CHAPTER 6

Cell Models for Stokes Flow Past an Assemblage of Reiner-Rivlin Liquid Drops

6.1 Introduction

Flow of an incompressible viscous fluid through a swarm of particles or liquid droplets occurs in many multiphase systems common in chemical and engineering processes. Due to its practical importance in many applications, where the particles are not isolated, such as fluidization, sedimentation, rheology of suspensions, motion of blood cells in a vein and micro fluidics, fluid flow through an assemblage of particles or liquid drops has been gaining scientific interest for more than five decades.

In the investigation of liquid motion through an assemblage of molecules, we come across cumbersome computations in the event that we consider the solution of the stream field over the whole accumulation by considering definite locations of the constituent parts. So as to dispose of the above intricacy, it is adequate to get the diagnostic expression by taking into account the impacts of the particles in the vicinity on the flow field around a solitary molecule of the assemblage. This has prompted the advancement of molecule in-cell models. However, the mathematical formulation of such problems is an unwieldy task as one has to analyze the complex interaction between numerous particles in order to receive information about the flow field. In this way, it is required to figure out if the vicinity of neighboring particles and/or boundaries influences the move of an individual molecule. Happel and Brenner (1983) introduced the theory that has been utilized to foresee the impact of molecule accumulation on the particle rate of sedimentation is the unit cell model. The model involves the
concept that random assemblage of particles can be divided into a number of identical cells where each particle or droplet assumed to have been enveloped by a cell. More precisely, the volume of the fluid cell is so chosen that the solid volume fraction in the cell equals to the solid volume fraction of the assemblage. Thus, the entire annoyance due to each particle is confined within the fluid cell associated with it. The problem then gets reduced to consideration of a single particle and its bounding envelope and the interaction effect is accounted by the application of suitably chosen boundary conditions at the envelope’s surface. Such problems of hydrodynamic interactions between two or more solid or fluid particles and between these particles and numerous boundaries have been used vastly in the recent past. The brief and constructive information in this area and some useful references can also be found in Kim and Karrila (1991).

The unit cell model has been generally used to solve boundary value problems for rigid particles in concentrated frameworks where the impact of compartment may be disregarded. To visualize the impact of molecule on the mean sedimentation rate in a limited suspension of indistinguishable spherical particles, the unit cell model for Newtonian fluids has been employed successfully and tested against the experimental outcomes. Although different shapes of cells can be used, the assumption of a spherical shape for the fictitious envelope of the suspending fluid surrounding each spherical particle is of great ease. The study of uniform flow of fluids around liquid or rigid bodies involving a variety of geometries has been well thought-out by numerous investigators taking into consideration of various analytical and numerical methods. The creeping motion of Newtonian liquid over a fluid sphere with a fluid layer of another incompressible viscous fluid of a different viscosity has been studied analytically and numerically owing to the fact that it serves as a reasonably good model for many applications to modern engineering and technology. For instance, utilizing spherical bipolar co-ordinates, Bart (1968) inspected the movement of a spherical fluid drop settling typical to a plane interface between two immiscible gooey liquids. Hetsroni and Haber (1970) examined the issue of a solitary
spherical droplet submerged in an unbounded gooey liquid of a different viscosity. Recently, Choudhuri and Padamavati (2010) have discussed the problem of axisymmetric/non-axisymmetric Stokes flow of an incompressible viscous fluid past a sphere coated with a thin fluid layer of a different viscosity, and later, Gupta and Deo (2013) solved the same problem for axisymmetric flow of a micropolar fluid. Most recently, Murthy and Kumar (2014) studied viscous fluid over a sphere with a uniform flow farther from the obstacle applying HAM technique.

Distinctive boundary conditions are recommended at the hypothetical surface. Uchida (1954) proposed a cell model for a sedimenting swarm of spherical particle encompassed by a liquid envelope with cubic external boundary, and later, the problem was accurately solved by Brenner (1957). Happel (1958, 1959) proposed cell models in which both particle and outer envelope are spherical assuming uniform velocity condition and no tangential stress at the cell surface. On the other hand, Kuwabara (1959) assumes in his model the vanishing of vorticity on the cell surface. Both the formulations yield almost the same results, but in Kuwabara’s case there takes place a slight exchange in mechanical energy between cell and environment. On the contrary, no such exchange takes place in Happel’s case. Two other boundary conditions are suggested by Cunningham (1910) [and later by Mehta and Morse (1975)] and Kvashnin (1979). Cunningham (1910) expected the tangential speed as a part of the average liquid speed signifying the homogeneity of the stream on the cell boundary. Kvashnin (1979) proposed the condition that the tangential component of velocity reaches a minimum at the cell surface with respect to radial distance signifying the symmetry on the cell.

Dassios et al. (1995) tackles creeping motion of viscous fluid in an approximate particle-in-cell model with Happel and Kuwabara conditions. The viscous flow with slip and Kuwabara boundary conditions has been concentrated on by Datta and Deo (2002), and they assessed the drag force over a rigid oblate spheroid in cell. Zholkovskiy et al. (2007) pointed out that if the particle volume
fraction is sufficiently low, the zero vorticity cell model performs better than
the free surface cell model does. However, if the volume fraction approaches
the value assigned to the close packing, the free surface cell model becomes
the better choice among other models including the zero vorticity cell model.
Vasin et al. (2008) used the cell method to model permeability of membrane
built up by solid spherical particles with a permeable porous shell and reviewed
all the four boundary conditions mentioned above. Deo and Gupta (2009) and
later, Deo (2009) extended the earlier work of Dassios et al. (1995) on a swarm
of solid spheroids to the porous spheroid swarms. The viscous motion over
an assemblage of permeable spherical particles by applying the Mehta-Morse
boundary condition on the cell surface was studied by Deo and Shukla (2009).
Keh and Keh (2010) have studied the slow motion of an assemblage of identi
cal porous spherical shells relative to a fluid using Happel and Kuwabara cell
models. Yadav et al. (2010) have contemplated viscous move through a swarm
of permeable spherical particles utilizing the stress jump condition at the liquid
permeable interface and considered every one of the four known limit condi
tions on the cell surface and looked at them.

Recently, Saad (2012b), Faltas and Saad (2012), and Faltas and Saad (2014)
have studied, respectively, the problems of Stokes flow past an assemblage of
axisymmetric porous spheroidal and eccentric spherical particles-in-cell mod-
els. Also, the issue of micropolar liquid past fluid droplets-in-cell models is ex-
plored scientifically by Saad (2012a). The quasi-steady axisymmetrical stream
of an incompressible viscous liquid past a gathering of permeable concentric
spherical shell-in-cell model is discussed considering all the four boundary con-
ditions by Saad (2013). Most recently, Datta and Raturi (2014) studied the
creeping flow through a swarm of spherical particles, where each particle con-
sists of a solid core covered by a liquid shell coated with monomolecular layer
of surfactant layer using the cell model technique and obtained an analytical
solution of the problem for four models discussed above.

This chapter concerns an analytical solution to the problem of flow of New-
tonian fluid through an assemblage of Reiner-Rivlin liquid droplets-in-cell with a brief survey of the four limit conditions on the fictitious boundary of the cell. Stokes equation is used to study the flow outside the liquid sphere, whereas inside the liquid sphere the flow field is determined by using the Stokes approximation and expanding the internal stream function in a power series of a parameter $S$ characterizing the cross-viscosity of the internal fluid. We have used the cell model technique to study the flow. In limiting cases, earlier results reported by Happel, Kuwabara, Kvashin and Cunningham/Mehta-Morse have been deduced. The drag coefficient and the normalized hydrodynamic drag force in each case are obtained and the effect of various parameters on these is presented in graphical form.

6.2 Mathematical Formulation of the Problem

![Diagram of the physical circumstance and the depiction of the direction framework for the cell model.]

**Figure 6.1:** The physical circumstance and the depiction of the direction framework for the cell model.

In the present model, we consider that all the Reiner-Rivlin liquid spherical droplets are randomly and homogeneously distributed in an incompressible viscous fluid of viscosity $\mu_1$ and have their axes along $z$-axis. The internal
fluid medium is also assumed to be homogeneous and isotropic. Further, we likewise accept that fluid spherical particles are stationary and an enduring axisymmetric stream has been set up around them by a uniform speed $U_z$ towards negative $z$-direction. The Reynolds number is thought to be adequately small so that the inertial terms in the force mathematical expression can be ignored in comparison with the viscous terms. We utilize a unit cell model in which each spherical molecule of radius $a$ is encompassed by concentric invented spherical envelope of suspending liquid having an external radius $b$ (see Fig. 6.1). Let $(r, \theta, \phi)$ denote the spherical coordinate system with the origin of coordinates at the center of the cell. The hypothetical cell’s radius is chosen to the point that the molecule/cell volume division (proportion) in the cell is equivalent to the molecule volume portion $\gamma$ all through the whole suspension, viz.,

$$\gamma = \frac{(4/3)\pi a^3}{(4/3)\pi b^3}. \quad (6.2.1)$$

We, now, assume that the flow outside and inside the liquid spherical particle to be Stokesian and Reiner-Rivlin flow field is obtained by expanding the stream function in a power of small parameter $S$. The external region $(a \leq R \leq b)$ and the internal region $(R \leq a)$ are denoted as the regions I and II, respectively. The parameters pertaining to the exterior and the interior of the liquid sphere to be distinguished, respectively, by the index in the superscripts under the bracket of an entity $\chi^{(i)}$, $i = 1, 2$. The equations of motion and continuity for external region are

$$\tilde{\nabla} \tilde{p}^{(1)} + \mu_1 \tilde{\nabla} \times \tilde{\nabla} \times \tilde{q}^{(1)} = 0 \quad (6.2.2)$$

$$\tilde{\nabla} \cdot \tilde{q}^{(1)} = 0 \quad (6.2.3)$$

where $\tilde{q}^{(1)}$ is the velocity, $\mu_1$ is the coefficient of viscosity and $\tilde{p}^{(1)}$ is the pressure. As the flow is axially symmetric and in the meridian plane, all the flow functions are independent of $\phi$. Thus, we choose both velocity fields $\tilde{q}^{(1)}$ and
\( \tilde{q}^{(2)} \), in spherical polar co-ordinates, in the forms

\[
\tilde{q}^{(i)} = \tilde{u}^{(i)}_R (R, \theta) \tilde{e}_R + \tilde{u}^{(i)}_\theta (R, \theta) \tilde{e}_\theta, \quad i = 1, 2. \tag{6.2.4}
\]

Rendering the problem in non-dimensional form by introducing the following dimensionless variables

\[
R = \ar, \quad \tilde{u}_R = U_z u_r, \quad \tilde{u}_\theta = U_z u_\theta, \quad \tilde{\tau}_{i j} = \mu \tilde{u}_R \tilde{d}_{i j}, \quad \tilde{\nabla} R = U_z \nabla_r, \quad \tilde{\nabla}_\theta = U_z \nabla_\theta, \quad \tilde{\nabla}_\phi = U_z \nabla_\phi
\tag{6.2.5}
\]

where \( U_z \) and \( a \) represent some typical velocity and length of the flow field, respectively. Therefore, the non-dimensional velocity components of the fluid away from the sphere in the directions of the unit vectors \((\tilde{e}_r, \tilde{e}_\theta, \tilde{e}_\phi)\) are

\[
V_r = -\cos \theta, \quad V_\theta = \sin \theta, \quad V_\phi = 0. \tag{6.2.6}
\]

Since, both the velocity fields satisfy the incompressibility condition \( \nabla \cdot q^{(i)} = 0, \quad i = 1, 2 \), we can write the velocity components in terms of Stokes’ stream function \( \psi^{(i)}(r, \theta), \quad i = 1, 2 \) through the relation given by the equation (1.4.56). We write the stream function and pressure in ascending powers of a parameter \( S \) for the internal flow within the liquid sphere as follows

\[
\psi^{(2)} = \psi_0 + \psi_1 S + \psi_2 S^2 + \ldots.,
\tag{6.2.7}
\]

\[
p^{(2)} = p_0 + p_1 S + p_2 S^2 + \ldots., \text{etc.}
\]

where \( S = \mu_e U / \mu_2 a \) is the sufficiently small number. The suffixes in (6.2.7) represent the zeroth, first, second and higher order approximations of the corresponding variables. Following Ramkissoon (1989b), the stream functions \( \psi_0, \quad \psi_1 \) and \( \psi_2 \) satisfy the following equations

\[
E^4 \psi_0 = 0, \quad E^4 \psi_1 = 8r \sin^2 \theta \cos \theta, \quad E^4 \psi_2 = \frac{32}{3} r^2 \sin^2 \theta. \tag{6.2.8}
\]

Particular solution of the equations mentioned in Eq. (6.2.8) are given by
\[
\psi_0 = (r^4 - r^2) \sin^2 \theta, \quad \psi_1 = \frac{2}{21} r^5 \sin^2 \theta \cos \theta, \quad \psi_2 = \frac{2}{63} r^6 \sin^2 \theta. \quad (6.2.9)
\]

After annihilating the pressure from Eq. (6.2.2) and making use of (6.2.4) and (1.4.56) in the resulting equation, we obtain a fourth-order linear partial differential equation for stream function \( \psi^{(1)} \)

\[
(E^4 \psi^{(1)}) = 0 \quad (6.2.10)
\]

where \( E^2 \) is the axisymmetric Stokesian operator

\[
E^2 = \frac{\partial^2}{\partial r^2} + \frac{1 - \zeta^2}{r^2} \frac{\partial^2}{\partial \zeta^2}. \quad (6.2.11)
\]

6.3 Boundary Conditions

To determine the arbitrary constants present in stream functions, the boundary conditions must be determined. Therefore, these conditions at the droplet surface are as follows:

**At the droplet surface** \( R = a \) (i.e. \( r = 1 \)):

\[
\begin{align*}
& \psi_1^{(1)} = 0, \quad \psi_1^{(2)} = 0 \\
& \psi_2^{(1)} = \psi_2^{(2)} \\
& \tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)}
\end{align*}
\]

(6.3.1)

(6.3.2)

(6.3.3)

where \( \tau_{r\theta}^{(1)} \) and \( \tau_{r\theta}^{(2)} \) are shear stresses of Newtonian and Reiner-Rivlin fluids, respectively.

**At the hypothetical surface** \( R = b \) (i.e. \( r = \ell^{-1} \)):

Key attention is more often paid to the boundary conditions on the outer cell surface. There are four frequently used versions of those boundary conditions, which are referred as the Happel, Kuwabara, Kvashnin and Cunningham Mod-
els (Mehta-Morse models) models. All four models assume the continuity of radial component of velocity on the fictitious surface of cell as follows:

\[ u_r^{(1)} = V_r \quad (6.3.4) \]

A second boundary condition is used in each of the mentioned models as follows:

- Happel model [Happel (1958)] assumes that the tangential viscous stress vanishes on the surface of cell
  \[ \tau_{r\theta}^{(1)} = 0 \quad (6.3.5) \]

- Kuwabara model [Kuwabara (1959)] assumes that the vorticity vanishes on the surface of cell i.e. the flow is assumed to be potential kind:
  \[ \nabla \times \mathbf{q} = 0 \quad (6.3.6) \]

- Kvashnin model [Kvashnin (1979)] assumes a symmetry condition at hypothetical cell surface
  \[ \frac{\partial u_\theta^{(1)}}{\partial r} = 0 \quad (6.3.7) \]

- Cunningham (1910), and afterwards Mehta and Morse (1975) assumes homogeneity at hypothetical cell surface
  \[ u_\theta^{(1)} = V_\theta \quad (6.3.8) \]

Which one of the four models mentioned above is the most appropriate one remains to be answered, because there have been reported no physical arguments in favour of any model neither in the literature nor in the previous published research work. However, in some limiting cases, all four discussed boundary conditions are satisfied. This is the reason why we consider and compare below all the four models.
6.4 Solution of the Problem and Application of Boundary Conditions

The regular solution to Eq. (6.2.10) for the motion of a Newtonian liquid over a Reiner-Rivlin droplet-in-cell is given by

\[
\psi^{(1)} = \left( a_2 \frac{r^2}{r} + b_2 + c_2 r^4 + d_2 r \right) G_2(\zeta), \quad 1 \leq r \leq \frac{1}{\ell} \quad (6.4.1)
\]

and the solution for the internal flow field within the Reiner-Rivlin liquid sphere can be expressed \cite{Ramkissoon1989b} as

\[
\psi^{(2)} = \psi_0 + \psi_1 S + \psi_2 S^2 + \sum_{n=2}^{\infty} \left( e_n r^n + f_n r^{n+2} \right) G_n(\zeta), \quad r \leq 1. \quad (6.4.2)
\]

With the aid of Eq. (6.2.9), we can now write Eq. (6.4.2) explicitly in the form:

\[
\psi^{(2)} = \left\{ (e_2 - 2) r^2 + (f_2 + 2) r^4 + \frac{4}{63} S^2 r^6 \right\} G_2(\zeta) + \left\{ e_3 r^3 + 5 \left( f_3 + \frac{4}{21} S \right) r^5 \right\} G_3(\zeta) + \sum_{n=4}^{\infty} \left( e_n r^n + f_n r^{n+2} \right) G_n(\zeta), \quad r \leq 1. \quad (6.4.3)
\]

where \( G_n(\zeta) \) are the Gegenbauer function defined in \cite{Abramowitz1970}, and connected to the Legendre polynomials \( P_n(\zeta) \) as given by the relation (1.4.81)

The unknown coefficients are to be evaluated by utilizing the limit conditions (6.3.1)–(6.3.8) lead to the followings:

**At the droplet surface** \( r = 1 \):

\[
\frac{\partial \psi^{(1)}}{\partial \theta} = 0, \quad \frac{\partial \psi^{(2)}}{\partial \theta} = 0 \quad (6.4.4)
\]

\[
\frac{\partial \psi^{(1)}}{\partial r} = \frac{\partial \psi^{(2)}}{\partial r} \quad (6.4.5)
\]

\[
\lambda \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \psi^{(1)}}{\partial r} \right) = \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial \psi^{(2)}}{\partial r} \right) \quad (6.4.6)
\]
On the surface $r = \ell^{-1}$ of cell:

$$\frac{\partial \psi^{(1)}}{\partial \theta} = r^2 \sin \theta \cos \theta \quad (6.4.7)$$

Happel boundary condition:

$$- \cot \theta \frac{\partial \psi^{(1)}}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \psi^{(1)}}{\partial \theta^2} - \frac{2}{r} \frac{\partial \psi^{(1)}}{\partial r} = 0 \quad (6.4.8)$$

Kuwabara boundary condition:

$$\frac{\cot \theta}{r^2} \frac{\partial \psi^{(1)}}{\partial \theta} - \frac{1}{r^2} \frac{\partial^2 \psi^{(1)}}{\partial \theta^2} - \frac{\partial^2 \psi^{(1)}}{\partial \theta^2} - \frac{\partial^2 \psi^{(1)}}{\partial r^2} = 0 \quad (6.4.9)$$

Kvashnin boundary condition:

$$\frac{1}{r} \frac{\partial \psi^{(1)}}{\partial r} - \frac{\partial^2 \psi^{(1)}}{\partial r^2} = 0 \quad (6.4.10)$$

Mehta-Morse (Cunningham) boundary condition:

$$\frac{\partial \psi^{(1)}}{\partial r} - r \sin^2 \theta = 0 \quad (6.4.11)$$

where $\ell = a/b$ and $\lambda = \mu_1/\mu_2$.

As a result of the substitution of the expressions (6.4.1) and (6.4.3) into the estimated boundary conditions (6.4.4)–(6.4.7) and one condition from (6.4.8)–(6.4.11) yields a system of algebraic equations to different models as follows:

**For Happel model:**

$$a_2 + b_2 + c_2 + d_2 = 0 \quad (6.4.12)$$

$$e_2 + f_2 + \frac{4}{63} S^2 = 0 \quad (6.4.13)$$

$$a_2 - 2b_2 - c_2 - 4d_2 + 2e_2 + 4f_2 + 4 + \frac{8}{21} S^2 = 0 \quad (6.4.14)$$

$$2\lambda a_2 - \lambda b_2 - \lambda c_2 + 2\lambda d_2 + e_2 - 2f_2 - 6 - \frac{4}{7} S^2 = 0 \quad (6.4.15)$$

$$a_2 + \gamma^{-1} b_2 + \gamma^{-2/3} c_2 + \gamma^{-5/3} d_2 - \gamma^{-1} = 0 \quad (6.4.16)$$
\[ a_2 - \gamma^{-5/3}d_2 = 0 \]  
\[ \text{(6.4.17)} \]

For Kuwabara model:

\[ a_2 + b_2 + c_2 + d_2 = 0 \]  
\[ \text{(6.4.18)} \]

\[ e_2 + f_2 + \frac{4}{63}S^2 = 0 \]  
\[ \text{(6.4.19)} \]

\[ a_2 - 2b_2 - c_2 - 4d_2 + 2e_2 + 4f_2 + 4 + \frac{8}{21}S^2 = 0 \]  
\[ \text{(6.4.20)} \]

\[ 2\lambda a_2 - \lambda b_2 - \lambda c_2 + 2\lambda d_2 + e_2 - 2f_2 - 6 - \frac{4}{7}S^2 = 0 \]  
\[ \text{(6.4.21)} \]

\[ a_2 + \gamma^{-1}b_2 + \gamma^{-2/3}c_2 + \gamma^{-5/3}d_2 - \gamma^{-1} = 0 \]  
\[ \text{(6.4.22)} \]

\[ c_2 - 5\gamma^{-1}d_2 = 0 \]  
\[ \text{(6.4.23)} \]

For Kvashnin model:

\[ a_2 + b_2 + c_2 + d_2 = 0 \]  
\[ \text{(6.4.24)} \]

\[ e_2 + f_2 + \frac{4}{63}S^2 = 0 \]  
\[ \text{(6.4.25)} \]

\[ a_2 - 2b_2 - c_2 - 4d_2 + 2e_2 + 4f_2 + 4 + \frac{8}{21}S^2 = 0 \]  
\[ \text{(6.4.26)} \]

\[ 2\lambda a_2 - \lambda b_2 - \lambda c_2 + 2\lambda d_2 + e_2 - 2f_2 - 6 - \frac{4}{7}S^2 = 0 \]  
\[ \text{(6.4.27)} \]

\[ a_2 + \gamma^{-1}b_2 + \gamma^{-2/3}c_2 + \gamma^{-5/3}d_2 - \gamma^{-1} = 0 \]  
\[ \text{(6.4.28)} \]

\[ 3a_2 - \gamma^{-2/3}c_2 + 8\gamma^{-5/3}d_2 = 0 \]  
\[ \text{(6.4.29)} \]

For Cunningum / Mehta-Morse model:

\[ a_2 + b_2 + c_2 + d_2 = 0 \]  
\[ \text{(6.4.30)} \]

\[ e_2 + f_2 + \frac{4}{63}S^2 = 0 \]  
\[ \text{(6.4.31)} \]

\[ a_2 - 2b_2 - c_2 - 4d_2 + 2e_2 + 4f_2 + 4 + \frac{8}{21}S^2 = 0 \]  
\[ \text{(6.4.32)} \]

\[ 2\lambda a_2 - \lambda b_2 - \lambda c_2 + 2\lambda d_2 + e_2 - 2f_2 - 6 - \frac{4}{7}S^2 = 0 \]  
\[ \text{(6.4.33)} \]
\[ a_2 + \gamma^{-1} b_2 + \gamma^{-2/3} c_2 + \gamma^{-5/3} d_2 - \gamma^{-1} = 0 \]  
(6.4.34)

\[ a_2 - 2 \gamma^{-1} b_2 - \gamma^{-2/3} c_2 - 4 \gamma^{-5/3} d_2 + 2 \gamma^{-1} = 0 \]  
(6.4.35)

where \( \gamma = \ell^3 \). The solution of these systems of Eqs. (6.4.12)-(6.4.35) for different models are obtained as

**For Happel model:**

\[ a_2 = - (189 + 32 S^2 - 32 S^2 \gamma^{1/3}) \Delta_1 \]  
(6.4.36)

\[ b_2 = -(378 + 32 S^2 \gamma^{1/3} - 64 S^2 \gamma - 567 \gamma^{5/3} + 32 S^2 \gamma^2 + 378 \lambda + \]
\[ + 378 \gamma^{5/3} \lambda) \Delta_1 \]  
(6.4.37)

\[ c_2 = -(567 - 32 S^2 + 64 S^2 \gamma + 378 \gamma^{5/3} - 32 S^2 \gamma^{5/3} - 378 \lambda - \]
\[ - 378 \gamma^{5/3} \lambda) \Delta_1 \]  
(6.4.38)

\[ d_2 = (189 - 32 S^2 + 32 S^2 \gamma^{1/3}) \Delta_1 \gamma^{5/3} \]  
(6.4.39)

\[ e_2 = \frac{2}{27} (27 + 2 S^2) + (189 + 32 S^2 - 32 S^2 \gamma^{1/3}) (\lambda + \gamma^{5/3} \lambda) \Delta_1 \]  
(6.4.40)

\[ f_2 = - \frac{2}{189} (189 + 20 S^2) + (189 + 32 S^2 - 32 S^2 \gamma^{1/3})(- \lambda - \]
\[ - \gamma^{5/3} \lambda) \Delta_1 \]  
(6.4.41)

where

\[ \frac{1}{\Delta_1} = 189(-1 + \gamma^{1/3})(2 - \gamma^{1/3} - \gamma^{2/3} + \gamma + \gamma^{4/3} - 2 \gamma^{5/3} + 2 \lambda + 2 \gamma^{5/3} \lambda). \]

**For Kuwabara model:**

\[ a_2 = - [-945 - 160 S^2 + 192 S^2 \gamma^{1/3} + 378 \gamma (1 - \lambda) - 32 S^2 \gamma] \Delta_1 \]  
(6.4.42)

\[ b_2 = - [-1890(1 + \lambda) - 192 S^2 \gamma^{1/3} - 945 \gamma + 32 S^2 \gamma(5 + \gamma)] \Delta_2 \]  
(6.4.43)

\[ c_2 = 5(-567 - 32 S^2 + 32 S^2 \gamma - 378 \lambda) \Delta_2 \]  
(6.4.44)

\[ d_2 = \gamma(-567 - 32 S^2 + 32 S^2 \gamma - 378 \lambda) \Delta_2 \]  
(6.4.45)
For Kvashnin model:

\[
e_2 = 2 + \frac{4S^2}{63} + \frac{3}{2(1+\gamma)} - (5 - 4\gamma^{1/3} - 4\gamma^{2/3} + \gamma + \gamma^{4/3} + \gamma^{5/3}) \times \\
(-567 - 32S^2 + 32S^2\gamma - 378\lambda)\Delta_2
\]

\[
f_2 = -2 - \frac{8S^2}{63} - \frac{3}{2(-1+\gamma)} - (-5 + 4\gamma^{1/3} + 4\gamma^{2/3} - \gamma - \\
\gamma^{4/3} - \gamma^{5/3})(-567 - 32S^2 + 32S^2\gamma - 378\lambda)\Delta_2
\]

where

\[
\frac{1}{\Delta_2} = 378(-1 + \gamma^{1/3})^2(5 + \gamma^{1/3} - 3\gamma^{2/3} - 2\gamma - \gamma^{4/3} + 5\lambda + 4\gamma^{1/3}\lambda + \\
3\gamma^{2/3}\lambda + 2\gamma\lambda + \gamma^{4/3}\lambda).
\]

For Kvashnin model:

\[
a_2 = 2(756 + 128S^2 - 144S^2\gamma^{1/3} - 189\gamma + 16S^2\gamma + 189\gamma\lambda)\Delta_3
\]

\[
b_2 = -(3024 - 288S^2\gamma^{1/3} - 945\gamma + 160S^2\gamma - 1701\gamma^{5/3} + \\
128S^2\gamma^2 - 3024\lambda + 1134\gamma^{5/3}\lambda)\Delta_3
\]

\[
c_2 = 2[-2268 - 128S^2 + 80S^2\gamma - 567\gamma^{5/3}(1 - \lambda) + 48S^2\gamma^{5/3} - \\
1512\lambda]\Delta_3
\]

\[
d_2 = -[567(\gamma + \gamma^{5/3}) + 32S^2(\gamma + 3\gamma^{5/3}) - 128S^2\gamma^2 + 378\gamma\lambda]\Delta_3
\]

\[
e_2 = 2 + \frac{4S^2}{63} + \frac{8(-\frac{1}{2} + \frac{\gamma}{8})}{8 - 9\gamma^{1/3} + \gamma} - (16 - 11\gamma^{1/3} - 11\gamma^{2/3} - \gamma - \gamma^{4/3} + \\
8\gamma^{5/3})(756 + 16S^2(8 - 9\gamma^{1/3} + \gamma) - 189\gamma(1 - \lambda)]\Delta_3
\]

\[
f_2 = -2 - \frac{8S^2}{63} + \frac{8(\frac{1}{2} - \frac{\gamma}{8})}{8 - 9\gamma^{1/3} + \gamma} - [-16 + 11(\gamma^{1/3} + \gamma^{2/3}) + \gamma + \gamma^{4/3} - \\
8\gamma^{5/3})(756 + 16S^2(8 - 9\gamma^{1/3} + \gamma) - 189\gamma(1 - \lambda)]\Delta_3
\]
where
\[
\frac{1}{\Delta_3} = 189(-1 + \gamma^{1/3})^2(16 + 5\gamma^{1/3} \gamma - 6\gamma^{2/3} - 7\gamma - 8\gamma^{4/3} + 16\lambda + 14\gamma^{1/3} \lambda + 12\gamma^{2/3} \lambda + 10\gamma \lambda + 8\gamma^{4/3} \lambda).
\]

For Cunningham/Mehta-Morse model:

\[
a_2 = -[32S^2(2 - 3\gamma^{1/3} + \gamma) + 378(1 + \gamma(-1 + \lambda))]\Delta_4 \tag{6.4.55}
\]
\[
b_2 = -(756 + 965^2\gamma^{1/3} + 945\gamma - 160S^2\gamma - 1701\gamma^{5/3} + 64S^2\gamma^2 + 756\lambda + 1134\gamma^{5/3}\lambda)\Delta_4 \tag{6.4.56}
\]
\[
c_2 = 2[567(1 - \gamma^{5/3} + \gamma^{5/3}\lambda) + 16S^2(2 - 5\gamma + 3\gamma^{5/3}) + 378\lambda]\Delta_4 \tag{6.4.57}
\]
\[
d_2 = -[-567(\gamma - \gamma^{5/3}) - 32S^2\gamma(1 - 3\gamma^{2/3} + 2\gamma) - 378\gamma\lambda)\Delta_4 \tag{6.4.58}
\]
\[
e_2 = 4S^2\{42\gamma^{2/3} + \gamma^{1/3}(7 - 6\lambda) + 4\gamma^{4/3}(-7 + 3\lambda) - 4(7 + 3\lambda) + \gamma(7 + 6\lambda)\} + 189\{-8 + 2\gamma^{1/3} + 8\gamma^{4/3}(-1 + \lambda) - 6\lambda + 6\gamma^{2/3} \times (2 + \lambda) + \gamma(2 + 7\lambda)\}]\Delta_4 \tag{6.4.59}
\]
\[
f_2 = -[8S^2\{30\gamma^{2/3} + \gamma^{1/3}(5 - 6\lambda) + 4\gamma^{4/3}(-5 + 3\lambda) - 4(5 + 3\lambda) + \gamma(5 + 6\lambda)\} + 189\{-8 + 2\gamma^{1/3} + 8\gamma^{4/3}(-1 + \lambda) - 6\lambda + 6\gamma^{2/3} \times (2 + \lambda) + \gamma(2 + 7\lambda)\}]\Delta_4 \tag{6.4.60}
\]

where
\[
\frac{1}{\Delta_4} = 189(-1 + \gamma^{1/3})^3[4\gamma(-1 + \lambda) + 4(1 + \lambda) + \gamma^{2/3}(-3 + 6\lambda) + \gamma^{1/3}(3 + 6\lambda)].
\]

Also
\[
e_3 = 0, \quad f_3 = \frac{4}{21}S, \quad e_n = f_n = 0, \quad \text{for } n \geq 4
\]

for all the four models.
6.5 Evaluation of Drag Force

Drag on the sphere is the force exerted on it by the moving fluid. It is evaluated by integrating the components of stress perpendicular and shear to sphere’s surface by using the following formula [Ramkissoon and Majumadar (1976)]

\[
F_z = 2\pi a^2 \int_0^\pi r^2 (\tau_{rr} \cos \theta - \tau_{r\theta} \sin \theta) |_{r=1} | \sin \theta d\theta = 4\pi a \mu_1 U_z d_2.
\] (6.5.1)

For comparison point of view, we present here some exact solutions. In case where the cell is absent, when \( b \to \infty \) (\( \ell = 0 \), i.e., \( \gamma = 0 \)), so that the fluid is in infinite expanse. The drag force applied on a spherical droplet moving in a viscous liquid is acquired as

\[
F_{z\infty} = -\frac{2a\pi U_z}{3(1+\lambda)} \left(9 + \frac{32}{63}S^2 + 6\lambda\right) \mu_1
\] (6.5.2)

6.6 Limiting Cases and Some Known Results

(a). For a Newtonian liquid \((S \to 0)\) sphere of radius \( a \) in a cell of radius \( b \), we have the following expressions for the hydrodynamic drag force

For Happel model:

\[
F_z = \frac{4a\pi U_z[3+2\gamma^{5/3}(-1+\lambda)+2\lambda]\mu_1}{(-1+\gamma^{1/3})[-\gamma^{1/3}+\gamma^{2/3}+\gamma+\gamma^{1/3}+2\gamma^{5/3}(-1+\lambda)+2(1+\lambda)]}
\] (6.6.1)

For Kuwabara model:

\[
F_z = -\frac{10a\pi U(3+2\lambda)\mu_1}{5\gamma+\gamma^2(-1+\lambda)+5(1+\lambda)-3\gamma^{1/3}(3+2\lambda)}
\] (6.6.2)
For Kvashnin model:

\[ F_z = \frac{8a\pi U[3\gamma^{5/3}(-1+\lambda)-4(3+2\lambda)]\mu_1}{10\gamma+\gamma^{5/3}(9-6\lambda)+8\gamma^2(-1+\lambda)+16(1+\lambda)-9\gamma^{1/3}(3+2\lambda)} \] (6.6.3)

For Mehta-Morse /Cunningham model:

\[ F_z = \frac{8a\pi U[3+3\gamma^{5/3}(-1+\lambda)+2\lambda]\mu_1}{(-1+\gamma^{1/3})^3[4\gamma(-1+\lambda)+4(1+\lambda)+\gamma^{2/3}(-3+6\lambda)+\gamma^{4/3}(3+6\lambda)]} \] (6.6.4)

these are the new results reported here for the viscous fluid past a Newtonian liquid droplet-in-cell.

(b). When \( \lambda \to 0 \), we recover the noted outcome for the drag force on a solid sphere of radius \( a \) in a cell of radius \( b \), and we have the following expressions for hydrodynamic drag force

For Happel model:

\[ F_z = \frac{4a\pi U(-3+2\gamma^{5/3})\mu_1}{(-1+\gamma^{1/3})^2(2+\gamma^{1/3}+\gamma+2\gamma^{4/3})} \] (6.6.5)

A well known result previously obtained by Happel and Brenner (1983).

For Kuwabara model:

\[ F_z = \frac{30a\pi U\mu_1}{-5+9\gamma^{1/3}-5\gamma+\gamma^2} \] (6.6.6)

a noted result earlier obtained by Kuwabara (1959).

For Kvashnin model:

\[ F_z = \frac{24a\pi U(4+\gamma^{5/3})\mu_1}{-16+27\gamma^{1/3}-10\gamma-9\gamma^{5/3}+8\gamma^2} \] (6.6.7)

a famous result of Kvashnin (1979).

For Mehta-Morse /Cunningham model:

\[ F_z = \frac{24a\pi U(1+\gamma^{1/3}+\gamma^{2/3}+\gamma+\gamma^{4/3})\mu_1}{(-1+\gamma^{1/3})^3(4+7\gamma^{1/3}+4\gamma^{2/3})} \] (6.6.8)
a result earlier obtained by Mehta and Morse (1975).

(c). When $\gamma \to 0$, we recover the noted result of drag in case of rigid body immersed in an infinite expanse of fluid. The expression of hydrodynamic drag force for all the four models comes out as

$$F_z = -6a \pi U_z \mu_1$$

(6.6.9)

a well known result earlier reported by Stokes for the drag experienced by a solid sphere in an unbounded medium.

The unit cell model technique is also applicable to the case of a fluid flow through a bed of particles. For the model under thought, the drag force $F_z$ divided by the cell volume $(4/3)\pi b^3$ will meet $-\Delta P/L$, the pressure drop per unit length of bed because of entry of liquid through it. Utilizing this connection with that given by Eq. (6.5.1) gives the superficial liquid speed through the bed as

$$U_z = \left(\frac{-a^2}{3\gamma d_2}\right) \frac{\Delta P}{\mu_1 L}$$

(6.6.10)

**Drag coefficient $C_D$:**

The drag coefficient can be defined as

$$C_D = \frac{F}{\frac{1}{2} \rho U_z^2 \pi a^2} = \frac{16 d_2}{Re}$$

(6.6.11)

where $Re = 2aU_z/\nu$ and $\nu = \mu_1/\rho$, respectively.

**Normalized hydrodynamic drag force $W_c$:**

The normalized hydrodynamic drag force $W_c$ is defined as the ratio of the actual hydrodynamic drag force acting on the droplet by the external fluid in enclosure to the drag experienced by a spherical droplet in an infinite expanse of viscous
fluid. With the aid of the Eqs. (6.5.1) and (6.5.2) this simplifies to

\[ W_c = \frac{F_z}{F_{z\infty}} = -\frac{6(1 + \lambda)d_2}{(9 + \frac{32}{63}S^2 + 6\lambda)}. \] (6.6.12)

We note that, \( W_c \to 1 \) as \( \gamma \to 0 \) (the cell surface is infinitely far away from the liquid particle) for any assigned estimations of \( \lambda \) and \( S \).

In the restricting situation when \( S = 0 \), the explanatory solutions depicting the moderate movement of a gooey fluid sphere situated at the focal point of a spherical cell containing Newtonian liquid, the accurate solution of its normalized drag is found expressly as follows:

**For the Happel model:**

\[ W_c = -\frac{378(1 + \lambda)[3 + 2\gamma^{5/3}(-1 + \lambda) + 2\lambda]}{(-1 + \gamma^{1/3})(567 + 378\lambda)} \left\{ -\gamma^{1/3} - \gamma^{2/3} + \gamma + \gamma^{4/3} + +2\gamma^{5/3}(-1 + \lambda) + 2(1 + \lambda) \right\} \] (6.6.13)

**For the Kuwabara model:**

\[ W_c = \frac{945(1 + \lambda)(3 + 2\lambda)}{(567 + 378\lambda)} \frac{5\gamma + \gamma^2(-1 + \lambda) + 5(1 + \lambda) - 3\gamma^{1/3}(3 + 2\lambda)}{5\gamma + \gamma^2(-1 + \lambda) + 5(1 + \lambda) - 3\gamma^{1/3}(3 + 2\lambda)} \] (6.6.14)

**For the Kvashnin model:**

\[ W_c = -\frac{756(1 + \lambda)[3\gamma^{5/3}(-1 + \lambda) - 4(3 + 2\lambda)]}{(-1 + \gamma^{1/3})^2(567 + 378\lambda)} \left\{ 8\gamma^{4/3}(-1 + \lambda) + 16(1 + \lambda) + +6\gamma^{2/3}(-1 + 2\lambda) + \gamma(-7 + +10\lambda) + +\gamma^{1/3}(5 + 14\lambda) \right\} \] (6.6.15)
For the Mehta-Morse /Cunningham model:

\[
W_c = -\frac{756(1 + \lambda)[3 + 3\gamma^{5/3}(-1 + \lambda) + 2\lambda]}{(-1 + \gamma^{1/3})^3(567 + 378\lambda)} \left\{ \frac{4\gamma(-1 + \lambda) + 4(1 + \lambda) + \gamma^{2/3} \times (-3 + 6\lambda) + \gamma^{1/3}(3 + 6\lambda)}{\left(-1 + \gamma^{1/3}\right)^3(567 + 378\lambda)} \right\}
\]

(6.6.16)

these are the new results of the normalized hydrodynamic drag force \(W_c\) in case of creeping motion of a liquid spherical body situated at the center of a spherical cell containing viscous fluid of different viscosity.

The drag coefficient \(C_D\) and the hydrodynamic drag force \(W_c\) acting on the spherical droplet-in-cell as well as the velocity profile and the pressure allotment outside of the liquid sphere for all the four models are depicted in Figs. 6.2–6.9 and Table 6.1 at low Reynolds number \(Re\) and \(\theta = \pi/4\), and for the numerous estimations of the accompanying parameters:

- The volume fraction of the particle \(\gamma\) (0 < \(\gamma\) < 1).
- The cross-viscosity parameter \(S\).
- The relative viscosity \(\lambda\).

The Figs. 6.2–6.4 correspond to the drag coefficient results for a viscous fluid past a Reiner-Rivlin liquid droplet-in-cell. The deviation in \(C_D\) for the motion of a suspension of Reiner-Rivlin liquid spheres (for \(\lambda = 0\), \(\lambda = 1\) and \(\lambda \to \infty\)) for all four cell models is depicted in Fig. 6.2. Drag coefficient increases monotonically with an increase in \(\gamma\) for any specified finite values of \(Re\) and \(S\) and all four models agree for low volume fraction. The curves with \(\lambda = 0\) represent the result for solid sphere-in-cell, the curves with \(\lambda = 1\) show the results for liquid sphere in-cell when inner and outer fluids have equal viscosities, whereas the curves with \(\lambda \to \infty\) denote the results for spherical bubbles-in-cell. It is observed that \(C_D\) increases with increasing solid volume fraction \(\gamma\) and decreasing viscosity ratio \(\lambda\). It is noted that for the case of smaller \(\gamma(\gamma < 0.001)\), the growth
rate of $C_D$ is very slow. However, for relatively high values of particle volume $\gamma (\gamma \geq 0.001)$, a significant increase in $C_D$ is observed for all four models. From the Fig.6.2, it is also observed that a solid sphere-in-cell experiences greater drag coefficient and spherical bubbles experiences less drag. When neighboring liquid particles are sufficiently close to one another, a large pressure gradient is developed in between them that causes more drag coefficient. Whereas for small $\gamma$, the flow around a liquid sphere is not very much influenced by neighboring particles. However, $C_D$ is slightly higher for the Cunningham/Mehta-Morse cell model when compared with the other models.

The effect of viscosity ratio on drag coefficient is depicted in Fig.6.3 for all four models at $S = 0.5$ and for very small Reynolds number $Re = 0.002$. It can be observed from the maps of Fig.6.3 that the drag decreases as viscosity ratio increases. In the beginning, drag coefficient $C_D$ is almost constant ($\lambda < 0.5$) for all the four models and then decreases very sharply ($0.5 \leq \lambda$) and then again becomes almost constant for larger values of $\lambda$. This variation in the drag coefficient is reported to have been decreased with almost equal amount for Kuwabara’s and Kvashnin’s models as compared with remaining two models.
Fig. 6.3: Variation of $C_D$ against $\lambda$ at $\gamma=0.3$, $S=0.5$ and $Re=0.2 \times 10^{-3}$.

Fig. 6.4: Variation of $C_D$ versus $S$ at $\gamma=0.2$, $\lambda=0.01$ and $Re=0.25 \times 10^{-3}$.

Fig 6.4 depicts the variation in $C_D$ with regard to the parameter $S$ for different models. It can be observed from the maps of the Fig. 6.4 that the drag on liquid sphere is constant for very small values of $S$ varies from 0 to 1 for all the four models and starts increasing from the values of ($S > 0.2$). Physically, it implies that drag coefficient on liquid sphere is not very much influenced by very small values of cross-viscosity in-cell. The calculated drag coef-
icient increases in the following direction from the lowest for Happel’s model, to Kvashnin’s, Kuwabara’s and reaches the highest possible value for Mehta-Morse/Cunningham’s model.

Figure 6.5: Radial velocity distributions for different values of $\lambda$ with $S = 0.5$, $\theta = \pi/4$ and $\gamma = 0.05$.

Figure 6.6: Tangential velocity distributions for different values of $\lambda$ with $S = 0.5$, $\theta = \pi/4$ and $\gamma = 0.05$. 
The velocity and pressure profiles against radius of vector $r$ are depicted in Figs. 6.5–6.7. It can be observed from these figures that in the given range of the values of radius of vector (1 to 2), the radial velocity decreases and tangential velocity increases as the value of $\lambda$ increases. However, velocity profile is slightly lower for radial velocity and slightly higher for tangential velocity for the Cunningham/Mehta-Morse cell model when compared with the other models. Also, the pressure distribution over the surface is the lowest and increases as the radial distance goes beyond the surface.

![Figure 6.7: Pressure distributions for different values of $\lambda$ with $S = 0.5$, $\theta = \pi/4$ and $\gamma = 0.05$.](image)

The normalized hydrodynamic force $W_c$ for the motion of a non-Newtonian Reiner-Rivlin liquid sphere-in-cell given by Eqs. (6.6.13)-(6.6.16) for all the four models are plotted in Figs. 6.8–6.9. It can be observed from Fig. 6.8 that over the entire range of the volume fraction, the normalized hydrodynamic force $W_c$ for each case is a monotonically increasing function of $\gamma$ for all four models with the decreasing $\lambda$ and approaches to infinity when the surface of the spherical droplet touches the cell surface, i.e., $\gamma = 1$ for any given fixed values of $\lambda$ and $S$. Here, it may be noted that the normalized hydrodynamic drag force on the liquid sphere of the same equatorial radius is less than that of a solid
Figure 6.8: Variations of the normalized hydrodynamic drag force against the particle volume fraction for different values of viscosities ratio $\lambda$ with $S = 0.5$ and $Re = 0.2 \times 10^{-3}$: (a) Hap-pel cell model calculations (b) Kvashnin cell model calculations (c) Kuwabara cell model calculations (d) Cunningham/Mehta–Morse cell model calculations.

From Fig. 6.9, it is obvious that the hydrodynamic drag force acting on the solid spherical particle or on the spherical gas bubble tends to a constant value for different values of $\gamma$, for any specified value of $S = 0.5$ and $Re = 0.2 \times 10^{-3}$ and the normalized drag decreases with decreasing particle volume fraction $\gamma$. As $\lambda$ decreases, in general, the drag force acting on the viscous liquid sphere increases, and decreases for a non-Newtonian liquid sphere, with keeping the other parameters unchanged for all four cell models. The values of $W_c$ are listed...
Figure 6.9: Variations of the normalized hydrodynamic drag force w.r.t. viscosities ratio $\lambda$ for different values of the particle volume fraction $\gamma$ with $S = 0.5$ and $Re = 0.2 \times 10^{-3}$: (a) Happel cell model calculations; (b) Kvashnin cell model calculations (c) Kuwabara cell model calculations (d) Cunningham/Mehta–Morse cell model calculations.

in Table 6.1 for different values $\lambda$ and $S$ with various values of $\gamma$ for all the four models. We observe the value of normalized drag force is a decreasing function of the parameter $S$ for a given value $\gamma$ and is an increasing function of the volume fraction for given values of $S$ for all four models. In addition, this drag force decreases with increasing $S$, for any fixed values of $\gamma$ and $\lambda$ and then reaches to a constant value for a solid particle in cell. It is interesting to note that when the radii of the droplet and cell are not too close, (say, $\gamma \leq 0.1$), the drag force act-
Table 6.1: Normalized hydrodynamic drag force of the droplet in a cell for numerous estimations of $\gamma$ and $\lambda$ with $S = 0.5$ and $Re = 0.2 \times 10^{-3}$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$W_c$</th>
<th>Proposed model</th>
<th>Saad's model (2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda \to \infty$</td>
<td>$\lambda = 1$</td>
<td>$\lambda = 0$</td>
</tr>
<tr>
<td></td>
<td>$S = 0$</td>
<td>$S = 0.5$</td>
<td>Micropolar liquid sphere</td>
</tr>
</tbody>
</table>

**Happel cell model**

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$W_c$</th>
<th>Proposed model</th>
<th>Saad's model (2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda \to \infty$</td>
<td>$\lambda = 1$</td>
<td>$\lambda = 0$</td>
</tr>
<tr>
<td></td>
<td>$S = 0$</td>
<td>$S = 0.5$</td>
<td>Micropolar liquid sphere</td>
</tr>
</tbody>
</table>

**Kvashnin cell model**

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$W_c$</th>
<th>Proposed model</th>
<th>Saad's model (2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda \to \infty$</td>
<td>$\lambda = 1$</td>
<td>$\lambda = 0$</td>
</tr>
<tr>
<td></td>
<td>$S = 0$</td>
<td>$S = 0.5$</td>
<td>Micropolar liquid sphere</td>
</tr>
</tbody>
</table>

**Kuwabara cell model**

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$W_c$</th>
<th>Proposed model</th>
<th>Saad's model (2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda \to \infty$</td>
<td>$\lambda = 1$</td>
<td>$\lambda = 0$</td>
</tr>
<tr>
<td></td>
<td>$S = 0$</td>
<td>$S = 0.5$</td>
<td>Micropolar liquid sphere</td>
</tr>
</tbody>
</table>

**Mehta-Morse/Cunningham cell model**

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$W_c$</th>
<th>Proposed model</th>
<th>Saad's model (2012)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda \to \infty$</td>
<td>$\lambda = 1$</td>
<td>$\lambda = 0$</td>
</tr>
<tr>
<td></td>
<td>$S = 0$</td>
<td>$S = 0.5$</td>
<td>Micropolar liquid sphere</td>
</tr>
</tbody>
</table>

$W_c = \frac{R}{\lambda} \left( \frac{S}{\infty} - \frac{S}{\lambda} \right)$
ing on Reiner-Rivlin droplet is almost constant with respect to \( \lambda \). This drag reduction occurs since the droplet surface becomes more slippery, with more fluid momentum being transferred in the tangential direction of the surface, leading to less momentum transfer in the normal direction of the droplet [Keh and Lee (2010)]. On the other hand, the calculated normalized drag increases in the following direction from the lowest for Happel model, to Kvashnin, Kuwabara and reaches the highest possible value for Cunningham/Mehta-Morse model. However, \( W_c \) is slightly higher for the Cunningham/Mehta-Morse cell model when compared with the other models.

It is also observed that as \( \gamma > 0.1 \), the normalized drag acting on a Reiner-Rivlin droplet-in-cell, for given values of \( S \), first decreases with an increase in \( \lambda \) from \( \lambda = 0 \), reaches to a minimum at some finite value of \( \lambda \) and then increases and becomes almost constant with an increase in \( \lambda \) to the limit \( \lambda \to \infty \) for all the four models. Also, the proposed model presents numerical values (see Table 6.1) with the model given by Saad (2012a) to the case when a viscous fluid past a micropolar droplet-in-cell. In Saad’s model the normalized hydrodynamic drag force \( W_c \) increases with the increasing viscosity ratio \( \lambda \), whereas in the present proposed model \( W_c \) is decreasing with the increasing \( \lambda \) which is clearly the effect of cross-viscosity, i.e., \( S \) of the Reiner-Rivlin fluid which has a significant impact in reducing the drag force in-cell. Also, the proposed model provides the minimum drag force for Reiner-Rivlin liquid droplet-in-cell as compared the case of a viscous droplet and micropolar droplet [Saad (2012a)] in-cell for all the four models.

### 6.7 Conclusion

In this chapter, the cell model technique is used to solve the problem of creeping motion of Newtonian liquid over Reiner-Rivlin liquid droplet-in-cell. The hydrodynamic drag force acting on the droplet-in-cell as a function of the relative viscosity \( \lambda \), cross-viscosity, i.e. \( S \), and the volume fraction of the par-
article $\gamma$ is obtained in closed forms for all the four models. It is found that the normalized hydrodynamic drag, all in all, is a diminishing function of the relative viscosity and cross-viscosity. For given an estimation of $S$ and $\lambda$, of course, the normalized drag force is a monotonic increasing function of $\gamma$ for all cases and becomes infinite in the touching limit. Further, the numerical results obtained here clearly show that the normalized hydrodynamic drag $W_c$ is less for the case of Reiner–Rivlin droplet-in-cell as compared with Newtonian droplet-in-cell for a given value of $\lambda$ and also observed that the $W_c$ is higher for micropolar droplet-in-cell [Saad (2012a)] when compared with viscous and Reiner–Rivlin droplet-in-cell. The calculated numerical values and the corresponding plots of $W_c$ increases in the following direction from the lowest for Happel model, to Kvashnin, Kuwabara and reaches the highest possible value for Cunningham/Mehta-Morse model. The Mehta–Morse/Cunningham boundary condition shows a weak depiction of the influence of particles in the vicinity on the flow in a preferred cell and, hence, cannot be suggested for applications. Also, the proposed model provides the minimum drag force for Reiner-Rivlin liquid droplet-in-cell as compared the case of a viscous droplet and micropolar droplet [Saad (2012a)] in-cell for all the four models.