INTRODUCTION

§1. Schlicht functions in the unit circle.

1.1 In problems of conformal representation the inverse of a mapping function is an automorphic function of the group which takes in the fundamental domain no value more than once. We always consider such functions only in the case of mapping of a simply-connected domain, where the fundamental domain coincides with the whole unit circle and the group consists only of the identical transformation. Thus there arises the question of analytic functions which are regular in the unit circle and take there no value more than once. We shall call them to form the family of schlicht functions in the unit circle.

It was first proved by Koebe [18] that if the analytic function
\[ w(z) = \sum_{0}^{\infty} a_n z^n , \]
regular for \(|z| = r < 1\), be schlicht in the unit circle, and if \(a_0 = 0, a_1 = 1\), then
\[ |w(z)| \leq \phi(r) , \quad 0 < r < 1 , \]
where \(\phi\) is a function of \(r\) only. In fact, the condition (1.2) is a direct consequence of the Koebe distortion theorem (see also [13,249],[26,209-52],[27,93]). We shall now show, how this condition holds for different classes of schlicht functions.

1.2. Classical basis for classification of schlicht functions.

The Koebe condition (1.2) presents the classical basis for the problem of classifying the totality of schlicht functions in the unit circle.

We consider the normal and compact family \(S\) of all analytic functions \(w = w(z)\) which are regular and schlicht in the unit circle.
and have a development of the Taylor's form

\[ w = w(z) = z + \sum_{v=2}^{\infty} a_v z^v \]

with the normalisation conditions \( w(0) = 0 \) and \( w'(0) = 1 \). Then there exist the following six classes of these functions:

- \( S_I \): The class of functions (1.3), analytic and schlicht for every pair of values \( z_1, z_2 \) in the unit circle, i.e.
  \[ \frac{w(z_1) - w(z_2)}{z_1 - z_2} \neq 0. \]

- \( S_{II} \): The class of functions (1.3), convex and schlicht in the unit circle, i.e. every circle \( |z| < 1 \) is mapped into a region in which no straight line can cut the contour enclosing the origin in more than two points.

- \( S_{III} \): The class of functions (1.3), schlicht and star-like in the unit circle, i.e. every circle \( |z| < 1 \) is mapped into a region in which no straight line through the origin cuts the contour enclosing the origin in more than two points.

- \( S_{IV} \): The class of functions (1.3), schlicht and convex in the direction of the imaginary axis for \( |z| < 1 \), i.e. no straight line parallel to the imaginary axis cuts the image of the circle \( |z| = r, 0 < r < 1 \), in more than two points.

- \( S_V \): The class of functions (1.3), analytic (and not necessarily schlicht) in the unit circle and such that the image of the circle \( |z| = r, 0 < r < 1 \), is cut by the real axis in not more than two points.

- \( S_{VI} \): The class of functions (1.3) with real coefficients, which are also members of the class \( S_V \). These functions are called typically real, a term introduced by Rogosinski[35]. The conformal image of the circle \( |z| < 1 \) as yielded by functions of this class, which need not be schlicht, covers the linear segment \(-1/4 < |w| < 1/4 \).
We remark that the class \( S^* \) is contained in the class \( S_\gamma \); those members of the class \( S_\gamma \) which are real on the real axis form a subclass of \( S^* \); the class \( S_{III} \) is contained in the class \( S_\gamma \); the class \( S_{II} \) is contained in both the classes \( S_{III} \) and \( S_{IV} \).

The functions of the class \( S_\gamma \) can be approximated by means of polygonal functions and thus we obtain an integral representation which involves an arbitrary function of bounded variation. The importance of this representation also includes the class \( S_{II} \) and \( S_{III} \), and by its means we can easily find most of the well-known properties of these functions. Hence, if \( w(z) \) belongs to the classes \( S_\gamma, S_{II}, S_{III} \), we can always find a sequence of polygonal functions of the form

\[
(1.4) \quad w_n(z) = \sum_{m=1}^{n} \int_{0}^{\infty} b_m (1 - e^{i\theta_m}) \, dz,
\]

where \( \theta_m \) and \( b_m \) are real numbers connected by the relations \( \sum_{m=1}^{n} b_m = 2 \), \( -1 \leq \sum_{m=j}^{n} b_m \leq 3 \), \( j, l = 1, 2, \ldots \), \( j \neq l \), so that \( w_n(z) \) converges uniformly to \( w(z) \) for \( |z| < 1 \).

If, in general, we consider the symmetric functions

\[
(1.5) \quad w_p(z) = z + \sum_{n=1}^{\infty} a_{pn+1} z^{pn+1},
\]

where \( p \) is a positive integer, we may replace the approximating polygonal functions of the representation (1.4) by functions \( w_n(z) \) of the form

1) such as (1.4) or in general (1.6) given below.

2) For the proof of this statement see Robertson [34, 176-77]; for the representation (1.4) also see Bieberbach [5, 301-2], Gronwall [13, 251], Löwner [23, 111-13]. For the representation (1.6) see Robertson [34].
\[
(1.6) \quad w_n(z) = \frac{z^n}{\prod_{m=1}^{n} \left(1 - (e^{\theta_m} z)^p\right)} - b_m \quad dz, \quad p = 1, 2, \ldots,
\]

where now \( \sum_{m=1}^{n} b_m = 2/p \), \(-1 \leq \sum_{m=j}^{1} b_m \leq 1 + 2/p \), \( j, l = 1, 2, \ldots, j \neq l \).

It is well known that the family \( S \) of functions \( (1.3) \) is normal and compact on account of the Koebe distortion theorem:

\[
\frac{1-r}{(1+r)^3} \leq |w'(z)| \leq \frac{1+r}{(1-r)^3}
\]
or, as Bieberbach \([6, 71-83]\) has shown,

\[
\frac{1}{1+|a_\infty|^2 |r|^2} \leq |w(z)| \leq \frac{r}{(1-r)^2}, \quad 0 < r < 1.
\]

Bieberbach's result is the best possible as is shown by the function

\[
w(z) = \frac{z}{(1-e^{i\theta} z)^2}, \quad \zeta \text{ real},
\]

which takes every value, except those on the negative real axis between \(-1/4\) and \(\infty\) once and once only. Further, if we take \( w(z) \) as the function

\[
(1.7) \quad w(z) = z/(1-e^{i\theta} z)^2, \quad \zeta \text{ real},
\]

then there holds the well-known Faber inequality \( |a_\infty| \leq 2 \) and the inequality

\[
\left| \frac{1-|z|^2}{2} \cdot \frac{w''}{w'} - \frac{z}{w'} \right| \leq 2.
\]

Now the necessary and sufficient condition for the functions \( (1.3) \) to belong to the class \( S_{III} \) is that (cf. \([26, 221]\))

3) The exact inequalities \( |a_{\infty}| \leq 2 \), \( d \geq 1/4 \), and the bounds for \( |w'| \), \( |w| \) and \( |w'/w'| \) were obtained in 1916 by Bieberbach, Faber, Gronwall, Pick and others. In each case the bounds are attained only for the functions \( (1.7) \) which Hayman \([15]\) has called the greatest schlicht functions (g.s.f.'s.); they map \( |z| < 1 \) onto the \( w \)-plane cut along a radial slit from \(-1/4, \exp(\theta)\) to \(\infty\). These results made plausible the famous Bieberbach conjecture that \( |a_n| \leq n \) holds for all positive integers \( n \).
Re \{z w'(z)/w(z)\} \geq 0,
and the same functions belong to the class $S_{II}$ if and only if $zw'(z)$
belongs to the class $S_{III}$, where according to Gronwall [13], Löwner
[23] we have

$$|zw'(z)| \leq r/(1-r)^2.$$  

In the general case, when $w(z)$, represented by (1.5), belongs to the class $S_{II}$, we have (see [34, 379-81])

$$\frac{1}{(1+r^p)^{2/p}} \leq \frac{1}{w_p(re^{i\theta})} \leq \frac{1}{(1-r^p)^{2/p}},$$

and

$$\frac{1}{(1+r^p)^{2/p}} \leq \frac{1}{w_p(re^{i\theta})} \leq \frac{1}{(1-r^p)^{2/p}},$$

and

$$|a_{np+1}| \leq \frac{1}{np+1} \frac{1}{n!} \sum_{k=0}^{n-1} II (k + 2/p), \quad z = re^{i\theta}, r < 1.$$  

The signs of equality are attained here by the convex function

$$w(z) = \int_0^r \frac{dz}{(1-z^p)^{2/p}}.$$  

If, instead of belonging to the class $S_{II}$, $w(z)$ belongs to the class $S_{III}$, then we obtain

$$\frac{1}{(1+r^p)^{2/p}} \leq \frac{1}{w_p(re^{i\theta})} \leq \frac{1}{(1-r^p)^{2/p}},$$

and

$$\frac{1}{np+1} \frac{1}{n!} \sum_{k=0}^{n-1} II (k + 2/p),$$

and

$$\text{Re} \left\{\frac{w_p(z)/z}{p/2} \right\} \geq 1/2.$$  

It would not be out of place to mention an extension of the
Koebe distortion theorem for functions (1.3) in the class $S_{II}$ by
Rakhmanov [33, 370-71]. By making use of the Schwarz lemma he has shown that if the functions (1.3) are regular in $|z| < 1$ and if there hold the inequalities

$$|\arg(zw'/w)| \leq \pi/2n, \quad n=1,2,\ldots, |z| < 1,$$

then $w(z)$ is schlicht in $|z| < 1$ and
vi

\[
\frac{(1-r)^{1/n}}{(1+r)} \leq \text{Re} \left( \frac{zw'/w}{w} \right) \leq \left( \frac{1+r}{1-r} \right)^{1/n},
\]

\[
\left| \arg \left( \frac{zw'/w}{w} \right) \right| \leq \frac{2}{\pi} \arctan r,
\]

\[
r \exp \int_0^\infty \left\{ \left( \frac{1-t}{1+t} \right)^{1/n} - 1 \right\} \frac{dt}{t} \leq |w| \leq r \exp \int_0^\infty \left\{ \left( \frac{1+t}{1-t} \right)^{1/n} - 1 \right\} \frac{dt}{t},
\]

\[
\left( \frac{1-r}{1+r} \right)^{1/n} \exp \int_0^\infty \left\{ \left( \frac{1-t}{1+t} \right)^{1/n} - 1 \right\} \frac{dt}{t} \leq |w'| \leq \left( \frac{1+r}{1-r} \right)^{1/n} \exp \int_0^\infty \left\{ \left( \frac{1+t}{1-t} \right)^{1/n} - 1 \right\} \frac{dt}{t},
\]

and the radius of convexity for these functions is the positive root of the equation

\[
\left( \frac{1-r}{1+r} \right)^{1/n} - \frac{2r}{n(1-r^2)} = 0.
\]

All these evaluations are exact as is shown by the functions

\[
w_0(z) = z \exp \int_0^z \left\{ \left( \frac{1-t}{1+t} \right)^{1/n} - 1 \right\} \frac{dt}{t} \in S_{II}.
\]

The results of Rakhmanov can, however, be extended yet to the symmetrical functions (1.5), if they satisfy the condition

\[
\left| \arg \left( \frac{zw'/w}{w} \right) \right| \leq \pi/2n, \quad n=1,2,\ldots, \quad |z| < 1.
\]

The functions (1.3) in the class $S_{IV}$ are related to the same functions in the class $S_V$ in the same way as those in the class $S_{II}$ are related to those in the class $S_{III}$ on the basis of the condition: If $w(z)$ belongs to the class $S_{II}$, then $zw'(z)$ belongs to the class $S_{III}$. Thus a necessary and sufficient condition for $w(z)$ to belong to the class $S_{IV}$ is that $zw'(z)$ belongs to the class $S_{V}$. Functions of the class $S_V$ which are real on the real axis are called typically real and form the class $S_{VI}$. Hence we also find that a necessary and sufficient condition that $w(z)$ belongs to the class $S_{IV}$ and has real coefficients is that $zw'(z)$ belongs to the class $S_{VI}$. Conversely we see that functions of the class $S_V$ are
associated to those of the class $S_{IV}$ exactly as functions of the class $S_{III}$ are associated to those of the class $S_{II}$.

It follows immediately that if $g(z) = \sum_{n=1}^{\infty} a_n z^n$ belongs to the class $S_{V}$, then $f(z) = \sum_{n=1}^{\infty} a_n z^n = \int_{0}^{z} g(z) \, dz/z$ belongs to the class $S_{IV}$. In particular, if $g(z)$ belongs to the class $S_{VI}$, then $f(z) = \int_{0}^{z} g(z) \, dz/z$ is likewise schlicht. It means as follows: The functions $g(z)$ of the class $S_{V}$ are not in general schlicht and therefore map $|z| = 1$ into a contour which may have loops in it. But from the above we observe that by dividing each coefficient of the power series representing $g(z)$ by $n$ the loops will be removed, thus giving a schlicht function which maps the unit circle onto a non-overlapping region.

Hence if the functions (1.3) belong to the class $S_{IV}$, we have

$$\text{Re} \left\{ \frac{w(z)}{z} \right\} > \frac{1}{2} ,$$

$$\frac{1+|a_2|^r}{1+2|a_2|^r (1-r^2)} \leq \text{Re} \left\{ \frac{w(z)}{z} \right\} \leq \frac{1+|a_2|^r}{1-r^2} ,$$

and

$$\frac{r}{1+r} \leq |w(z)| \leq \frac{r}{1-r} , \quad |z|=r < 1 .$$

Also, if the same functions belong to the class $S_{VI}$, then for $|z| < 1$ we have

$$1 + \int_{0}^{2\pi} |w(re^{i\theta})| \, d\theta \leq \frac{r}{(1-r^2)} .$$

1.3. A generalisation of the class $S_{III}$. Špaček [42] has introduced a class $S(\gamma)$ of schlicht functions in the unit circle in view of the following consideration: Let $S(\gamma)$ be the subclass of $S_{III}$ whose members $w(z)$, defined by (1.3), satisfy, for some real constant $\gamma$ ($0 \leq \gamma \leq \pi/2$) and $|z| < 1$, the inequality

$$\text{Re} \left\{ e^{i\gamma} zw'(z)/w(z) \right\} \geq 0 .$$

Then the inequality (1.8) is a sufficient condition for $w(z)$ of (1.3) to be schlicht in $|z| < 1$. In general, a member of $S(\gamma)$ maps $|z| < 1$ onto a spiral-like domain. We shall call $w(z)$ 'spiral-like'
and schlicht if it belongs to the class $S(\gamma)$.

When $\gamma = 0$, the subclass of $S(\gamma)$ has been denoted by $S_{III}^\gamma$, where $w(z)$ of (1.3) maps $|z| < 1$ onto a star domain. We have as such called $w(z)$ 'star-like' if it belongs to the class $S_{III}^\gamma$. In fact every starlike function is also spiral-like, since $S_{III}^\gamma \subset S(\gamma)$.

1.4. Further classification of schlicht functions.

Besides the six main classes mentioned above we can form their subclasses by taking into consideration various combinations of two classes at a time. A simple calculation gives the total number of such subclasses to be 15. This naturally suggests that the theory of schlicht functions can be developed comprehensively on the classical lines; but there appears only one mathematician B.N. Rakhmanov [33] who has recently examined the subclass $\Sigma = S_{II}^\gamma + S_{III}^\gamma$, represented in terms of the functions

\[(1.9) \quad \Sigma : \quad w = f(z) = \frac{1}{2} \{w(z) + zw'(z)\},\]

schlicht in $|z| < 1$, where $w(z) \in S_{II}^\gamma$, so that obviously $zw'(z) \in S_{III}^\gamma$. A number of his results in this direction is as follows:

(i) $|\arg f'(z)| \leq 3 \arcsin r$, $|z| = r$,

where this evaluation is exact as the function

\[(1.10) \quad f(z) = z (2-z)/(2 (1-z)^2)\]

attains the upper bound along the curve $\cos \varphi = r$, $z = re^{i\varphi}$.

(ii) $(3-2r+4r^2+2r^3)/(4(1-r)^2) \leq r \leq (2+r)(1+r)^2 \leq \text{Re}\{f(z)/z\} \leq (2-r)/(2(1-r)^2),$

where the upper and lower bounds are exact for $r < 1/2$, as is shown by the functions (1.10).

4) These fifteen subclasses are: $S_{I}^\gamma + S_{II}^\gamma$, $S_{I}^\gamma + S_{III}^\gamma$, $S_{I}^\gamma + S_{IV}^\gamma$, $S_{I}^\gamma + S_{V}^\gamma$, $S_{I}^\gamma + S_{VI}^\gamma$, $S_{II}^\gamma + S_{III}^\gamma$, $S_{II}^\gamma + S_{IV}^\gamma$, $S_{II}^\gamma + S_{V}^\gamma$, $S_{II}^\gamma + S_{VI}^\gamma$, $S_{III}^\gamma + S_{IV}^\gamma$, $S_{III}^\gamma + S_{V}^\gamma$, $S_{III}^\gamma + S_{VI}^\gamma$, $S_{IV}^\gamma + S_{V}^\gamma$, $S_{IV}^\gamma + S_{VI}^\gamma$, $S_{V}^\gamma + S_{VI}^\gamma$. 
(iii) If the functions (1.3) belong to the subclass $\Sigma$, then the functions with a finite series development

$$S_n(z) = z + \sum_{v=2}^{n} a_v z^v \in \Sigma$$

are schlicht in the circle \(|z| < 1 - 3 \log \frac{n}{n}\) for \(n > 11\). Further, the functions $S_n(z)$ do not possess zeros (except $z = 0$) in the circle \(|z| < 1 - 2 \log \frac{n}{n}\) for \(n \geq 9\).

(iv) If the functions $w(z) \in S_{II}$, then the functions

$$f^*(z) = \frac{(c+1) w(e^{i\gamma} z) - w(e^{-i\gamma} z)}{(c+1) e^{i\gamma} - e^{-i\gamma}}$$

and

$$w^*(z) = z + \frac{c}{c+2} \sum_{v=2}^{\infty} a_v z^v, \quad c > 0, \quad 0 < \gamma \leq \pi$$

are each schlicht in \(|z| < 1\).

1.5. The Schwarzian derivative. Introduction of the Schwarzian derivative of $w = w(z)$ with respect to $z$, defined as

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2} \left(\frac{w''}{w'}\right)^2,$$

has also solved the problem of existence of schlicht functions in the unit circle. In fact it has served as an alternative means to solve the question of schlicht conformal mappings. Nehari [25,545] has proved that if for the analytic functions (1.1), schlicht in the unit circle \(|z| < 1\), the two normalisations are used: (a) $w(z)$ is finite in \(|z| < 1\), $w(0) = 0$, $w'(0) = 1$, and (b) $w(z)$ has a pole in $z = 0$ with residue +1, then the inequalities

\[(1.11) \quad \{w, z\}^2 \leq 6/(1-|z|^2)^2 \quad \text{and} \quad \{w, z\}^2 \leq 2/(1-|z|^2)^2\]

give the necessary and sufficient condition, respectively, for the analytic functions $w = w(z)$ to be schlicht in \(|z| < 1\). He has also shown [25,549] that if

\[(1.12) \quad |w, z| \leq \pi/2\]

in \(|z| < 1\), then $w = w(z)$ is schlicht in the unit circle, and that the constant $\pi^2/2$ is the best possible. As an example for the
application of the condition (1.12) we can take the error function

\[ w = \zeta(z) = \frac{2}{z} \exp(-z^2) \, dz, \]

which is schlicht in the unit circle with \( \{w, z\} = -2 (1+z^2) \).

But Hille[17] has mentioned the insufficient information contained in Nehari's necessary condition (1.11) by considering the function

\[ w(z) = \left(\frac{1-z}{1+z}\right)^{\gamma}, \]

where \( \gamma \) is a real constant, and \( w(0) = 1 \). In this case \( \{w, z\} \leq c/(1-|z|^2)^2 \), where \( c = 2 (1 + \gamma^2) \), and therefore it follows that the first inequality in (1.11) does not give a necessary condition in prospect.

We can, however, mention here the exact inequality in this respect, due to Alenitsin[2,863], [3,329], according to which

\[ \frac{|w(z)|^6 + |w'(z)|^2}{(1-|w(z)|^2)^2} \leq \frac{1}{(1-|z|^2)^2} \]

gives for \( |w(z)| < 1 \) and \( |z| < 1 \) the necessary condition for the existence of schlicht functions \( w = w(z) \) in the unit circle.

1.6. An abstract basis for classification of schlicht functions.

Publication of a recent work by Rakhmanov[32] has orientated the entire theory of schlicht functions in view of the following considerations:

Let \( D \) be a simply-connected domain in the complex \( w = u+iv \)-plane, which does not contain the point at infinity as interior point. Then the function

\[ v = \chi(u, a), \quad (\kappa < u < \beta, \quad \kappa_1 < a < \beta_1), \]

defines a family of curves depending on the parameter \( a \) and possessing the property that through each finite point of a finite or infinite domain \( G \) in the \( z \)-plane there passes only one curve of the family (1.13). We suppose that the function (1.13) is differentiable with respect to both the arguments in the intervals \((\kappa, \beta) \) and \((\kappa_1, \beta_1) \), which can be infinite on both sides, and \( \partial \chi / \partial a \neq 0 \). Then the
functions \( w = w(z) \), analytic in the circle \(|z| < 1\), map the circle \(|z| \leq 1\) onto a domain whose boundary \( \Gamma \) lies completely inside \( G \).

The functions \( w(z) = u + iv \) and \( v = \chi(u,a) \) determine a correspondence between the points of the circle \( z = e^{i\varphi} \) \((0 \leq \varphi \leq 2\pi)\) and values of the parameter \( a \). If this correspondence is 1-1, the boundary \( \Gamma \) is entirely circular, and therefore the functions \( w = w(z) \) are schlicht in \(|z| < 1\). Also it means that the function \( a = a(\varphi) \), defined by the relations

\[
(1.14) \quad v = \chi(u, a) \quad , \quad u + iv = w(e^{i\varphi}) ,
\]

is monotone. Hence by evaluating the derivative \( da/d\varphi \) of the corresponding function of the form \((1.14)\) and putting \( da/d\varphi \geq 0 \), we can obtain necessary and sufficient conditions for the existence of each of the following six classes of schlicht functions in the unit circle, where emphasis is laid on the form of the domain \( D \) under different conformal mappings, defined as follows:

1. **The class** \( D_{px} \). The domain \( D \) is called a domain of the form \( D_{px} \), if for any point \( w = w(z) \) on the boundary of \( D \), not lying on the real u-axis, there exists an \( a \), such that a branch of the parabola \( v = \sqrt{2p(u-a)} \) \((p \text{ const})\) passes through that point and that the segment of this branch between that point and the u-axis lies entirely in the domain \( D \).

2. **The class** \( D_{py} \). The domain \( D \) is called a domain of the form \( D_{py} \), if for any point \( w = w(z) \) on the boundary of the domain \( D \), not lying on the u-axis, there exists an \( a \), such that a branch of the parabola \( \sqrt{2pv} = a-u \) \((p \text{ const})\) passes through that point and that the segment of this branch between that point and the u-axis lies completely in the domain \( D \).

3. **The class** \( D_{p\theta} \). The domain \( D \) is called a domain of the form \( D_{p\theta} \), if for each point \( w = w(z) \) on the boundary of the domain \( D \) there exists a \( \theta \), such that a branch of the parabola \( \xi \sin^2 (\psi - \theta) = \)
\[ \gamma = \rho \cos (\phi - \theta) \quad (\rho \text{ const}), \]
where \( \rho = \rho (\phi - \theta) \) denotes the family of curves (1.5) in terms of the polar coordinates, passes through that point and that the segment of this branch between that point and the origin of coordinates in the \( w \)-plane lies in \( D \).

4. The class \( D_{rk} \). Let a circle of radius \( r \) and centre in the origin of coordinates in the \( w \)-plane lie in the domain \( D \) and each point \( w = w(z) \) on the boundary of \( D \) lie on a tangent to this circle. For the sake of definiteness we choose a positive direction on the tangent and assume the point \( w = w(z) \) always lying on the tangent in the positive direction from the apex of the parabola (1.13). Then the domain \( D \) is called a domain of the form \( D_{rk} \), if the segment of the tangent between the point \( w = w(z) \) and the apex lies in \( D \).

5. The class \( D_{pk} \). Let the parabola \( v^2 = 2p \rho \) (\( \rho \) const) lie in the domain \( D \) and each point \( w = w(z) \) on the boundary of \( D \) lie on a tangent to this parabola. We choose a positive direction on the tangent and suppose the point \( w = w(z) \) always lying in the positive direction from the apex of the parabola. Then the domain \( D \) is called a domain of the form \( D_{pk} \), if the segment of the tangent between the point \( w = w(z) \) and the apex lies in \( D \).

6. The class \( D_{pqk} \); Let the parabola \( v^2 = 2p(p-q) \) (\( p,q \) const, \( p > 0, q > 0 \)) lie in the complement to the domain \( D \) and each point \( w = w(z) \) on the boundary of the domain \( D \) lie on the tangent to this parabola. Choosing a positive direction on the tangent we suppose the point \( w = w(z) \) always lying in the positive direction from the apex of the parabola. Then the domain \( D \) is called a domain of the form \( D_{pqk} \), if the segment of the tangent between the point \( w = w(z) \) and the apex lies completely in \( D \).

It is assumed in the above classification that the boundaries of the connected domain in the classes \( D_{px}, D_{py}, D_{pk} \) are closed Jordan curves lying in the extended plane and passing through the
point at infinity, whereas in the classes $D_{rk}$, $D_{p0}$ they can be closed Jordan curves situated in the finite plane as well as closed Jordan curves of the extended plane. For the sake of simplicity we consider these domains as finite infinite domains in view of the fact that the necessary and sufficient conditions for their existence are also correct for finite domains.

Let the analytic functions $w_n = w_n(z)$, $n = 1, \ldots, 6$, be continuous in $|z| < 1$ (also in the point at infinity) and regular analytic everywhere in $|z| < 1$, with the exception of a finite number of boundary points. We can relax here the requirement of analyticity on $|z| = 1$. We then suppose that all these six functions are normalised, i.e. $w_n(0) = 0$, $w_n'(0) = 1$, $n = 1, \ldots, 6$. Then Rakhmanov [32, 974] has given six necessary and sufficient conditions, one for schlichtness of each function $w_n(z)$, $n = 1, \ldots, 6$, in the unit-circle. Thus

1. The function $w = w_1(z)$ is schlicht in $|z| < 1$ and maps the unit circle onto a domain of the form $D_{p1}$, if and only if

$$\text{Im } w(z) \cdot \text{Re } z w'(z) + p \text{Im } zw'(z) \geq 0, \quad p > 0,$$

for all $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, with the exception of a finite number of points.

2. The function $w = w_2(z)$ is schlicht in $|z| < 1$ and maps the unit circle onto a domain of the form $D_{p2}$, if and only if

$$\sqrt{2} \text{Im } w_2(z) \cdot \text{Im } zw'_2(z) - \sqrt{v} \text{Re } zw'_2(z) \geq 0, \quad p > 0,$$

for all $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, with the exception of a finite number of points.

3. The function $w = w_3(z)$ is schlicht in $|z| < 1$ and maps the unit circle onto a domain of the form $D_{p3}$, if and only if

$$\text{Re } \left( \sqrt{p^2 + |w_3(z)|^2} + i \sqrt{p^2 + |w_3(z)|^2 - p^2} \cdot \frac{zw'_3(z)}{w_3(z)} \right) \geq 0.$$

$p > 0$, for all $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, with the exception of a
finite number of points.

(4) The function \( w = w_4(z) \) is schlicht in \( |z| < 1 \) and maps the unit circle onto a domain of the form \( D_{r_k} \), if and only if

\[
\text{Re} \left( 1 + \frac{ir}{\sqrt{1 - w_4(z)^2 - r^2}} \right) \frac{zw_4'(z)}{w_4(z)} \geq 0, \quad r > 0,
\]

for all \( z = e^{i\theta}, \ 0 \leq \theta < 2\pi \), with the exception of a finite number of points.

(5) The function \( w = w_5(z) \) is schlicht in \( |z| < 1 \) and maps the unit circle onto a domain of the form \( D_{p_k} \), if and only if

\[
p \text{Im} zw_5'(z) + (\text{Im} w_5(z) + \sqrt{\text{Im} w_5(z)^2 - 2p (\text{Re} w_5(z))}) \text{Re}zw_5'(z) \leq 0, \quad p > 0,
\]

for all \( z = e^{i\theta}, \ 0 \leq \theta < 2\pi \), with the exception of a finite number of points.

(6) The function \( w = w_6(z) \) is schlicht in \( |z| < 1 \) and maps the unit circle onto a domain of the form \( D_{pq_k} \), if and only if

\[
(\text{Im} w_6(z) + \sqrt{\text{Im} w_6(z)^2 - 2p \text{Re} w_6(z) + 2pq}) \text{Re}zw_6'(z) + p \text{Im} zw_6'(z) \geq 0, \quad p > 0, \quad q > 0,
\]

for all \( z = e^{i\theta}, \ 0 \leq \theta < 2\pi \), with the exception of a finite number of points.

Further, it is so far only known that the classes \( D_{p,q} \) and \( D_{r,k} \) coincide with the class \( S_{III} \) (§1.2), and can therefore be associated to the class \( S_{II} \); their corresponding conditions of schlichtness can also be compared.

1.7. The coefficient problem. The so-called coefficient problem in the theory of schlicht functions originates as follows: The power series (1.3) defines a 1-1 conformal mapping of the circle \( |z| < 1 \) onto some simply-connected region in the \( w \)-plane, precisely according to the classes \( S_I - S_{VI} \) (§1.2). This property of the power series (1.3) restricts the coefficients \( a_v \) \((v = 2, 3, \ldots)\) in the said
power series and the problem of evaluating their exact value is very difficult. Since the coefficients $a_v$ form a system of complex numbers we always aim, while solving the coefficient problem, at finding a region $V_n$ of points $a_v (v = 2, 3, \ldots, n)$ in the $(n-1)$-dimensional complex space which corresponds to the power series (1.3).

It is well known, Bieberbach [6], Faber (see e.g. [7]), Gronwall [13], Pick [28] and others proved that $|a_2| \leq 2$. Löwner [23, 118] obtained a partial differential equation (known as the Löwner classical formula) satisfied by the functions (1.3) and showed that $|a_2| \leq 3$; he has also found the exact upper bounds $|a_n| \leq \frac{1.3 \ldots (2n-1)}{1.2 \ldots (n+1)} 2^n$ for all coefficients of the inverse function $z = z(w)$, again with the sign of equality only for the g.s.f's. (1.7). His method has also led to numerous other results, notably in the first place the exact bounds for $\arg(w'(z))$ and $\arg(w(z)/z)$, and in the second place a result due to Krawzowblocki [19] that $|a_4|$ has a local maximum with $|a_4| \leq 4$; Schaffer and Spencer [36] have also proved by an alternative method that $|a_3| \leq 3$, whereas Garabedian and Schiffer have shown that $|a_4| \leq 4$.

As regards behaviour of $a_n$ for large $n$, we know that the Bieberbach conjecture $|a_n| \leq n$ has been proved for the case where the

---

5) This formula is defined as:

$$a_{n+1}(t) = e^{-\sum_{i=1}^{n} \alpha_i t_i} \prod_{i=1}^{n} \int \cdots \int k_i(\tau_i) d\tau_1 \cdots d\tau_n,$$

with

$$|k(\tau_i)| = 1,$$

$$C = 2^n (n+1-\alpha_1) (n+1-\alpha_2) \cdots (n+1-\alpha_1-\alpha_2-\cdots-\alpha_v),$$

and

$$\sum_{i=1}^{v} \alpha_i = n, \quad v \leq n, \quad n = 0, 1, \ldots.$$
coefficients are real. Littlewood [21] has proved that $|a_n| < e^n$. A result due to Landau [20, 635] states that $\lim_{n \to \infty} \frac{|a_n|}{n!} \leq (1/2 + 1/\pi)$ e, and due to Robertson [34, 398] that $\lim_{n \to \infty} \frac{|a_n|}{n!} \leq 1 + 2/\pi$, but the sharpest result up-to-date is due to Bazilevitch [4] that $\sum_{n=1}^{\infty} \frac{|a_n|}{n!} < e^{e^{2/e}}$. Also Hayman [15, 105] has recently shown that if the functions (1.3) are mean schlicht in $|z| < 1$, then $\lim_{n \to \infty} \frac{|a_n|}{n!} \leq 1$, where the sign of equality holds only for the g.s.f.'s. (1.7).

For coefficients of typically real functions (the class $S_{VI}$ of §1.2) we know that Rogosinski [35] and Goluzin [12] proved that $|a_n| + |a_{n-1}| \leq 2$, $n = 2, 3, \ldots$, $-1 \leq a_1 \leq 1$, and $-1/2 \leq a_2 \leq 3/2$. But very recently Gelfer [11] has extended these evaluations to include $-1 - \sqrt{3}/18 \leq a_3 \leq 1 + \sqrt{3}/18 = 1.09$, $a_{2k} \leq 3/2$ (k = 1, 2, \ldots), $a_4 \geq -2/3$, and $a_6 \geq -5/16 - 121/48\sqrt{3} \approx -3/4$.

1.8. Curvature of level curves and orthogonal trajectories.

We shall confine ourselves to the classes $S_I$, $S_{II}$, $S_{III}$ of §1.2. If a point $z$ moves with uniform angular velocity 1 on the circumference $|z| = r$, then the images of the circle $|z| = r < 1$ are sketched by the functions (1.3) in the $w$-plane are called the level curves of $w(z)$, and they have in the point $w$ the measure of curvature (see [30, 105 and 276])

\[
K = \frac{1 + \text{Re}(zw''/w')}{|zw'(z)|}.
\]

This value of $K$ is positive or negative according as the sense of

8) See §2.3 for the definition of mean valency of analytic functions in the unit circle.

9) We define more precisely as follows: The function $w(z)$ is called typically real in $|z| < 1$, if it is real for real $z$ and satisfies in the remaining points of the unit circle the conditions: $\text{Im} (w(z)) \cdot \text{Im}(z)$, and $\text{Re}(w(z)) \cdot \text{Re}(z) > 0$.

In other words, we say $w(z) \in S_{VI}$ for $|z| < 1$, if $w(z)$ is regular in $|z| < 1$, is real on the real axis and if $\text{Im} (w(z)) = 0$ for $|z| < 1$ only when $z$ is real. As an example we can represent it as $w(z) = z + a_1 z^3 + a_2 z^5 + \ldots + a_n z^{2n+1} + \ldots$. 
rotation of the velocity vector \( izw'(z) \) is positive or negative.

Study [44] and R. Nevanlinna (see [31, 205]) have obtained for \( |z| < r \) the inequality

\[
\text{Re} \ (zw''/w') > 2r(r-2)/(1-r^2),
\]

where the right hand side is \( \geq -1 \), when \( r \) does not exceed the smaller root of the equation \( r^2-4r+1 = 0 \).

Bieberbach [7] has also defined the boundary limits of \( K \) by means of the inequality

\[
\text{Re} \ (zw''/w') \geq 2r(r-2)/(1-r^2),
\]

and so

\[
(1.16) \quad 1 + \text{Re} \ (zw''/w') \geq (1-4r+r^2)/(1-r^2),
\]

with the conclusion that for \( |z| < 2 - \sqrt{3} \) the expression on the right hand side of (1.6) is positive. Therefore in every schlicht mapping of \( |z| < 1 \) every circle \( |z| < 2 - \sqrt{3} \) is mapped convex (the class \( S_{II} \)).

Hence the lower bound for \( K \) follows from (1.15) as

\[
K \geq (1-4r+r^2) (1-r)/r (1+r), \quad 0 < r < 1.
\]

Miroshnitchenko [24] has evaluated the exact lower limit for \( K \) for the functions (1.3) defined in the ring domain \( 2 - \sqrt{3} \leq |z| < 1 \).

Zmorovitch [48] has also found exact lower and upper limits of \( K \) for the subclass of functions (1.3), belonging to the class \( S_{II} \), and represented by

\[
(1.17) \quad w = \phi(z) = z + c_0 + \sum_{n=1}^{\infty} c_n z^{-n},
\]

which are regular in the domain \( |z| > 1 \), except the simple pole \( z=\infty \).

In the paper (A) we have considered the functions (1.4) of the class \( S_{II} \), and by applying a formula of Study [43, 14] and following the method suggested by Gronwall [13] and Zmorovitch [48] we have

10) See also L. Bieberbach [6].
11) This formula states that

\[
K = \frac{1}{|z|^1 |w|^1} \text{Re} \ (1 + zw''/w'),
\]

\[
\text{Im} \ (z''/z') \neq 0.
\]
found the best possible bounds for $K$ as follows:

$$K \geq \begin{cases} 
\frac{1-4r+r^2}{r} \frac{\tau(r)}{e} & \text{for } r < 1, \\
\frac{(1-r)^2}{r} \frac{\tau(r)}{2} & \text{for } r \geq 0,
\end{cases}$$

$$K \leq \begin{cases} 
\frac{2^3 \sqrt{r}}{(1+\tau)(\tau(r))^{3/2}} \frac{(1-\alpha)\tau(r)}{e} & \text{for } r \leq r_0 < 1, \\
\frac{1+r}{r} \frac{\tau(r)}{e} & \text{for } r_0 < 1, \\
\frac{1+4r+r^2}{r} \frac{\tau(r)}{e} & \text{for } 0 \leq r \leq r_0,
\end{cases}$$

where

$$\tau(r) = 2 \log \frac{1+r}{1-r}$$

$$\alpha = \frac{3}{2} \left\{ \frac{1}{\tau(r)} - \frac{(1-r)^2}{4r} \right\},$$

and $r_0$ is the root of the equation $\tau(r) = 4r/(1+r)^2$, $0 \leq r \leq r_0 \leq 1$.

We have shown in Appendix 1 that $\tau(r)$ represents a homogeneous Legendre equation of second kind and zero degree in $r$.

Further, for the functions (1.6) we find in general the following result:

$$\frac{1-2(p+1)r^p+r^{2p}}{r} \cdot \frac{(1+r^p)^{2/p}}{(1-r^p)^{2/p}} \leq K \leq \frac{1+2(p+1)r^p+r^{2p}}{r} \cdot \frac{(1-r^p)^{2/p}}{(1+r^p)^{2/p}}, \quad 0 < r < 1.$$ 

Now we define the orthogonal trajectories of $w = w(z)$ as images of those radii of the circle $|z| < 1$ onto the convex domains, which intersect there orthogonally. By applying another formula of Study [43,13] we obtain for the functions (1.6) of the class $S_{II}$ the following result on the measure of curvature $K_r$ of these orthogonal trajectories:

12) This formula states that $K = \frac{1}{Iz''/z'''} \Im \left( \frac{vw'''}{w'} \right)$, \Im \left( \frac{z''/z'}{z''/z'} \right) = 0.
\[ 0 \leq K_r \leq \frac{2 r^{p-1} \sin \psi}{(1 - 2 r^p \cos \psi + r^{2p})^{1-1/p}}, \]

where

\[ \cos \psi = \frac{p(1+r^{2p}) - \sqrt{p^2 (1+r^{2p})^2 - 16(p-1)r^{2p}}}{4r^p}, \quad 0 < r < 1. \]

In particular, for \( p = 1 \), i.e. for the functions (1.4), we have \( \Psi = \pi/2 \) and \( 0 \leq K_r \leq 2 \).

1.9. The Faber theory in the complex plane.

We consider a schlicht analytic function \( w = \phi(z) \), which has in the neighbourhood of \( z = \infty \) a development of the form (1.17). Then by the Faber polynomials \( \Phi_n(z) \) of the n-th degree with respect to \( w \) we understand the totality of magnitudes with non-negative powers of \( z \) in the Laurent series of \( \phi^n(z) \) in the neighbourhood of the point at infinity, so that

\[ \Phi_n(z) = z^n + \sum_{v=1}^{\infty} c_{nv} z^{-v}. \]

The existence and uniqueness of all the Faber polynomials with respect to the given function \( w = \phi(z) \) is easily shown by recursion 13).

In the paper (B) we have examined a question, suggested by Lokhin[22], on the representation of analytic functions, regular in a domain of finite connectivity, by means of the Faber polynomials. The question is: What is the representation of such functions inside and outside their regularity domain? Solution of this problem comprises the contents of the so-called Faber theory in the complex plane.

The outline of the solution is as follows: Let \( G \) be a domain of finite connectivity with contours \( C_i \) (\( i = 0,1,2,\ldots \)) which are assumed to be closed analytic curves nonreducible to a point. We shall denote the total boundary of \( G \) by \( \partial G \).

13) See Lokhin[22], Schiffer[37], Schur[38], Szego[45,363].
We know that a regular analytic function \( f(z) \) can be represented in \( G \) by a series with the Faber polynomials as
\[
(1.18) \quad f(z) = \sum_{n=1}^{\infty} A_n \Phi_n(z),
\]
where
\[
(1.19) \quad A_n = \frac{1}{2\pi i} \int \frac{f(z(w))}{w^{n+1}} \, dw, \quad n = 0, 1, 2, \ldots,
\]
z\( \Phi \) being the inverse function of \( w = \Phi(z) \) and the integration extended over a closed analytic curve around the origin. If \( C_{r_0}, r_0 \leq r \), be the first of the closed analytic curves, in whose closure \( f(z) \) is regular analytic, then the series (1.18) will converge inside and diverge outside \( C_{r_0} \). But if \( f(z) \) were continued analytically across \( C_{r_0} \), then naturally the summation of the series (1.18) outside \( C_{r_0} \) would arise. We have followed the method suggested by Dienes [10,308] which is based on the analytic continuation of \( f(z) \) to the Mittag-Leffler's star domain \( M \).

Let \( D \subset G \) be the regularity domain of \( f(z) \). Then there holds the following theorem on the representation of \( f(z) \) inside \( D \):

The regular analytic function \( f(z) \) can be uniquely represented inside its regularity domain \( D \) by means of an auxiliary summatory function \( \Psi(t) \), \( t = \Phi(\zeta) \), as
\[
(1.20) \quad f(z) = \frac{1}{2\pi i} \int_{C_{r_1}} \frac{\Psi(\Phi(\zeta))}{\zeta - z} \, d\zeta,
\]
where
\[
(1.21) \quad \Psi(t) = \sum_{n=0}^{\infty} A_n t^n.
\]
Since the choice of the closed analytic curve \( C_{r_1} \) is arbitrary, the relation (1.20) represents \( f(z) \) uniquely inside \( D \).

The relations (1.20) and (1.21) make the summation of the series (1.18) outside \( D \) possible, since the analytic continuation of \( f(z) \) across the boundary of \( D \) gives rise to \( M \) with respect to the origin of coordinates of \( G \). Hence we have the following result on the representation of \( f(z) \) outside \( D \):

Inside any closed domain \( \bar{M}_0 \subset M \) there holds
(1.22) \[ f(z) = \lim_{n \to \infty} \sum_{v=0}^{n} A_k \Phi_v(z), \]
where \( k \) denote complex numbers independent of \( f(z) \) and \( \Phi(z) \). The series on the right hand side of (1.22) converges uniformly in \( M_0 \).

The only defect of all representations of the form (1.22) is that none of them converges in the whole star domain \( M \) but only in closed domains \( M_0 \subset M \), which consequently tend to \( M \) if the arbitrary closed analytic curve \( C_0 \) is taken very close to the boundary of \( M \).

We can, however, represent \( f(z) \) in the whole star domain \( M \) by considering the absolute convergence of the series (1.22) in \( M \), so that we find that \( f(z) \) has the form

(1.23) \[ f(z) = \lim_{n \to \infty} \sum_{v=0}^{n} A_n k \Phi_v(z) \]
in the whole \( M \), since the series on the right in (1.23) converges absolutely in the whole \( M \).

We can, however, use an alternative method, independent of that of Dienes [10], for effecting the analytic continuation of \( f(z) \) to the star domain \( M \) with respect to \( G \) as follows: We consider all kinds of orthogonal trajectories to \( C_0 \), so that \( f(z) \) is continued analytically across \( G \) along each one of these orthogonal trajectories. The totality of sections of these trajectories together with the domain \( G \) is called the star \( M \) of \( f(z) \) with respect to the domain \( G \).

The general star \( M_r \) of the function \( f(z) \) with respect to \( G \) is the union of the star \( M \) of \( f(z) \) with respect to \( G \) and the star of \( f(z) \) with respect to the domain bounded by \( C \). Thus \( M_\infty \) will coincide with the Star of \( f(z) \) with respect to \( G \), if \( G \) contains \( z = \infty \) as an interior point. Further, if \( G \) be a circle, the star of \( f(z) \) is the Mittag-Leffler rectangular domain of \( f(z) \) with respect to circular \( G \). Thus we have proved in Appendix 2 the following result on the representation of \( f(z) \) outside its regularity domain: In the general star \( M_r \), there holds

(1.24) \[ f(z) = \sum_{n=0}^{\infty} A_n \gamma_n \Phi_n(z), \]
where \( \gamma_n > 0, n = 0, 1, 2, \ldots \), denotes a sequence of positive numbers.

A comparison of the representations (1.23) and (1.24) shows that the series on the right hand side in (1.24) also converges absolutely in the whole star \( M_r \).

\[ \sum \]

\section{2. p-valent functions in the unit circle}

2.1. We understand by regular p-valent functions in the unit circle the functions which take there no value more than p times, so that there exist \( p \) values which these functions take in the unit circle exactly \( p \) times. Thus the functions (1.1):

\[ w = w(z) = \sum_{n=0}^{\infty} a_n z^n, \]

regular analytic for \( 0 < |z| = r < 1 \), are said to be p-valent, if \( w = w(z) \) has atmost \( p \) zeros in the unit circle.

Evaluations for p-valent functions began to be studied almost after Löwner's classical approach to the theory of schlicht functions in the unit circle. The first significant work in this direction was compatible by Hardy and Littlewood[14] with the establishment of the evaluations:

\[ |w(z)| < A(p) \frac{p}{(1 - |z|)} , \quad |z| < 1, \]

and

\[ |a_n| < A_1(p) \frac{p_n}{|z|}, \quad n = p+1, p+2, \ldots, \]

where \( A(p), A_1(p), \beta(p) \) and \( \beta_1(p) \) are constants depending only on \( p \), and \( p = \max \{ |a_1|, \ldots, |a_p| \} \). Later on Cartwright[9] has shown that for p-valent functions(1.1)

\[ M(r,w) = \max_{|z|=r} |w(z)| < A(p) \frac{2}{(1-r)^{2p}}, 0 < r < 1. \]

Further, Biernacki[8] has established the following inequalities:

\[ I(r,w) = \frac{1}{2\pi} \int_{0}^{2\pi} |w(re^{i\theta})| d\theta < A(p) \frac{2}{(1-r)^{1-2p}}. \]
The function

\[ w(z) = z^p / (1-z)^{2p} \]

which takes every value, except those on the real axis, between \((-1)^p 2^p\) and \((-1)^p \infty\) exactly \(p\) times, shows that the exponent \(2p\) in (2.1) cannot be replaced by any smaller number.

Hayman [15] has defined a positive constant \(\kappa\) by means of the relation

\[ \kappa = \lim_{r \to 1} (1-r)^{2p} M(r,w) > 0, \quad r < 1, \]

and called it the limit constant of the \(p\)-valent function \(w(z)\); in fact, this definition of \(\kappa\) is a direct consequence from his following result:

\[ M(r,w) \sim \kappa / (1-r)^{2p} \quad \text{as} \quad r \to 1. \]

Thus he has modified the above results of Biernacki as follows:

\[ \lim_{r \to 1} (1-r)^{2p-1} I(r,w) = \kappa / \Gamma(p-1/2) / 2 \Gamma(1/2) \Gamma(p) \]

for \(p > 1/2\), and

\[ \lim_{n \to \infty} a_n / n^{2p-1} = \kappa / \Gamma(2p) \quad \text{for} \quad p > 1/4 . \]

Thus we have directly from (2.1) and (2.3) the following corollary:

There holds for the limit constant \(\kappa\) the asymptotic relation:

\[ \kappa \sim A(p) p . \]

2.2. The coefficient problem. The relations (2.2) and (2.4), proved by Biernacki and Hayman respectively, solve the coefficient problem of \(p\)-valent functions in the unit circle. A similar problem, developed and solved in the paper (C), arises as follows: Let \(w(re^{i\theta}) = u(r,\theta) + iv(r,\theta)\), \(z = re^{i\theta}, 0 < r < 1\). Assuming \(v(r,\theta) \neq 0\), let \(z = re^{i\theta_0}\). Thus \(v(r,\theta)\) is a continuous function of \(\theta\) from any point \(z_0\), if \(z = re^{i\theta}\) traverse the circle \(|z| = r\) once, starting from \(z_0\). As \(z\) makes one complete rotation around the circle \(|z| = r\) from \(z_0\) back again to \(z_0\), so \(v(r,\theta)\) has either a constant sign or changes sign an even number of times, if the number of changes in signs is assumed to be finite. If there is an interval \(0 < r_0 < r < 1\),
in which \( v(r, \theta) \) changes its signs at most \( 2p \) times on \( |z| = r \) for any value of \( r \) of the given interval, then we can consider instead of \( w(z) \) the function

\[
F(z) = \sum_{v=-p}^{+p} A_v z^v,
\]

which is regular in \( 0 < r < 1 \) and has a pole of order at most \( p \) in the origin. We further assume that \( \text{Re} F(r e^{i\theta}) \geq 0 \) and \( p > 0 \).

Then we prove the following result on the coefficients of \( F(z) \):

\[
|A_n| \leq \left( 2^{p-1} \left\{ 1 + \frac{(2p-1) \Gamma(p-1/2)}{2p \Gamma(p) \Gamma(1/2)} \right\} \right)^{1/2} \text{ for } p > 1/2.
\]

As a corollary we note that in case \( p = 1/2, |A_n| \leq n \), and thus in these cases we can connect the coefficients \( |a_n| \) and \( |A_n| \) by means of (2.2) and (2.4).

2.3. Mean valency of analytic functions. We shall say that a regular analytic function \( w(z) \) is mean \( p \)-valent in a domain \( D \), if the average number \( p(R) \) of roots of the equation \( w = w(z) \), when \( w \) ranges over the circumference \( |w| = R \), always satisfies the inequality \( p(R) \leq p \).

Here the total length of the arcs in the Riemann surface of \( w = w(z) \) lying over the circle \( |w| = R \) is equal to \( 2\pi p(R) \). Alternatively,

\[
p(R) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\theta}) \, d\theta,
\]

where \( n(Re^{i\theta}) \) denotes the number of roots of the equation \( w(z) = Re^{i\theta} \) in \( D \). Further, we consider those analytic functions \( w(z) \) which are analytically continuable with one-valued modulus throughout the ring domain \( r_0 < |z| < 1 \). If \( w_2(z) \) is the branch obtained from \( w_1(z) \) by continuing it analytically once around the said ring, then \( |w_2/w_1| = 1 \), and therefore by the maximum modulus theorem we shall have \( w_2(z) = w_1(z) e^{i\lambda} \), where \( \lambda \) is a real constant. Thus \( w(z)/z \) remains single-valued in that ring domain and possesses a Laurent series development.

14) Because if \( p = 0 \), then \( v(r,\theta) \) would be of constant sign inside the unit circle.
\( w(z) = z^\lambda \sum_{n=0}^{\infty} a_n z^n \), \( r_0 < |z| < 1 \).

We define \( p(R) \) as above for a fixed branch of \( w(z) \) in the ring domain cut along a radial slit. Since replacement of \( w(z) \) by a different branch gives rise to a rotation in the \( w \)-plane, so \( p(R) \) is independent of the radial cut and the particular branch taken, so that the above definition of mean \( p \)-valency of analytic functions is meaningful.

If \( D \) is the unit circle, let us denote by \( V_p \) the class of mean \( p \)-valent functions

\[ w_p(z) = z^p (1+a_1 z+a_2 z^2+\ldots), \quad p > 1, \quad 0 < |z| = r < 1. \]

Then \( w_p \neq 0 \) in \( 0 < |z| < 1 \), since the immediate neighbourhood of \( z = 0 \) contributes \( p \) to \( p(R) \) for small positive \( R \) (see [15,105]). Thus especially

\[ w_1(z) = w(z) = z (1+a_1 z/p + \ldots) = z (1+b_1 z + \ldots) \]

denotes the class of mean schlicht functions in the unit circle, since \( w_1(z) \) is obviously single-valued. It is not difficult to verify that the class \( V_1 \) coincides with the class \( S_{\text{III}} \) (§1.2).

Further, if \( \alpha \) is the limit constant of \( w_1(z) \), the limit constant of \( w_p(z) \) is \( \alpha_p \leq 1 \), and

\[ \lim_{n \to \infty} \frac{|a_n|/n^{2p-1}}{\Gamma(2p)} = \alpha_p / \Gamma(2p) \leq 1 / \Gamma(2p) \]

with equality only for the function \( w^*_p(z) = z^p/(1-e^{i\lambda} z)^{2p} \), \( \lambda \) real, which is in fact the \( p \)-th power of the g.s.f.'s. (1.7).

Now we define the mean value of the function \( w(z) \in V_p \) as

\[ I(r,w_p) = \frac{1}{2\pi r} \int_{|z|=1} \log |w_p(z)| \, d \arg w_p(z). \]

We prove in the paper (D) the following result:

Let \( w_p(z) \in V_p \) in \( 0 < |z| = r < 1 \). Then there holds the inequality

\[ I(r,w_p) \leq A \left\{ \log M(r) + p \log \frac{r(1-r)}{1+r} \right\}, \]

where \( A \) is a positive constant depending on \( r \) and \( M(r) = \max \{1 + a_1 z + a_2 z^2 + \ldots\} \). Equality holds here for the function

\[ w^*_p(z) = z^p (1-e^{i\lambda} z)^{-2p} \in V_p, \quad \lambda \text{ real.} \]
We generalise the class \( V_p \) as follows [15,106]: Let \( p, k \) be positive integers, \( 1 \leq k < 4p \), and let
\[
\omega_{p,k}(z) = z^p (1 + a_1 z^k + a_2 z^{2k} + \cdots) \in V_p
\]
be mean \( p \)-valent in \( |z| < 1 \). Then by means of the transformation
\[
z = \xi
\]
we get a class of functions
\[
V_{p/k} : \omega_{p,k}(z) = \xi^{1/k} (1 + a_1 \xi + a_2 \xi^2 + \cdots),
\]
which are mean \( p/k \)-valent in \( 0 < |\xi| < 1 \).

We define the mean value
\[
I(r, \omega_{p,k}) = \frac{k}{2\pi \log r} \int_{|z|=1} \log |\omega_{p,k}(z)| \, d \arg \omega_{p,k}(z).
\]
Then there holds the following inequality:
\[
(2.7) \quad I(r, \omega_{p,k}) \leq A \left\{ \log M(r) + \frac{1}{k} \log \frac{r^k(1-r^k)}{1+r^k} \right\},
\]
where \( A \) is again a positive constant depending on \( r \) and \( M(r) = \max_{|\xi|=1} \{1 + a_1 \xi + a_2 \xi^2 + \cdots\} \). Sign of equality holds here for the functions \( \omega^*(z) = z^p (1 - e^{i\lambda} z)^{-2p/k} \), \( \lambda \) real.

We remark that the inequalities (2.6) and (2.7), proved in the paper (D), are significant in giving a close link between the maximum modulus of a schlicht function in the unit circle, which is usually easy to deal with, and the mean of a \( p \)-valent function in the unit circle, which is frequently less tractable. These inequalities do not, however, give the best possible results which are possible e.g. by applying highly developed variational technique; but our results are interesting only on account of the elementary method used in their proof.

§ 3. Entire and meromorphic functions.

3.1. Derivatives of entire functions. Let
\[
f(z) = \sum_{n=0}^{\infty} c_n z^n
\]
be an entire function of order \( \rho \) and lower order \( \lambda \) for \( c_n \geq 0 \), \( 0 < |z| = r < R \leq \infty \). Then Valiron [46, 103-5] has given asymptotic formulas for the growth of derivatives of \( f(z) \) for all ordinary values of \( r < R \) as
Another result in this direction is due to Vijayaraghvan\cite{47}, which states that
\[ M'(r) > \frac{r \log r}{\log M(r)} \]
for \( r < r_0 \), where \( r_0 \) is a number depending on \( f(z) \). Shah\cite{39} has further proved a number of results:
\[ \lim_{r \to 0} \frac{\log \left( \frac{M'(r)}{M(r)} \right)}{\log r} = \frac{\eta}{\lambda}, \quad 0 < \eta \leq \alpha, \quad 0 < \lambda \leq \alpha, \]
and in general
\[ \lim_{r \to 0} \frac{M'(r)}{M(r)} \leq \lim_{r \to 0} \frac{N(r)}{r} \leq \lim_{r \to 0} \frac{N(r)/r}{M'(r)/M(r)}, \]
\[ \lim_{r \to 0} \frac{M^{(j)}(r)}{M(r)} \leq \lim_{r \to 0} \frac{N(r)}{r} \leq \lim_{r \to 0} \frac{N(r)/r}{M^{(j)}(r)/M(r)}, \]
for \( j = 1, 2, \ldots \).

3.2. Growth of derivatives of admissible functions.

Hayman\cite{16} has defined admissible functions as follows: The class of functions (3.1) is called admissible in \( |z| < R \), if they are regular for \( |z| < R \), \( 0 < R < \infty \), are real for real \( z \), and if \( M(r) = f(r) \) for all \( r < R \). Now if we define a function
\[ \alpha(r) = \frac{d \log M(r)}{d \log r} = \frac{M'(r)}{M(r)} = \frac{f(r)}{f'(r)} \]
for \( r < R \), then according to a theorem of Hayman\cite{16,69,Cor.III} we have
\[ N(r) \sim \alpha(r). \]

In view of this asymptotic relation we find that Valiron's result (3.2) reduces in the case of admissible entire functions to
\[ \frac{M^{(j)}(r)}{M(r)} \sim \frac{\alpha(r)}{r^j}, \]
which holds for a sequence of values \( r < R \) and for any fixed positive integer \( j \) as \( r \to \infty \). We also note that on account of the condition \( M(r) = f(r) \) for admissible functions, the relation (3.5) coincides with Hayman's definition of admissibility of \( f(z) \) in \( |z| < R \) the condition \( c_0 \geq 0 \) is not necessary to show that \( M(r) = f(r) \).
with another theorem of Hayman [16,76, theo. III] on the growth of derivatives of admissible functions, according to which
\[ f^{(j)}(r) \sim f(r) \left\{ \frac{\zeta(r)}{r} \right\}^j \quad \text{as } r \to \infty. \]

Further, the relation (3.5) means that
\[ \left\{ \frac{\zeta(r)}{r} \right\}^j M(r) (1 - \varepsilon_j(r)) \leq M(r) \leq \left\{ \frac{\zeta(r)}{r} \right\}^j M(r) (1 + \varepsilon_j(r)), \]
where \( 0 \leq \varepsilon_j(r) \to 0 \) as \( r \to \infty \).

In the paper (1) we have proved the following results:

Let \( f(z) \) be admissible in \( |z| < R \). Then for a sequence of values \( r < R \) and any fixed positive integer \( j \) we have
\[ f^{(j)}(r) = \left\{ \frac{\zeta(r)}{r} \right\}^j f(r) (1 + \eta_j(r)) \quad \text{as } r \to \infty, \]
with \( \eta_j(r) = 0(\zeta(r)) \), \( \varepsilon > 0 \).

Since \( M(r) = f(r) \) for admissible functions, we obtain from (3.5) and (3.6)
\[ M(r) \leq \left\{ \frac{\zeta(r)}{r} \right\}^j f(r) (1 + \eta_j(r)) = f^{(j)}(r). \]

Again, if \( \zeta^{(j)}(r) \) denotes the \( j \)-th derivative of \( \zeta(r) \), then by successive differentiation on both sides of (3.3) we find that
\[ \zeta^{(j)}(r) = \zeta^{(j)}(r) (1 + \eta_j(r)) \]
with \( \eta_j(r) = 0(\zeta(r)) \), \( \varepsilon > 0 \).

Also in view of the asymptotic relation (3.4) and the result (3.7) we have
\[ N^{(j)}(r) \sim \zeta^{(j)}(r). \]

A number of applications of the above results to the theory of entire and meromorphic functions is given in Appendix 3.

§ 4. A class of bounded harmonic functions.

4.1. It is well known that if \( f(z) \) is a single-valued analytic function in a domain \( G \) containing \( z = \infty \) as an interior point, then logarithm of the absolute value of \( f(z) \) is a harmonic function, except for logarithmic poles. The converse statement that every harmonic function can be represented as \( \log |f(z)| \) is, however, not true (see [1, 1]
Let \( g(z, \zeta) \) denote the Green's function of \( G \) with finite positive pole \( \zeta \) and \( \overline{g} \) its harmonic conjugate. Then

\[
(4.1) \quad f(z) = f(z, \zeta) = \exp \left[ - \sum_{v=1}^{\infty} \left\{ g(z, \zeta_v) - i\overline{g}(z, \zeta_v) \right\} \right]
\]

will be a single-valued analytic function in \( G \), have there zeros \( \zeta_1, \zeta_2, \ldots \) which may be finite (but we shall a priori assume here that these zeros are enumerably infinite), and possess a uniquely determined limit value on the boundary of \( G \). Thus

\[
(4.2) \quad u(z) = \log |f(z)| = \text{Re} \left( \log f(z) \right) = - \sum_{v=1}^{\infty} g(z, \zeta_v), \quad \zeta_\infty = \infty,
\]

will represent a bounded harmonic function in \( G \), except for logarithmic poles. Totality of all harmonic functions \( u(z) \) makes a class which we have considered in the paper (F).

4.2. Variation problem. If the poles \( \zeta_v \) be shifted to \( \zeta_v^* (v = 1, 2, \ldots) \) infinitesimally close to their original positions, such that the Green's function with positive finite pole \( \zeta^* \) becomes \( g(z^*, \zeta^*) \), then there arises a variation in the Green's function which we define precisely by the formula

\[
\partial g = \sum_{v=1}^{\infty} \left\{ g(z, \zeta_v) - g(z^*, \zeta_v^*) \right\}, \quad z, z^* \in G.
\]

This variation is positive or negative according as \( \partial g \) is 0, and it would not exist if \( \partial g = 0 \).

We note that the star-marked letters denote here as elsewhere the points induced by variation.

---

16) Since the periods of \( f(z) \) are integral multiples of \( 2\pi i \), so it is free of periods along the boundary of \( G \), and therefore poles of the Green's functions in \( G \) become simple zeros of \( f(z) \).

17) The representation (4.2) of \( u(z) \) as a linear combination of all the Green's functions of \( G \), which are harmonic functions bounded in \( G \) outside their poles, follows directly from (4.1), or from Ahlfors [1, 3-7] if we assume that the number of zeros of \( f(z) \) is enumerably infinite.
In order to find conditions under which the variation in the Green's functions of \( G \) would exist, we have the following result:

A positive variation for the Green's function of \( G \) exists, if

\[
\sum \{ g(z, \zeta_v) - g(z, \zeta_v^*) \} > 0
\]

and

\[
\sum \{ \overline{g(z, \zeta_v)} - \overline{g(z^*, \zeta_v)} \} > 0.
\]

We note that in these conditions the poles \( \zeta_v \) and \( \zeta_v^* \) appear as parameters rather than independent variables. If we introduce the Neumann's function \( N(z; \xi, \eta) \) of \( G \) with the logarithmic poles of opposite signs in the points \( \xi, \eta \), then we have shown that the variation for the Green's functions leads to the Neumann's function in \( G \): A positive variation for the Neumann's functions of \( G \) exists, if

\[
\sum \{ \overline{N(\zeta_v; \xi_v, \eta_v)} - \overline{N(\zeta_v^*; \xi_v, \eta_v)} \} > 0,
\]

where \( \overline{N} \) denotes the harmonic conjugate of the Neumann's function \( N \). We find that in the condition (4.5) the poles \( \zeta_v, \zeta_v^* \) of the Green's functions of \( G \) appear as independent complex variables.

Further, the variation for the bounded harmonic function \( u(z) \), represented by (4.2), can be likewise defined by

\[
\partial u = u(z^*) - u(z) = \sum \{ g(z, \zeta_v) - g(z^*, \zeta_v^*) \}, \ z, z^* \in G.
\]

Then the existence of the relations \( \partial u > < 0 \) would mean

\[
\partial g(z, \zeta) = \sum \{ g(z, \zeta_v) - g(z^*, \zeta_v^*) \} > < 0,
\]

and the variation problem reduces to the existence of variation for the Green's functions, so that the above conditions are in general valid for the class of bounded harmonic functions \( u(z) \).

§ 5. A short survey of the work.

In the previous sections we have discussed the theory and results connected with the six papers \((A)-(F)\) and three appendices 1-3 on schlicht, \( p \)-valent, entire, meromorphic and bounded harmonic functions. The remaining work deals with the following topics:

5.1. In the paper \((G)\) we have proved three theorems on zeros of a
class of polynomials

\[ P(z) = \sum_{v=0}^{n} a_v z^v \]

with real or complex coefficients, such that for \( n \geq 2 \)

\[ |a_0| + |a_1| + \cdots + |a_{n-1}| \leq n |a_n|. \]

Then at least one zero of \( P(z) \) lies outside the circle

\[ |z| = r \leq \left\{ 2 \frac{(n+1)}{la_n/a_0} \right\}^{-1/n}, \quad r > 1/2. \]

Further, if for the polynomial (5.1)

\[ \min_{v=0}^{n-1} |a_v| > 1, \quad \max_{v=0}^{n-1} |a_v| > |a_n|, \]

then

\[ n(R/k) \leq \log \left\{ \left( \frac{(n+1)}{la_n/a_0} \right)^{\frac{R}{n}} \right\} \quad \log k \]

where \( R = \max \{ |a_{n-1}/a_n|, |a_{n-2}/a_n|^{1/2}, \cdots \} \), and \( n(r) \) denotes the number of zeros of \( P(z) \) in \( |z| < r \). Again, if the coefficients of

the polynomial (5.1) are such that \( |a_{n-1}/a_n| > \max \{ |a_{n-2}/a_n|^{1/2}, \)

\( |a_{n-3}/a_n|^{1/3}, \cdots \} \), then all the zeros of \( P(z) \) lie outside the ring

domain \( 1/2 |a_n/a_0| < |z| < 2 |a_{n-1}/a_n| \).

Singh[41] has also proved similar results by using the Bernstein lemma, but we have made use of the Jensen theorem (see e.g. [46, 48]) to prove our results.

5.2. In the paper (H) we have proved two results, one on giving an alternative proof of a well-known result of Valiron[46, 32] which states that

\[ m(r) < M(r) < m(r) \left\{ 2 N(r + r/N(r)) + 1 \right\}, \]

the terms \( m(r), M(r) \) and \( N(r) \) having their meaning as in [46], and the other result on maximum term \( t(r) \) of a quasianalytic series in close link to the maximum modulus \( M(r) \) of an entire series associated to the given quasianalytic series, according to which

\[ \log t(r) \geq k \log M(r), \quad 0 < k \leq 1. \]

If \( k = 1/\log 2p < 1 \) for \( r > r_0 \), where \( p = N(r + r/N(r)) + 1 \), then, being known that \( \log M(r) \geq \log t(r) \), we obtain from (5.2) the inequalities

\[ \log t(r) \leq \log M(r) \leq \log t(r) + \log (2p-1). \]
5.3. In the symposium article (A) we have discussed the three cases of conformal mapping of the universal covering surface $G$ of a $p$-fold connected domain ($p \geq 2$) and thus defined the type of $G$. The three cases are: (i) $p = 2$, (ii) $p = 3$, and (iii) $p > 3$. In the first case $G$ is of the parabolic type, in the second it is of the hyperbolic type and in the last case it is again of the hyperbolic type.

5.4. In the expository article (B) we have given an up-to-date survey of the developments in the theory of Riemann surfaces, emphasising on the recent trends in the study of various connected problems, some of which still lay unsolved. The survey is self-contained and comprehensive, detailing the theory from the early period of Riemann's innovation down to the year 1957. We have also suggested at places directions in which the theory can yet be developed.

BIBLIOGRAPHY


