CHAPTER-V

PARANORMED $I$-CONVERGENT SEQUENCE SPACES

5.1 INTRODUCTION

The notion of $I$-convergence was studied at the initial stage Kostyrko, Šalát and Wilczyński [69]. Later on it was studied by Šalát, Tripathy and Ziman ([109],[110]), Demirci [28] and others.

The notion of paranormed sequence space was studied at the initial stage by Nakano [94] and Simons [121]. Later on it was further investigated by Maddox [78], Lascarides ([71], [72]), Tripathy and Sen [138] and others.

5.2 DEFINITIONS AND NOTATIONS

A sequence space $E$ is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

A sequence space $E$ is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi$ is a permutation of $\mathbb{N}$.

A sequence space $E$ is said to be a sequence algebra if $(x_k)(y_k) = (x_k \ast y_k) \in E$, whenever if $(x_k), (y_k) \in E$.

A sequence space $E$ is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$. 
Let $K = \{k_1 < k_2 < \ldots \} \subseteq N$ and $E$ be a sequence space. A $K$-step space of $E$ is a sequence space $\lambda^E_K = \{(x_{k_n}) \in \mathcal{w}: (k_n) \in E\}$.

A canonical preimage of a sequence $\{x_{k_n}\} \in \lambda^E_K$ is a sequence $\{y_k\} \in \mathcal{w}$ defined as

$$y_k = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda^E_K$ is a set of canonical preimages of all elements in $\lambda^E_K$, i.e., $y$ is in canonical preimage of $\lambda^E_K$ if and only if $y$ is canonical preimage of some $x \in \lambda^E_K$.

A sequence space $E$ is said to be monotone if it contains the canonical preimages of its step spaces.

Let $(x_k)$ and $(y_k)$ be two sequences. We say that $x_k = y_k$, for almost all $k$ relatively $I$ (a.a.r.I), if $\{k \in N: x_k \neq y_k\} \in I$.

In this chapter we introduce the following sequence spaces. Let $p = (p_k)$ a sequence of non-negative real numbers. Then for given $\varepsilon > 0$,

$$c_1^I(p) = \{(x_k) \in \mathcal{w}: \{k \in N: |x_k - L|^{p_k} \geq \varepsilon\} \in I, \text{ for some } L \in \mathbb{C}\};$$

$$c_0^I(p) = \{(x_k) \in \mathcal{w}: \{k \in N: |x_k|^{p_k} \geq \varepsilon\} \in I\};$$

$$\ell_\infty(p) = \{(x_k) \in \mathcal{w}: \sup_k |x_k|^{p_k} < \infty\}.$$

We write $m^I(p) = c^I(p) \cap \ell_\infty(p)$ and $m_0^I(p) = c_0^I(p) \cap \ell_\infty(p)$.

**Lemma 5.2.1 (Kamthan and Gupta [62] page 53).** A sequence space $E$ is solid implies $E$ is monotone.
LEMMA 5.2.2 (Šalát, Tripathy and Ziman [110] Lemma 2.5). Let $K \in \mathcal{F}(l)$ and $M \subseteq N$. If $M \in I$, then $M \cap K \subseteq I$.

LEMMA 5.2.3 (Kostyrko, Šalát and Wilczyński [69], Lemma 5.1). If $I \subset 2^N$ is a maximal admissible ideal, then for each $A \subset N$ we have $A \in I$, or $N-A \in I$.

LEMMA 5.2.4 (Lascarides [72], Proposition 1). Let $h = \inf_k p_k$, $H = \sup_k p_k$.

Then the following conditions are equivalent:

(i) $H < \infty$ and $h > 0$;
(ii) $c_0(p) = c_0$ or $\ell_\infty$ is equivalent to
(iii) $\ell_\infty$ is equivalent to
(iv) $c_0(p)$ is equivalent to
(v) $\ell_0(p)$ is equivalent to

5.3 MAIN RESULTS

THEOREM 5.3.1. Let $(p_k) \in \ell_\infty$. Then $c^0(p)$, $c_0(p)$, $m_0(p)$ and $m_0^0(p)$ are linear spaces.

THEOREM 5.3.2. Let $(p_k) \in \ell_\infty$, then the spaces $m_0(p)$ and $m_0^0(p)$ are paranormed spaces, paranormed by

$$g((x_k)) = \sup_k |x_k|^{p_k}, \text{where } M = \max(1, \sup_k p_k).$$

THEOREM 5.3.3. $m_0^0(p)$ is a closed subspace of $\ell_\infty(p)$.

PROPOSITION 5.3.4. The spaces $m_0(p)$ and $m_0^0(p)$ are nowhere dense subsets of $\ell_\infty(p)$.

RESULT 5.3.5. The spaces $c_0(p)$ and $m_0^0(p)$ are both solid and monotone.
RESULT 5.3.6. The spaces $c'(p)$ and $m'(p)$ are neither monotone nor solid, if $I$ is neither maximal nor $I = I'$.

RESULT 5.3.7. If $I$ is neither maximal nor $I = I'$, then the spaces $Z(p)$ are not symmetric, where $Z = c_0', c', m_0'$ and $m'$.

THEOREM 5.3.8. For any sequences $(p_k)$ and $(q_k)$, $m_0'(p) \supseteq m_0'(q)$ if and only if $\liminf_{k \in K} \frac{p_k}{q_k} > 0$, where $K \subseteq N$ such that $K \subseteq I$.

COROLLARY 5.3.9. For any two sequences $(p_k)$ and $(q_k)$, $m_0'(p) = m_0'(q)$ if and only if $\liminf_{k \in K} \frac{p_k}{q_k} > 0$ and $\liminf_{k \in K} \frac{q_k}{p_k} > 0$, where $K \subseteq N$ such that $K \subseteq I$.

THEOREM 5.3.10. Let $h = \inf_k p_k$ and $G = \sup_k p_k$, then the following results are equivalent:

(a) $G < \infty$ and $h > 0$;

(b) $c_0'(p) = c_0'$. 

RESULT 5.3.11. The spaces $m_0'(p)$ and $m'(p)$ are not separable.

THEOREM 5.3.12. Let $G = \sup_k p_k < \infty$ and $I$ is a maximal admissible ideal.

Then the following are equivalent:

(a) $(x_k) \in c'(p)$;

(b) there exists $(y_k) \in c'(p)$ such that $x_k = y_k$ for $a.a.k.r.I$;

(c) there exists $(y_k) \in c'(p)$ and $(z_k) \in c_0'(p)$ such that $x_k = y_k + z_k$, for all $k \in N$ and $\{k \in N : |y_k - L|^{|p_k|} \geq s\} \subseteq I$;

(d) there exists a subset $K = \{k_1 < k_2 < \cdots \}$ of $N$ such that $K \subseteq I$ and $\lim_{n \to \infty} |x_{k_n} - L|^{|p_n|} = 0$. 

5.4 THE PROOF OF THE RESULTS OF 5.3.

PROOF OF THE THEOREM 5.3.1. Let \((x_k), (y_k) \in c'(p)\) and \(\alpha, \beta\) be two scalars. Then for a given \(\varepsilon > 0\), we have

\[
\begin{align*}
\{k \in \mathbb{N} : |x_k - L_1|^p &> \frac{\varepsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \} \in I; \\
\{k \in \mathbb{N} : |y_k - L_2|^p &> \frac{\varepsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \} \in I,
\end{align*}
\]

where \(M_1 = D. \max \{1, \sup_k |\alpha|^p\}\); \(M_2 = D. \max \{1, \sup_k |\beta|^p\}\), \(D = \max (1, 2^{G_i})\) and \(G = \sup p_k \geq 0\).

Let \(A_1 = \{k \in \mathbb{N} : |x_k - L_1|^p < \frac{\varepsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \}\); \(A_2 = \{k \in \mathbb{N} : |y_k - L_2|^p < \frac{\varepsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \}\) be such that \(A_1, A_2 \in I\).

Then
\[
\begin{align*}
A_3 &= \{k \in \mathbb{N} : |(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|^p < \varepsilon \} \\
&\supseteq \{k \in \mathbb{N} : |\alpha|^p |x_k - L_1|^p < \frac{\varepsilon}{2M_1} |\alpha|^p D \} \\
&\cap \{k \in \mathbb{N} : |\beta|^p |y_k - L_2|^p < \frac{\varepsilon}{2M_2} |\beta|^p D \}.
\end{align*}
\]

Thus \(A_3^c = A_1^c \cup A_2^c \in I\).

Hence \((\alpha(x_k) + \beta(y_k)) \in c'(p)\).

Therefore \(c'(p)\) is a linear space.

The rest of the results follows similarly.

NOTE 5.4.1. For \((p_k) \in \ell_m\). Consider a set \(J = \{k_i: k_{i+1} \geq k_i + 1\} \text{ and } p_k > i, \text{ for } i = 1, 2, 3, \ldots\). Let \((x_k) \in c'(p)\) be defined by
Then for any scalar \( \lambda \geq 2 \); \( \lambda x_k \to \lambda \). We need to show that \( \lambda(x_k) \not\in c'(p) \).

If possible suppose \( \lambda(x_k) \in c'(p) \). Then there exists \( L \in C \) such that \( |\lambda x_k - L|^{p_k} \to 0 \), as \( k \to \infty \). Since \( p_k \to \infty \), as \( i \to \infty \). Therefore \( |\lambda - L|^{p_k} \leq 1 \), which is a contradiction, because \( |L|^{p_k} < 1 \). Thus \( \lambda(x_k) \not\in c'(p) \).

**Proof of Theorem 5.3.2.** The proof of this result is a routine work.

**Proof of Theorem 5.3.3.** Let \( (x_k^{(n)}) \) be a Cauchy sequence in \( m'(p) \) such that \( x^{(n)} \to x \). To show that \( x \in m'(p) \).

Since \( (x_k^{(n)}) \in m'(p) \), then there exists \( a_n \) such that

\[
\{k \in N : |x_k^{(n)} - a_n|^{p_k} \geq \varepsilon \} \in I.
\]

We need to show that

(i) \( (a_n) \) converges to \( a \) (ii) if \( U = \{k \in N : |x_k - a|^{p_k} < \varepsilon \} \), then \( U^c \in I \).

(i) Since \( (x_k^{(n)}) \) is a Cauchy sequence of \( m'(p) \) then for given \( \varepsilon > 0 \), there exists \( k_0 \in N \) such that

\[
\sup_k |x_k^{(n)} - x_k^{(m)}|^{p_k} < \frac{\varepsilon}{3}, \text{ for all } n, m \geq k_0.
\]

Given \( \varepsilon > 0 \), we have

\[
B_{mn} = \left\{ k \in N : |x_k^{(n)} - x_k^{(m)}|^{p_k} < \left( \frac{\varepsilon}{3} \right)^M \right\};
\]
\[ B_m = \{ k \in \mathbb{N} : |x_k^{(m)} - a_m|^p \leq \left( \frac{\varepsilon}{3} \right)^M \} \]

and \[ B_n = \{ k \in \mathbb{N} : |x_k^{(n)} - a_n|^p \leq \left( \frac{\varepsilon}{3} \right)^M \} \].

Then \[ B^e_m, B^e_m, B^e_n \in \mathcal{I} \].

Let \[ B^e = B^e_m \cup B^e_m \cup B^e_n \), where \[ B = \{ k \in \mathbb{N} : |a_m - a_n|^p < \varepsilon \}. \ Then \[ B^e \in \mathcal{I} \].

We choose \[ k_0 \in B^e \]. Then for each \( n, m \geq k_0 \), we have

\[ \{ k \in \mathbb{N} : |a_m - a_n|^p < \varepsilon \} \supseteq \left[ \{ k \in \mathbb{N} : |a_m - x_k^{(m)}|^p < \left( \frac{\varepsilon}{3} \right)^M \} \right] \cap \left[ \{ k \in \mathbb{N} : |x_k^{(m)} - x_k^{(n)}|^p < \left( \frac{\varepsilon}{3} \right)^M \} \right] \cap \left[ \{ k \in \mathbb{N} : |x_k^{(n)} - a_n|^p < \left( \frac{\varepsilon}{3} \right)^M \} \right]. \]

Then \( (a_n) \) is a Cauchy sequence of scalars in \( C \), so there exists a scalar 'a' in \( C \) such that \( a_n \to a \) as \( n \to \infty \).

(ii) Let \( 0 < \delta < 1 \) be given. To show that if \( U = \{ k \in \mathbb{N} : |x_k - a|^p < \delta \} \), then \( U^e \in \mathcal{I} \).

Since \( x^{(n)} \to x \), then there exists \( q_0 \in \mathbb{N} \) such that

\[ P = \{ k \in \mathbb{N} : |x_k^{(q_0)} - x_k)|^p < \left( \frac{\delta}{3D} \right)^M \} \quad (5.4.1) \]

implies \( P^e \in \mathcal{I} \).

The number \( q_0 \) can be so chosen that together with (5.4.1), we have

\[ Q = \{ k \in \mathbb{N} : |a_{q_0} - a|^p < \left( \frac{\delta}{3D} \right)^M \} \text{ such that } Q^e \in \mathcal{I} \].

Again since \( \{ k \in \mathbb{N} : |x_k^{(q_0)} - a_{q_0}|^p \geq \delta \} \in \mathcal{I} \). Then we have a subset \( S \) of \( N \) such that

\( S^e \in \mathcal{I} \), where \( S = \{ k \in \mathbb{N} : |x_k^{(q_0)} - a_{q_0}|^p < \left( \frac{\delta}{3D} \right)^M \} \).

Let \( U^e = P^e \cup Q^e \cup S^e \), where \( U = \{ k \in \mathbb{N} : |x_k - a|^p < \delta \} \).
Therefore for each \( k \in U \), we have

\[
\{ k \in \mathbb{N} : |x_k - a|_{p_k} < \delta \} = \left[ \left\{ k \in \mathbb{N} : (x_k) - (x_k^{(q_k)}) |_{p_k} < \left( \frac{\delta}{3D} \right)^M \right\} \cap \left\{ k \in \mathbb{N} : |a_{q_k} - a|_{p_k} < \left( \frac{\delta}{3D} \right)^M \right\} \right].
\]

Then the result follows.

**Proof of the Proposition 5.3.4.** The proof of this result is a routine work, so omitted.

**Proof of the Result 5.3.5.** Let \((x_k) \in c_0^I(p)\) and \((\alpha_k)\) be a sequence of scalars with \(|\alpha_k| \leq 1\), for all \( k \in \mathbb{N} \).

Since \(|\alpha_k x_k|_{p_k} \leq |x_k|_{p_k}\), for all \( k \in \mathbb{N} \).

Therefore the space \( c_0^I(p) \) is solid follows from the following inclusion relation

\[
\{ k \in \mathbb{N} : |x_k|_{p_k} \geq \varepsilon \} \supseteq \{ k \in \mathbb{N} : |\alpha_k x_k|_{p_k} \geq \varepsilon \}.
\]

Then the space \( c_0^I(p) \) is monotone by Lemma 5.2.1.

The other result follows similarly.

**Proof of the Result 5.3.6.** We prove this result with the help of the following example.

**Example 5.4.1.** Let \( I = I_\sigma \). Let \( p_k = 1 \), if \( k \) is even and \( p_k = 2 \), if \( k \) is odd.

Consider the \( K^{th} \) - step space \( W_K \) of \( W \) defined as follows:

Let \((x_k) \in W \) and \((w_k) \in W_K \) be such that

\[
w_k = \begin{cases} x_k, & \text{if } k \text{ odd;} \\ 1, & \text{otherwise.} \end{cases}
\]

Consider the sequence \((x_k)\) as \( x_k = k^{-1} \), for all \( k \in \mathbb{N} \).
Then \( (x_k) \in Z(p) \), but its \( K^{th} \) step space preimage does not belong to \( Z(p) \), where \( Z = c' \) and \( m' \).

Thus \( Z(p) \) is not monotone. Hence \( Z(p) \) is not solid, by Lemma 5.2.1.

**Proof of Theorem 5.3.7.** We prove this result with the help of the following example

**Example 5.4.2.** Let \( I = I_0 \). Let \( A_0 = \{ k: k = s^2 \text{ or } t^3, \text{ for } s, t \in \mathbb{N} \} \), then

\[
\sum_{a_0 \in A_0} a_0^{-1} < \infty.
\]

Let \( p_k = 1 \), if \( k \) is even and \( p_k = 2 \), if \( k \) is odd.

Consider the sequence \( (x_k) \) as follows:

\[
x_k = \begin{cases} 
  k^{-1}, & \text{if } k = t^3, t \in \mathbb{N}; \\
  0, & \text{otherwise}.
\end{cases}
\]

Then the rearrangement \( (y_k) \) of \( (x_k) \) defined as

\[
(y_k) = (x_1, x_2, x_3, x_8, x_4, x_5, x_{27}, x_6, x_7, x_{64}, x_8, x_9, \ldots)
\]

Then \( (y_k) \notin Z(p) \), but \( (x_k) \in Z(p) \), where \( Z = c_0', c', m_0' \) and \( m' \).

**Proof of Theorem 5.3.8.** Let \( \liminf_{k \to \infty} \frac{p_k}{q_k} > 0 \) and \( (x_k) \in m'_0(q) \).

Then there exists \( \beta > 0 \) such that \( p_k > \beta q_k \), for all sufficiently large \( k \in K \).

Since \( (x_k) \in m'_0(q) \). For given \( \varepsilon > 0 \), we have

\[
B_0 = \{ k \in \mathbb{N}: |x_k|^q_k \geq \varepsilon \} \in I.
\]

Let \( G_0 = K \cup B_0 \). Then \( G_0 \subseteq I \).

Then for all sufficiently large \( k \in G_0 \),
Therefore, \((x_k) \in m'_0(p)\).

The converse part of the result follows obviously.

**Proof of the Corollary 5.3.9.** The proof of the result is easy, so omitted.

**Proof of the Theorem 5.3.10.** Suppose first that \(h > 0\) and \(G < \infty\), then the inequality

\[
\min (1, s^h) \leq s^p \leq \max (1, s^G)
\]

hold for any \(s > 0\) and for all \(k \in \mathbb{N}\).

Therefore the equivalent of (a) and (b) is obvious.

**Proof of the Result 5.3.11.** Let \(M = \{m_1 < m_2 < \cdots\}\) be a subset of \(N\) such that \(M \in I\).

Let \(p_k = 1\), if \(k \in M\); and \(p_k = 2\), otherwise.

Let \(P_0 = \{(x_k): x_k = 0\ or\ 1, \ for\ k = m_j, j \in N\ and\ x_k = 0,\ otherwise\}\).

Then \(M\) is uncountable.

Consider the class of open balls \(B_1 = \{B(z, \frac{1}{2}): z \in P_0\}\). Let \(C_1\) be an open cover of \(m'_0(p)\) or \(m'(p)\) containing \(B_1\). Since \(B_1\) is uncountable, so \(C_1\) cannot be reduced to a countable sub cover for \(m'_0(p)\) as well as \(m'(p)\). Thus \(m'_0(p)\) and \(m'(p)\) are not separable.

**Proof of the Theorem 5.3.12.** (a) \(\Rightarrow\) (b). Let \((x_k) \in c'(p)\). Then there exists \(L \in C\) such that
Let \((m_i)\) be an increasing sequence with \(m_i \in \mathbb{N}\) such that
\[
\{m > m_i : \{k \leq m : |x_k - L|^{p_k} \geq \frac{1}{i}\}\} < \frac{1}{i}.
\]
Define a sequence \((y_k)\) as follows:
\[
y_k = x_k, \text{ for all } k \leq m_1; \text{ for } m_i < k \leq m_{i+1}, i \in \mathbb{N}.
\]
Let \(y_k = x_k\), if \(|x_k - L|^{p_k} < \varepsilon\) and \(y_k = L\), otherwise.

Then \((y_k) \in c'(p)\) and from the following inclusion
\[
\{k \leq m : x_k \neq y_k\} \subseteq \{k \in \mathbb{N} : |x_k - L|^{p_k} \geq \varepsilon\} \in I.
\]
We get \(x_k = y_k\), for a.a.k.r.I.

\((b) \Rightarrow (c)\). For \((x_k) \in c'(p)\), then there exists \((y_k) \in c'(p)\) such that \(x_k = y_k\), for a.a.k.r.I.

Let \(K = \{k \in \mathbb{N} : x_k \neq y_k\}\), then \(K \in I\).

Define \((z_k)\) as follows:
\[
z_k = \begin{cases} x_k - y_k, & \text{if } k \in K; \\ 0, & \text{if } k \notin K \end{cases}
\]
Then \((z_k) \in c'_0(p)\) and so \((y_k) \in c'(p)\).

\((c) \Rightarrow (d)\). Suppose \((c)\) holds. Let \(\varepsilon > 0\) be given.

Let \(P_1 = \{k \in \mathbb{N} : |z_k|^{p_k} \geq \varepsilon\}\) and \(K = P_1 = \{k_1 < k_2 < \cdots\} \in I\).

Then we have
\[
\lim_{n \to \infty} |x_{k_n} - L|^{p_{k_n}} = 0.
\]

\((d) \Rightarrow (a)\). Let \(K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}\) be such that \(K \in I\) and \(\lim_{n \to \infty} |x_{k_n} - L|^{p_{k_n}} = 0\).

Then for any \(\varepsilon > 0\), and Lemma 5.2.2, we have
\[
\{k \in \mathbb{N} : |x_k - L|^{p_k} \geq \varepsilon\} \subseteq K^c \cup \{k \in K : |x_k - L|^{p_k} \geq \varepsilon\}.
\]
Thus \((x_k) \in c'(p)\).