cases} 2q - 1 & \text{if } i = 1 \\ 2i + 6 & \text{if } 2 \leq i \leq m. \end{cases}$$

If $n > 4$,

$$f(u_i) = \begin{cases} i - 1 & \text{if } i \text{ is odd} \\ 2q - i - 1 & \text{if } i \text{ is even and } i \leq 2k - 2 \\ 2m + 1 & \text{if } i = 2k \end{cases}$$
and
\[
 f(v_i) = \begin{cases} 
 2q - 1 & \text{if } i = 1 \\
 2n + 2i - 4 & \text{if } 2 \leq i \leq k - 1 \\
 2n + 2i - 2 & \text{if } k - 1 < i \leq m.
\end{cases}
\]

The induced edge labeling \( f^* : E \rightarrow \{ 1,3,5,\ldots, 2q - 1 \} \) is given below.

If \( n = 4 \),
\[
 f^*(u_1u_2) = 2q - 3, \quad f^*(u_2u_3) = 2q - 5, \\
 f^*(u_3u_4) = 2q - 9, \quad f^*(u_4u_1) = 2q - 7, \\
 f^*(u_1v_1) = 2q - 1, \quad \text{and} \quad f^*(v_1v_i) = 2q - 2i - 2n + 1 \text{ if } 2 \leq i \leq m.
\]

If \( n > 4 \),
\[
 f^*(u_iu_{i+1}) = 2q - (2i + 1) \text{ if } 1 \leq i \leq n - 2, \\
 f^*(u_{n-1}u_n) = 2m - n + 3, \\
 f^*(u_nu_1) = 2q - 2n + 1,
\]
and
\[
 f^*(v_1v_i) = \begin{cases} 
 2q - 2i - 2n + 3 & \text{if } 2 \leq i \leq k - 1 \\
 2q - 2i - 2n + 1 & \text{if } i > k - 1.
\end{cases}
\]

Clearly \( f^*(E) = \{ 1,3,5,\ldots,2q - 1 \} \) and hence \( G \) is odd graceful. \[\square\]

**Example 4.9** An odd graceful labeling of \( C_8 \otimes K_{1,5} \) is given in Figure 4.5.
Theorem 4.10 Let $G = C_n * K_{1,m}$ be the graph obtained by joining a vertex of an even cycle $C_n$ to a pendant vertex of a star $K_{1,m}$. Then $G$ is odd graceful if $m \geq (n - 2)/2$.

Proof. Suppose $n = 2k$ and $m \geq k - 1$.

Let $V(G) = \{ u_1, u_2, \ldots, u_{2k}, v_0, v_1, \ldots, v_m \}$ and

$$E(G) = \{ u_1 u_2, u_2 u_3, \ldots, u_{2k} u_1 \} \cup \{ u_1 v_1, v_0 v_1, v_0 v_i \mid 2 \leq i \leq m \}.$$ 

Define $f : V \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ as follows.

$$f(u_i) = \begin{cases} 2q - i - 2 & \text{if } i \text{ is odd} \\ i & \text{if } i \text{ is even and } i \leq n - 2 \\ i + n - 2 & \text{if } i = n \end{cases}$$

and

$$f(v_i) = \begin{cases} 2q - 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ 2i + 2n - 2 & \text{if } 2 \leq i \leq k - 1 \\ 2i + 2n & \text{if } i > k - 1. \end{cases}$$

The induced edge labeling $f^* : E \rightarrow \{1, 3, \ldots, 2q - 1\}$ is given by

$$f^*(u_i u_{i+1}) = 2q - (2i + 3), \quad 1 \leq i \leq n - 2$$

$$f^*(u_{n-1} u_n) = 2m - n + 3,$$ 

$$f^*(u_n u_1) = 2q - (2n + 1)$$

$$f^*(u_1 v_1) = 2q - 3,$$ 

and

$$f^*(v_0 v_i) = \begin{cases} 2q - 2i - 2n + 1 & \text{if } 2 \leq i \leq k - 1 \\ 2q - 2i - 2n - 1 & \text{if } i > k - 1. \end{cases}$$

Clearly $f^*(E) = \{1, 3, 5, \ldots, 2q - 1\}$ and hence $C_n * K_{1,m}$ is odd graceful.
Example 4.11  An odd graceful labeling of $C_6 \ast K_{1,7}$ is given in Figure 4.6

![Figure 4.6](image)

Theorem 4.12  $G = rC_n$, where $n = 4k$, is odd graceful for any positive integer $r$.

Proof. Let $n = 4k$.

Let $V(G) = \{ a_{i,1}, a_{i,2}, \ldots, a_{i,4k} \mid i = 1, 2, \ldots, r \}$

and $E(G) = \{ a_{i,1}a_{i,2}, a_{i,2}a_{i,3}, \ldots, a_{i,4k}a_{i,1} \mid i = 1, 2, \ldots, r \}$.

Define $f : V \to \{0, 1, 2, \ldots, 2q - 1\}$ as follows.

$$f(a_{i,1}) = i - 1 \text{ and }$$

$$f(a_{i,j}) = \begin{cases} (2j - 2)r - i + 1 & \text{if } j \text{ is odd and } 3 \leq j \leq 2k - 1 \\ nr + i - 1 & \text{if } j = 2k + 1 \\ (2n - 2j + 4)r - i - 1 & \text{if } j \text{ is odd and } 2k + 3 \leq j \leq 4k - 1 \\ (2n - 2j + 4)r - 3i + 2 & \text{if } j \text{ is even and } 2 \leq j \leq 2k \\ 2jr - 3i & \text{if } j \text{ is even and } 2k + 2 \leq j \leq 4k. \end{cases}$$

The induced edge labeling $f^* : E \to \{1, 3, 5, \ldots, 2q - 1\}$ is given by

$$f^*(a_{i,1}a_{i,2}) = 2nr - 4i + 3,$$

$$f^*(a_{i,2}a_{i,3}) = (2n - 4)r - 2i + 1.$$
\[ f^* (a_{i,4}) = (2n - 8) r - 2i + 1, \]

\[ f^* (a_{i,2k-1} a_{i,2k}) = 8r - 2i + 1, \]

\[ f^* (a_{i,2k} a_{i,2k+1}) = 4r - 4i + 3, \]

\[ f^* (a_{i,2k+1} a_{i,2k+2}) = 4r - 4i + 1, \]

\[ f^* (a_{i,2k+2} a_{i,2k+3}) = 6r - 2i + 1, \]

\[ f^* (a_{i,2k+3} a_{i,2k+4}) = 10r - 2i + 1, \]

\[ f^* (a_{i,n-1} a_{i,n}) = (2n - 6) r - 2i + 1, \]

and \[ f^* (a_{i,n} a_{i,1}) = 2nr - 4i + 1. \]

Clearly \( f^*(E) = \{1, 3, 5, ..., 2q - 1\} \) and hence \( G \) is odd graceful.
Example 4.13 An odd graceful labeling of $3C_{16}$ is given in Figure 4.7.

**Theorem 4.14** $G = C_n \cup K_{1,m}$ where $n = 4k$ and $m$ is any positive integer is odd graceful.

**Proof.** Let $V(G) = \{ u_1, u_2, \ldots, u_{4k}, v_0, v_1, \ldots, v_m \}$ and

$$E(G) = \{ u_i u_{i+1}, u_n u_1 / 1 \leq i \leq n-1 \} \cup \{ v_0 v_i / 1 \leq i \leq m \}.$$

**Case(i)** $m \leq 2k$.

Define $f : V(G) \rightarrow \{ 0, 1, 2, \ldots, 2q-1 \}$ as follows

$$f(u_i) = \begin{cases} 
2m+i-3 & \text{if } i \text{ is even and } 2 \leq i \leq 2k \\
2m+i-1 & \text{if } i \text{ is even and } 2k+2 \leq i \leq n \\
2q - i - 1 & \text{if } i \text{ is odd}
\end{cases}$$

and $f(v_i) = \begin{cases} 
0 & \text{if } i=0 \\
2q - 2i + 1 & \text{if } i \geq 1.
\end{cases}$

The induced edge labeling $f^* : E \rightarrow \{ 1, 3, 5, \ldots, 2q-1 \}$ is given by

$$f^*(u_i u_{i+1}) = \begin{cases} 
2q - 2m - 2i + 1 & \text{if } 1 \leq i \leq 2k \\
2q - 2m - 2i - 1 & \text{if } 2k + 1 \leq i \leq n - 1
\end{cases}$$

$$f^*(u_n u_1) = n - 1$$
and \( f^*(v_0v_1) = 2q - 2i + 1 \) if \( 1 \leq i \leq m \).

Clearly \( f^*(E) = \{1, 3, \ldots, 2q - 1\} \) and hence \( G \) is odd graceful.

**Case (ii) \( 2k < m \leq 3k \).**

Define \( f : V(G) \rightarrow \{1, 2, \ldots, 2q - 1\} \) as follows.

\[
\begin{align*}
  f(u_i) &= \begin{cases} 
    2q - i - 1 & \text{if } i \text{ is odd and } 1 \leq i \leq 2k - 1 \\
    2m + i - 1 & \text{if } i \text{ is odd and } 2k - 1 < i \leq 4k \\
    i + 2j - 1 & \text{if } i \text{ is even, } 2 \leq i \leq 2k, m = 2k + j, 1 \leq j \leq k - 2 \\
    i + 2k - 3 & \text{if } i \text{ is even, } 2 \leq i \leq 2k \text{ and } m = 3k - 1 \text{ or } 3k \\
    2m + 2n - 1 - i & \text{if } i \text{ is even and } i > 2k \\
  \end{cases}
\end{align*}
\]

\[
\begin{align*}
  f(v_i) &= \begin{cases} 
    0 & \text{if } i = 0 \\
    2q - 2i + 1 & \text{if } 1 \leq i \leq k + 1 \text{ and } 1 \leq m \leq 3k - 2 \\
    2m + n + 3 - 2i & \text{if } k + 2 \leq i \leq 3k - 1 \text{ and } 1 \leq m \leq 3k - 2 \\
    2q - 2i + 1 & \text{if } 1 \leq i \leq k, m = 3k - 1 \text{ or } 3k \\
    2m + n + 3 - 2i & \text{if } k + 1 \leq i \leq 3k - 1 \text{ and } m = 3k - 1 \text{ or } 3k \\
    n + 1 & \text{if } i = 3k.
  \end{cases}
\end{align*}
\]

The induced edge labeling \( f^* : E \rightarrow \{1, 3, \ldots, 2q - 1\} \) is given by

\[
\begin{align*}
  f^*(u_iu_{n+1}) &= \begin{cases} 
    2q - (2j + 2i + 1) & \text{if } 1 \leq i \leq 2k - 1, m = 2k + j, 1 \leq j \leq k - 1 \\
    2q - (2k + 2i - 1) & \text{if } 1 \leq i \leq 2k - 1 \text{ and } m = 3k \\
    2i + 1 & \text{if } i = 2k \text{ and } 1 \leq m \leq 3k - 1 \\
    2i + 3 & \text{if } i = 2k \text{ and } m = 3k \\
    2n - 2i - 1 & \text{if } 2k + 1 \leq i \leq n - 1, 2k + 1 \leq m \leq 3k \\
    n - 1 & \text{if } i = n \text{ and } 2k + 1 \leq m \leq 3k
  \end{cases}
\end{align*}
\]
Clearly $f^*(E) = \{1, 3, \ldots, 2q - 1\}$ and hence $G$ is odd graceful.

**Case (iii) $m > 3k$**

Define $f : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$ as follows.

$$f(u_i) = \begin{cases} 
2q - i - 1 & \text{if } i \text{ is odd and } 1 \leq i \leq 2k - 1 \\
2m + i - 1 & \text{if } i \text{ is odd and } 2k - 1 < i \leq 4k \\
i + 2k - 1 & \text{if } i \text{ is even and } 2 \leq i \leq 2k \\
2m + 2n - 1 - i & \text{if } i \text{ is even and } i > 2k 
\end{cases}$$

$$f(v_i) = \begin{cases} 
0 & \text{if } i = 0 \\
2q - 2i + 1 & \text{if } 1 \leq i \leq k + 1 \\
2m + n + 3 - 2i & \text{if } k + 1 < i \leq 3k \\
2m + n + 1 - 2i & \text{if } 3k < i \leq m.
\end{cases}$$

The induced edge labeling $f^* : E(G) \to \{1, 3, \ldots, 2q - 1\}$ is given by

$$f^*(u_i u_{i+1}) = \begin{cases} 
2q - (2k + 2i + 1) & \text{if } 1 \leq i \leq 2k - 1 \\
2m - 2k + 1 & \text{if } i = 2k \\
2n - 2i - 1 & \text{if } 2k + 1 \leq i \leq n - 1,
\end{cases}$$

$$f^*(u_n u_1) = n - 1.$$
and \( f^*(v_0, v_i) = \begin{cases} 2q - 2i + 1 & \text{if } 1 \leq i \leq k + 1 \\ 2m + n + 3 - 2i & \text{if } k + 1 < i \leq 3k \\ 2m + n + 1 - 2i & \text{if } 3k < i \leq m. \end{cases} \)

Clearly \( f^*(E) = \{1, 3, \ldots, 2q - 1\} \) and hence \( G \) is odd graceful.

**Example 4.15** Odd graceful labelings of \( C_8 \cup K_{1,4} \), \( C_{12} \cup K_{1,8} \) and \( C_{12} \cup K_{1,11} \) are given in Figures 4.8, 4.9 and 4.10.
Theorem 4.16 $G = C_n \cup P_m$ where $n = 2k$ is odd graceful.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{m+1}\}$

where $C_n = (u_1, u_2, \ldots, u_n, u_1)$ and $P_m = (v_1, v_2, \ldots, v_{m+1})$

Case (i) $n \equiv 2 \pmod{4}$. Let $n = 4k + 2$, $k \geq 1$.

Define $f: V(G) \to \{0,1,2,\ldots,2q-1\}$ as follows.

$$f(u_i) = \begin{cases} 
i - 1 & \text{if } i \text{ is odd} \\ 2q - (i - 1) & \text{if } i \text{ is even and } i < n \\ 2q - 2i - 3 & \text{if } i = n \\ 2q - (i + 1) & \text{if } i \text{ is odd and } i \leq 2k - 1 \\ 2q - (i + 3) & \text{if } i \text{ is odd and } i > 2k - 1 \\ i + 2n - 5 & \text{if } i \text{ is even} \end{cases}$$

Case (ii) $n \equiv 0 \pmod{4}$. Let $n = 4k$, $k \geq 1$.

Define $f: V(G) \to \{0,1,2,\ldots,2q-1\}$ as follows.

$$f(u_i) = \begin{cases} 
i - 1 & \text{if } i \text{ is odd} \\ 2q - (i - 1) & \text{if } i \text{ is even and } i < n \\ 2q - 2i - 3 & \text{if } i = n \\ 2q - (i + 1) & \text{if } i \text{ is odd and } i \leq 2k - 1 \\ 2q - (i + 3) & \text{if } i \text{ is odd and } i > 2k - 1 \\ i + 2n - 5 & \text{if } i \text{ is even} \end{cases}$$

If $n = 4$,

$$f(v_i) = \begin{cases} 2q - (i + 1) & \text{if } i \text{ is odd} \\ i + 2n - 3 & \text{if } i \text{ is even} \end{cases}$$

If $n > 4$,

$$f(v_i) = \begin{cases} 2q - (i + 1) & \text{if } i \text{ is odd} \\ i + 2n - 5 & \text{if } i \text{ is even and } i \leq 2k - 2 \\ i + 2n - 3 & \text{if } i \text{ is even and } i > 2k - 2. \end{cases}$$
Let \( f^* \) be the induced edge labeling of \( f \).

Then \( f^*(u_i u_{i+1}) = 2q - (2i - 1) \) if \( 1 \leq i \leq n - 2 \),
\[
\begin{align*}
  f^*(u_{n-1} u_n) &= 2q - (3n - 5), \\
  f^*(u_n u_1) &= 2q - (2n - 3), \\
  f^*(v_1 v_2) &= 2q - (2n - 1), \\
  f^*(v_2 v_3) &= 2q - (2n + 1), \\
  f^*(v_3 v_4) &= 2q - (2n + 3), \\
  \cdots &
\end{align*}
\]

\[
\begin{align*}
  f^*(v_{2k-2} v_{2k-1}) &= 2q - (3n - 7), \\
  f^*(v_{2k-1} v_{2k}) &= 2q - (3n - 3), \\
  \cdots &
\end{align*}
\]

and \( f^*(v_m v_{m+1}) = 1 \).

Clearly \( f^*(E) = \{1, 3, 5, \ldots, 2q - 1\} \) and hence \( G \) is odd graceful. \( \square \)
Example 4.17 An odd graceful labeling of $C_8 \cup P_{19}$ is given in Figure 4.11.

Example 4.18 An odd graceful labeling of $C_{10} \cup P_{16}$ is given in Figure 4.12.

Theorem 4.19 Let $G_1 = K_{1,m}$ with $V(G_1) = \{u, u_1, \ldots, u_m\}$ and $\deg u = m$. Let $G_2 = \overline{K}_n$ with $V(\overline{K}_n) = \{v_1, v_2, \ldots, v_n\}$. Let $G$ be the graph obtained by joining each vertex of $K_{1,n}$ except $u$ with all the vertices of $V(\overline{K}_n)$. Then $G$ is odd graceful.

Proof. Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ as follows.

$$f(u) = 2q - 1,$$

$$f(u_i) = 2(i - 1), \quad 1 \leq i \leq m$$
and \( f(v_j) = 2mj - 1, \ 1 \leq j \leq n \)

The induced edge labeling \( f^* : E(G) \to \{ 1, 3, \ldots, 2q - 1 \} \) is given by

\[
f^*(uw_i) = 2q - 2i + 1, \ 1 \leq i \leq m
\]

and \( f^*(u_iv_j) = 2mj - 2i + 1, \ 1 \leq i \leq m, \ 1 \leq j \leq n. \)

Clearly \( f^*(E(G)) = \{ 2q - 1, 2q - 3, \ldots, 3, 1 \} \) and hence \( G \) is odd graceful.

**Example 4.20** An odd graceful labeling of the graph \( G \) which is obtained by joining each vertex of \( K_{1,4} \) except the centre with all the vertices of \( K_3 \) is given in Figure 4.13.

![Fig 4.13](image)

**Theorem 4.21** Let \( G_1 = K_{1,m} \) with vertex set \( \{ u, u_1, u_2, \ldots, u_m \} \) and \( \text{deg} \ u = m \). Let \( G_2 = K_{1,n} \) with vertex set \( \{ v, v_1, v_2, \ldots, v_n \} \) and \( \text{deg} \ v = n \). Let \( G \) be the graph obtained by joining \( u_i \) to \( v_j, 1 \leq i \leq m, \ 1 \leq j \leq n \). Then \( G \) is odd graceful.

**Proof.** Define \( f : V(G) \to \{ 0, 1, 2, \ldots, 2q - 1 \} \) as follows.

\[
f(u) = 2q - 1, \ f(u_i) = 2(i - 1), \ 1 \leq i \leq m
\]
and \( f(v) = 2m, f(v_j) = 2mj + 2j - 1, \ 1 \leq j \leq n. \)

The induced edge labeling \( f^* : E(G) \rightarrow \{1, 3, ..., 2q - 1\} \) is given by

\[
f^*(u_i v_j) = 2mj + 2j - 2i + 1, \ 1 \leq i \leq m \text{ and } 1 \leq j \leq n,
\]

\[
f^*(u_i u_i) = 2q - 2i + 1, \ 1 \leq i \leq m
\]

and \( f^*(v_j v_j) = 2mj + 2j - 2m - 1, \ 1 \leq j \leq n. \)

Clearly \( f^*(E(G)) = \{1, 3, 5, ..., 2q - 1\} \) and hence \( G \) is odd graceful □.

**Example 4.22.** An odd graceful labeling of the graph \( G \) obtained from \( K_{1, 4} \) and \( K_{1, 3} \) is given in Figure 4.14.

![Fig 4.14](image-url)
Theorem 4.23 Let \( P = (u_1, u_2, u_3) \) be a path. Let \( G_1 = K_n \) with \( V(G_1) = \{v_1, v_2, \ldots, v_n\} \). Let \( G \) be the graph obtained by joining \( u_1 \) to \( v_i \) and \( u_3 \) to \( v_i \), \( 1 \leq i \leq n \). Then \( G \) is odd graceful.

Proof. Define \( f: V(G) \to \{0, 1, 2, \ldots, 2q - 1\} \) as follows.

\[
\begin{align*}
    f(u_1) &= 0, \\
    f(u_2) &= 2q - 1, \\
    f(u_3) &= 2,
\end{align*}
\]

and \( f(v_i) = 4i - 1 \) if \( 1 \leq i \leq n \).

The induced edge labeling \( f^*: E(G) \to \{1, 3, \ldots, 2q - 1\} \) is given by

\[
\begin{align*}
    f^*(u_1u_2) &= 2q - 1, \\
    f^*(u_2u_3) &= 2q - 3, \\
    f^*(u_1v_i) &= 4i - 1 \quad \text{if} \quad 1 \leq i \leq n
\end{align*}
\]

and \( f^*(u_3v_i) = 4i - 3 \) if \( 1 \leq i \leq n \).

Clearly \( f^*(E(G)) = \{1, 3, 5, \ldots, 2q - 1\} \) and hence \( G \) is odd graceful.

Example 4.24 Odd graceful labeling of \( G \) which is obtained from \( P_2 \) and \( K_5 \) is given in Figure 4.15.

---

Fig 4.15
Theorem 4.25 Let $P = (u_1, u_2, u_3, u_4, u_5)$ be a path. Let $G_1 = K_n$ with $V(G_1) = \{ v_1, v_2, \ldots, v_n \}$. Let $G$ be the graph obtained by joining $u_i$ to $v_i$ and $u_5$ to $v_i$, $1 \leq i \leq n$. Then $G$ is odd graceful.

Proof. Define $f : V(G) \to \{ 0, 1, 2, \ldots, 2q - 1 \}$ as follows.

$$f(u_i) = i - 1 \quad \text{if} \quad i = 1, 3, 5$$

$$f(u_2) = 2q - 1, \quad f(u_4) = 2q - 3.$$ 

If $n$ is odd,

$$f(v_i) = \begin{cases} 4i - 1 & \text{if } i \text{ is odd} \\ 4i + 1 & \text{if } i \text{ is even} \end{cases}.$$

If $n$ is even,

$$f(v_i) = \begin{cases} 4i + 1 & \text{if } i \text{ is odd} \\ 4i - 1 & \text{if } i \text{ is even} \end{cases}.$$

The induced edge labeling $f^* : E(G) \to \{ 1, 3, \ldots, 2q - 1 \}$ is as follows.

$$f^*(u_i u_{i+1}) = 2q - (2i - 1) \quad \text{if} \quad 1 \leq i \leq 4.$$ 

If $n$ is odd,

$$f^*(u_1 v_i) = \begin{cases} 4i - 1 & \text{if } i \text{ is odd} \\ 4i + 1 & \text{if } i \text{ is even} \end{cases}.$$

and

$$f^*(u_5 v_i) = \begin{cases} 1 & \text{if} \quad i = 1 \\ 4i - 5 & \text{if } i \text{ is odd and } i \geq 3 \\ 4i - 3 & \text{if} \quad i \text{ is even} \end{cases}.$$
If $n$ is even,

$$f^*(u_1v_1) = \begin{cases} 
4i + 1 & \text{if } i \text{ is odd} \\
4i - 1 & \text{if } i \text{ is even}
\end{cases}$$

and $f^*(u_3v_i) = \begin{cases} 
4i - 3 & \text{if } i \text{ is odd} \\
4i - 5 & \text{if } i \text{ is even}.
\end{cases}$

Hence $f^*(E(G)) = \{1, 3, 5, \ldots, 2q - 1\}$ so that $G$ is odd graceful.

**Example 4.26** Odd graceful labeling of a graph $G$ which is obtained from $P_4$ and $\overline{K}_6$ is given in Figure 4.16.

![Figure 4.16](image)

**Example 4.27** Odd graceful labeling of a graph $G$ which is obtained from $P_4$ and $\overline{K}_5$ is given in Figure 4.17.

![Figure 4.17](image)
Theorem 4.28 Quadrilateral snakes are odd graceful.

Proof. Let $G$ be a quadrilateral snake with

$$V(G) = \{u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{2n}\} \quad \text{and}$$

$$E(G) = \{u_iu_{i+1} / 1 \leq i \leq n \} \cup \{v_{2j-1}v_{2j}, u_{j+1}v_{2j-2}, u_jv_{2j-1} / 1 \leq j \leq n\}.$$

Define $f: V(G) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ as follows.

$$f(u_i) = \begin{cases} 
4i - 4 & \text{if } i \text{ is odd} \\
2q - 4i + 7 & \text{if } i \text{ is even}
\end{cases}$$

and $f(v_i) = \begin{cases} 
2q - 2i - 1 & \text{if } i \equiv 1 \pmod{4} \\
2q - 2i + 3 & \text{if } i \equiv 0 \pmod{4} \\
2i & \text{if } i \equiv 2 \pmod{4} \\
2i + 4 & \text{if } i \equiv 3 \pmod{4}.
\end{cases}$

The induced edge labeling $f^*: E(G) \rightarrow \{1, 3, \ldots, 2q - 1\}$ is given by

$$f^*(u_iu_{i+1}) = 2q - 8i + 7, 1 \leq i \leq n$$

$$f^*(v_{2j-1}v_{2j}) = 2q - 8j + 1, 1 \leq j \leq n$$

$$f^*(u_{j+1}v_{2j-2}) = 2q - 8j + 3, 1 \leq j \leq n$$

and $f^*(u_jv_{2j-1}) = 2q - 8j + 5, 1 \leq j \leq n.$

Clearly $f^*(E(G)) = \{2q - 1, 2q - 3, \ldots, 5, 3, 1\}.$

Hence $f$ is an odd graceful labeling of $G.$
Example 4.29  Odd graceful labeling of a quadrilateral snake with 5 blocks is given in Figure 4.18

![Figure 4.18](image)

**Theorem 4.30.** Let $G_n$ be the graph with $V(G_n) = \{a_{ij} / i = 1, 2, ..., n \text{ and } j = 1, 2, 3, 4, \}$ and $E(G_n) = \{a_{i, i+1} / i = 1, 2, ..., n \}$

$\cup \{a_{i,3} a_{i+1,3} / i = 1, 2, ..., n-1 \} \cup \{a_{1, i, 2} a_{i, i, 3} a_{i, 3, i, 4} a_{4, i, 1} / i = 1, 2, ..., n \}$. Then $G_n$ is odd graceful.

**Proof.** Define $f: V(G_n) \rightarrow \{0, 1, 2, ..., 2q-1 \}$ as follows.

- $f(a_{i,1}) = \begin{cases} 2q - 6i + 5 & \text{if } i \text{ is odd} \\ 6i - 2 & \text{if } i \text{ is even} \end{cases}$
- $f(a_{i,2}) = \begin{cases} 6(i-1) & \text{if } i \text{ is odd} \\ 2q - 6i + 5 & \text{if } i \text{ is even} \end{cases}$
- $f(a_{i,3}) = \begin{cases} 2q - 6i + 3 & \text{if } i \text{ is odd} \\ 6(i-1) & \text{if } i \text{ is even} \end{cases}$

and

- $f(a_{i,4}) = \begin{cases} 6(i-1) + 4 & \text{if } i \text{ is odd} \\ 2q - 6i + 3 & \text{if } i \text{ is even} \end{cases}$

The induced edge labeling $f^*: E(G_n) \rightarrow \{1, 3, ..., 2q - 1 \}$ is as follows.

- $f^*(a_{i,1} a_{i+1,1}) = 2q - 2n - 12i + 13, i = 1, 2, ..., n - 1$.
- $f^*(a_{i,3} a_{i+1,3}) = 2q - 2n - 12i + 15, i = 1, 2, ..., n - 1$. 
\[ f^*(a_{i,1}, a_{i,2}) = \begin{cases} 2q - 12i + 11 & \text{if } i \text{ is odd} \\ 2q - 12i + 7 & \text{if } i \text{ is even} \end{cases} \]

\[ f^*(a_{i,2}, a_{i,3}) = \begin{cases} 2q - 12i + 9 & \text{if } i \text{ is odd} \\ 2q - 12i + 11 & \text{if } i \text{ is even} \end{cases} \]

\[ f^*(a_{i,3}, a_{i,4}) = \begin{cases} 2q - 12i + 5 & \text{if } i \text{ is odd} \\ 2q - 12i + 9 & \text{if } i \text{ is even} \end{cases} \]

\[ f^*(a_{i,4}, a_{i,1}) = \begin{cases} 2q - 12i + 7 & \text{if } i \text{ is odd} \\ 2q - 12i + 5 & \text{if } i \text{ is even} \end{cases} \]

Clearly \( f^* (E(G_n)) = \{2q - 1, 2q - 3, \ldots, 5, 3, 1\} \) and hence \( G_n \) is odd graceful.

**Example 4.31** An odd graceful labeling of \( G_6 \) is given in Figure 4.19.

\[
\begin{array}{cccccccccccc}
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
0 & 4 & 61 & 51 & 12 & 16 & 49 & 47 & 24 & 38 & 33 & 34 & 30 \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
& & & & & & & & & & & & \\
67 & 10 & 55 & 22 & 43 & 41 & 37 & 35 & & & & & \\
\end{array}
\]

**Fig 4.19**

**Theorem 4.32** Let \( G_n \) be the graph with vertex set \( v(G_n) = \{a_i, b_i \mid i = 1, 2, \ldots, n\} \) and edge set \( E(G_n) = \{a_i, a_{i+1}, b_i, b_{i+1}, a_i b_i, a_{i+1} b_{i+1} \mid 1 \leq i \leq n - 1\} \). Then \( G_n \) is odd graceful.

**Proof.** Define \( f: V(G_n) \to \{0, 1, 2, \ldots, 2q - 1\} \) as follows.

\[ f(a_i) = \begin{cases} 4 & \text{if } i = 1 \\ 4i - 4 & \text{if } i \text{ is odd and } 3 \leq i \leq n \\ 2q - 4i + 7 & \text{if } i \text{ is even} \end{cases} \]

\[ f(b_i) = \begin{cases} 0 & \text{if } i = 1 \\ 4i & \text{if } i \text{ is odd and } i > 1 \\ 2q - 4i + 5 & \text{if } i \text{ is even} \end{cases} \]

The induced edge labeling \( f^\ast \) is given by

\[ f^\ast (a_{i+1}) = \begin{cases} 2q - 5 & \text{if } i = 1 \\ 2q - 8i + 7 & \text{if } 2 \leq i \leq n - 1 \end{cases} \]

\[ f^\ast (b_{i+1}) = \begin{cases} 2q - 8i + 5 & \text{if } i = 1 \\ 2q - 8i + 1 & \text{if } 2 \leq i \leq n - 1 \end{cases} \]

\[ f^\ast (a_{i+1}) = \begin{cases} 2q - 7 & \text{if } i = 1 \\ 2q - 8i + 5 & \text{if } i \text{ is odd and } i > 1 \\ 2q - 8i + 3 & \text{if } i \text{ is even} \end{cases} \]

and \( f^\ast (b_{i+1}) = \begin{cases} 2q - 1 & \text{if } i = 1 \\ 2q - 8i + 3 & \text{if } i \text{ is odd and } i > 1 \\ 2q - 8i + 5 & \text{if } i \text{ is even} \end{cases} \)

Clearly \( f^\ast (E) = \{1, 3, \ldots, 2q - 1\} \) and hence \( G_n \) is odd graceful.

Example 4.33 Odd graceful labeling of \( G_7 \) is given in Figure 4.20

![Figure 4.20](image)

**Theorem 4.34** Let \( G_n \) be the graph with vertex set \( \{a_i, b_i/1 \leq i \leq n\} \cup \{c_i/1 \leq i \leq n + 1\} \) and edge set \( \{a_i c_i, b_i c_i, a_i c_i, b_i c_i/1 \leq i \leq n\} \). Then \( G_n \) is odd graceful.
Proof. Define \( f: V(G_n) \rightarrow \{0,1,2,...,2q-1\} \) as follows.

\[
f(a_i) = 2q - 4i + 1, 1 \leq i \leq n,
\]

\[
f(b_i) = 2q - 4i + 3, 1 \leq i \leq n \text{ and }
\]

\[
f(c_i) = 4i - 4, 1 \leq i \leq n + 1.
\]

The induced edge labeling \( f^* \) is given by

\[
f^*(a_i c_i) = 2q - 8i + 5, 1 \leq i \leq n,
\]

\[
f^*(b_i c_i) = 2q - 8i + 7, 1 \leq i \leq n,
\]

\[
f^*(a_i c_{i+1}) = 2q - 8i + 1, 1 \leq i \leq n \text{ and }
\]

\[
f^*(b_i c_{i+1}) = 2q - 8i + 3, 1 \leq i \leq n.
\]

Clearly \( f^*(E) = \{1,3,...,2q-1\} \) and hence \( G_n \) is odd graceful.

Example 4.35 Odd graceful labeling of \( G_4 \) is given in Figure 4.21.

![Figure 4.21](image)

Theorem 4.36 \( P_n \times P_2 \) is odd graceful where \( P_n \) is a path of length \( n - 1 \).
Proof. Let $G = P_n \times P_2$, $V(G) = \{a_i, b_i/1 \leq i \leq n\}$ and $E(G) = \{a_i a_{i+1}, b_i b_{i+1}/1 \leq i \leq n-1\} \cup \{a_i b_i/1 \leq i \leq n\}$.

Define $f: V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ by

$$f(a_i) = \begin{cases} 2q - 2i - 1 & \text{if } i \text{ is odd} \\ 4i - 4 & \text{if } i \text{ is even} \end{cases}$$

and $f(b_i) = \begin{cases} 2q - 2i + 3 & \text{if } i \text{ is even} \\ 4i - 4 & \text{if } i \text{ is odd} \end{cases}$

The induced edge labeling $f^*$ is given by

$$f^*(a_i a_{i+1}) = \begin{cases} 2q - 6i - 1 & \text{if } i \text{ is odd} \\ 2q - 6i + 1 & \text{if } i \text{ is even} \end{cases}$$

and $f^*(b_i b_{i+1}) = \begin{cases} 2q - 6i + 3 & \text{if } i \text{ is odd} \\ 2q - 6i + 5 & \text{if } i \text{ is even} \end{cases}$

Clearly $f^*(E) = \{1, 3, 5, \ldots, 2q-1\}$ so that $G$ is odd graceful.

Example 4.37 Odd graceful labeling of $P_7 \times P_2$ is given in Figure 4.22.

![Fig 4.22](image_url)