CHAPTER VI

I-CONVERENT SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

6.1 INTRODUCTION

The notion of $I$-convergence was studied at the initial stage Kostyrko, Šalát and Wilczyński [69]. Later on it was studied by Šalát, Tripathy and Ziman [109], Demirci [28] and others.

An Orlicz function is a function $M: [0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

If the convexity of Orlicz function $M$ is replaced by

$$M(x + y) \leq M(x) + M(y),$$

then this function is called modulus function. The notion was introduced by Nakano [94] and Ruckle [107] further studied with application to sequence space by Maddox [83] and many others.

**Remark 6.1.1.** It is well known if $M$ is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$, for all $\lambda$ with $0 < \lambda < 1$.

6.2 DEFINITIONS AND NOTATIONS

A sequence space $E$ is said to be solid (or normal) if $(\alpha_k, x_k) \in E$, whenever $(x_k) \in E$ and for all sequence $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$, for all $k \in N$.

A sequence space $E$ is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi$ is a permutation of $N$. 
A sequence space $E$ is said to be a sequence algebra if $(x_k)(y_k) = (x_k \ast y_k) \in E$, whenever $(x_k), (y_k) \in E$.

A sequence space $E$ is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k \to 0$ implies $y_k \to 0$.

Let $K = \{k_1 < k_2 < \cdots \subseteq N$ and $E$ be a sequence space. A $K$-step space of $E$ is a sequence space $\lambda^E_K = \{(x_{k_n}) \in w : (k_n) \in E \}.$

A canonical preimage of a sequence $\{x_{k_n}\} \in \lambda^E_K$ is a sequence $\{y_k\} \in w$ defined as

$$y_k = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise}. \end{cases}$$

A canonical preimage of a step space $\lambda^E_K$ is a set of canonical preimages of all elements in $\lambda^E_K$, i.e., $y$ is in canonical preimage of $\lambda^E_K$ if and only if $y$ is canonical preimage of some $x \in \lambda^E_K$.

A sequence space $E$ is said to be monotone if it contains the canonical preimages of its step spaces.

In this chapter we introduce the following classes of sequence:

$$c'(M) = \left\{(x_k) \in w : I - \lim M \left( \frac{|x_k - L|}{\rho} \right) = 0, \text{for some } L \text{ and } \rho > 0 \right\};$$

$$c'_0(M) = \left\{(x_k) \in w : I - \lim M \left( \frac{|x_k|}{\rho} \right) = 0, \text{for some } \rho > 0 \right\};$$

$$\ell_\infty(M) = \left\{(x_k) \in w : \sup_k M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{for some } \rho > 0 \right\}.$$ 

Also we write $m'_M = c'(M) \cap \ell_\infty(M)$ and $m'_0(M) = c'_0(M) \cap \ell_\infty(M)$. 
**Lemma 6.2.1** (Kamthan and Gupta [62], page 53). A sequence space $E$ is solid implies $E$ is monotone.

**Lemma 6.2.2** (Šalát, Tripathy and Ziman [110], Lemma 2.5). Let $K \in \mathcal{S}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

**Lemma 6.2.3** (Kostyrko, Šalát and Wilczyński [69], Lemma 5.1). If $I \subseteq 2^N$ is a maximal admissible ideal, then for each $A \subseteq N$ we have $A \in I$, or $N-A \in I$.

### 6.3 MAIN RESULTS

**Theorem 6.3.1.** For any Orlicz function $M$, the classes of sequences $c^1(M), c^0(M), m^1(M)$ and $m^0(M)$ are linear spaces.

**Theorem 6.3.2.** The spaces $m^0(M)$ and $m^1(M)$ are Banach spaces normed by

$$
\| (x_k) \| = \inf \left\{ \rho > 0 : \sup_k M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
$$

**Theorem 6.3.3.** Let $M_1, M_2$ be Orlicz functions those satisfy $\Delta_2$-condition. Then

(i) $W(M_2) \subseteq W(M_1 \ast M_2)$;

(ii) $W(M_1) \cap W(M_2) \subseteq W(M_1 + M_2)$, for $W = c^1, c^0, m^1, m^0$.

**Corollary 6.3.4.** $W \subseteq W(M)$, where $W = c^1, c^0, m^1, m^0$.

**Result 6.3.5.** The spaces $c^0(M)$ and $m^0(M)$ are solid and monotone.

**Result 6.3.6.** The space $c^1(M)$ and $m^1(M)$ are neither monotone nor solid in general.
RESULT 6.3.7. The spaces \( c^1(M) \) and \( c'^0(M) \) are not convergence free in general.

RESULT 6.3.8. The spaces \( c^1(M) \) and \( c'^0(M) \) are sequence algebra.

THEOREM 6.3.9. Let \( M \) be Orlicz function. Then \( c'^0(M) \subset c^1(M) \subset \ell_+^i(M) \) and the inclusion is proper.

6.4 THE PROOF OF THE RESULTS OF 6.3.

PROOF OF THE THEOREM 6.3.1. We shall prove the result only for \( c^1(M) \). The others can be treated similarly.

Let \( (x_k), (y_k) \in c^1(M) \) and \( \alpha, \beta \) be scalars. Then there exist positive numbers \( \rho_1 \) and \( \rho_2 \) such that

\[
\begin{align*}
I\text{-}lim M\left( \frac{|x_k - L_1|}{\rho_1} \right) &= 0, \text{ for some } L_1 \in C, \\
I\text{-}lim M\left( \frac{|y_k - L_2|}{\rho_2} \right) &= 0, \text{ for some } L_2 \in C.
\end{align*}
\]

Let \( \rho_3 = \max \{2|\alpha| \rho_1, 2|\beta| \rho_2 \} \)

Since \( M \) is non-decreasing and convex function, we have

\[
M\left( \frac{\alpha x_k + \beta y_k - (\alpha L_1 + \beta L_2)}{\rho_3} \right) \\
\leq M\left( \frac{\alpha |x_k - L_1|}{\rho_3} + \frac{\beta |y_k - L_2|}{\rho_3} \right) \\
\leq M\left( \frac{|x_k - L_1|}{\rho_1} \right) + M\left( \frac{|y_k - L_2|}{\rho_2} \right).
\]
Therefore

\[ I\lim M \left( \frac{|αx_k + βy_k|}{ρ_3} - (αL_1 + βL_2) \right) \leq I\lim M \left( \frac{|x_k - L_1|}{ρ_1} \right) + I\lim M \left( \frac{|y_k - L_2|}{ρ_2} \right) \]

= 0.

Therefore (αx_k + βy_k) ∈ c^1(M).

Hence c^1(M) is a linear space.

**PROOF OF THE THEOREM 6.3.2.** The proof of this result is easy, so omitted.

**PROOF OF THE THEOREM 6.3.3.** (i) Let (x_k) ∈ c^1(M). Then there exists ρ>0 such that

\[ I\lim \rho \left( \frac{|x_k|}{ρ} \right) = 0. \quad (6.4.1) \]

Let ε>0 and choose δ with 0<δ<1 such that M(t) < ε for 0 ≤ t ≤ δ. Write

\[ y_k = M_2 \left( \frac{|x_k|}{ρ} \right) \]

and consider

\[ \lim_{k \to N} M_1(y_k) = \lim_{k \to N} M_1(y_k) + \lim_{k \to N} M_1(y_k). \]

By the Remark 6.1.1, we have

\[ \lim_{k \to N} M_1(y_k) \leq M_1(2 \lim_{k \to N} (y_k)). \quad (6.4.2) \]

For y_k > δ, we have

\[ y_k < \frac{y_k}{δ} < 1 + \frac{y_k}{δ}. \]

Since M_1 is non-decreasing and convex, it follows that
\[ M_i \left( y_k \right) < M_i \left( 1 + \frac{y_k}{\delta} \right) < \frac{1}{2} M_i(2) + \frac{1}{2} M_i \left( \frac{2y_k}{\delta} \right). \]

Since \( M_i \) satisfies \( \Delta_2 \)-condition, we have
\[ M_i \left( y_k \right) < \frac{1}{2} K \frac{y_k}{\delta} M_i(2) + \frac{1}{2} K \frac{y_k}{\delta} M_i(2) = K \frac{y_k}{\delta} M_i(2). \]

Hence
\[ \lim_{k \in \mathbb{N}} M_i \left( y_k \right) \leq \max \left( 1, K \delta^{-1} M_i(2) \right) \lim_{k \in \mathbb{N}} \left( y_k \right). \]  

(6.4.3)

From (6.4.1), (6.4.2) and (6.4.3), we have
\[ (x_k) \in c_0^i \left( M_1 \ast M_2 \right) \]

Thus \( c_0^i \left( M_2 \right) \subseteq c_0^i \left( M_1 \ast M_2 \right) \).

The other cases can be proved similarly.

\( ii \) Let \( (x_k) \in c_0^i \left( M_1 \right) \cap c_0^i \left( M_2 \right) \). Then there exists \( \rho > 0 \) such that
\[ I - \lim_{k} M_1 \left( \frac{|x_k|}{\rho} \right) = 0 \]
and
\[ I - \lim_{k} M_2 \left( \frac{|x_k|}{\rho} \right) = 0. \]

The rest of the proof follows from the following equality
\[ \lim_{k \in \mathbb{N}} (M_1 + M_2) \left( \frac{|x_k|}{\rho} \right) = \lim_{k \in \mathbb{N}} M_1 \left( \frac{|x_k|}{\rho} \right) + \lim_{k \in \mathbb{N}} M_2 \left( \frac{|x_k|}{\rho} \right). \]

**PROOF OF THE COROLLARY 6.3.4.** The proof of the result easy, so omitted.

**PROOF OF THE RESULT 6.3.5.** We shall prove the result for \( c_0^i \left( M \right) \). The other can be treated similarly.
Let \((x_k) \in c_0'(M)\). Then there exists \(p > 0\) such that
\[
I - \lim M\left(\frac{|x_k|}{\rho}\right) = 0.
\]
(6.4.4)

Let \((\alpha_k)\) be a sequence of scalars with \(|\alpha_k| \leq 1\), for all \(k \in \mathbb{N}\). Then the rest follows from (6.4.4) and the following inequality
\[
M\left(\frac{|\alpha_k x_k|}{\rho}\right) \leq |\alpha_k| M\left(\frac{|x_k|}{\rho}\right), \text{ for all } k \in \mathbb{N}, \text{ [by the remark 6.1.1.]} 
\]
\[
\leq M\left(\frac{|x_k|}{\rho}\right), \text{ for all } k \in \mathbb{N}.
\]
The monotone of the spaces follows from the Lemma 6.2.1.

**PROOF OF THE RESULT 6.3.6.** The prove of this result follows from the following example.

**EXAMPLE 6.4.1.** For \(I = I_6\). Let \(M(x) = x^2\) for \(x \in [0, \infty)\). Consider the \(K^{th}\)-step space \(Z_K\) of \(Z\) defined as follows:

Let \((x_k) \in Z\) and \((y_k) \in Z_K\) be such that
\[
y_k = x_k, \text{ for } k \text{ even;}
\]
\[
y_k = 0, \text{ otherwise.}
\]
Consider the sequence \((x_k)\) as \(x_k = 1\), for all \(k \in \mathbb{N}\).

Then \((x_k) \in c'(M)\) but its \(K^{th}\)-step space preimage does not belong to \(c'(M)\).

Thus \(c'(M)\) is not monotone. Hence \(c'(M)\) is not solid by Lemma 6.2.1.
**PROOF OF THE RESULT 6.3.7.** The prove of this result follows from the following example.

**EXAMPLE 6.4.2.** Let \( M(x) = x^3 \), for \( x \in [0, \infty) \). Consider the sequences \((x_k)\) and \((y_k)\) defined as

\[
x_k = \frac{1}{k} \quad \text{and} \quad y_k = k, \quad \text{for all } k \in \mathbb{N}.
\]

Then \((x_k)\) belong to \( c' (M) \) and \( c'_0 (M) \), but \((y_k)\) does not belong to both \( c' (M) \) and \( c'_0 (M) \).

Hence the spaces are not convergence free.

**PROOF OF THE RESULT 6.3.8.** We prove that \( c'_0 (M) \) is sequence algebra. Rest of the result follows similarly.

Let \((x_k), (y_k) \in c'_0 (M)\). Then

\[
\lim_{k \to \infty} M\left(\frac{|x_k|}{\rho_1}\right) = 0, \text{ for some } \rho_1 > 0
\]

and

\[
\lim_{k \to \infty} M\left(\frac{|y_k|}{\rho_2}\right) = 0, \text{ for some } \rho_2 > 0.
\]

Therefore, for \( \rho = \rho_1 \rho_2 > 0 \), we have

\[
\lim_{k \to \infty} M\left(\frac{|x_k \cdot y_k|}{\rho}\right) = 0.
\]

Then \((x_k \cdot y_k) \in c'_0 (M)\).
Hence $c'_0(M)$ is sequence algebra.

**PROOF OF THE THEOREM 6.3.9.** Let $(x_k) \in c^I(M)$. Then we have

$$M \left( \frac{|x_k - L|}{\rho} \right) \leq \frac{1}{2} M \left( \frac{|x_k|}{\rho} \right) + M \frac{1}{2} \left( \frac{|L|}{\rho} \right),$$

(by the remark 6.1.1).

Taking supremum over $k$ on both sides. Therefore we get $(x_k) \in \ell^I_{\infty}(M)$.

The inclusion $c'_0(M) \subset c^I(M)$ is obvious.

The inclusion is proper follows from the following example:

**EXAMPLE 6.4.3.** For $I = I_d$. Let $M(x) = x^2$, for all $x \in [0, \infty)$.

(a) Consider the sequence $(x_k)$ defined by $x_k = 1$, for all $k \in \mathbb{N}$. Then $(x_k) \in c^I(M)$, but $(x_k) \notin c'_0(M)$.

(b) Consider the sequence $(y_k)$ defined by:

$$y_k = \begin{cases} 2, & \text{if } k \text{ even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in \ell^I_{\infty}(M)$, but $(y_k) \notin c^I(M)$.