CHAPTER VI
ON (K,1,\(\alpha\)) METHOD OF SUMMABILITY

6.1 Let \( \Sigma a_n \) be a given infinite series with \( s_n \) as its \( n \)-th partial sum. We denote by \( S_n^k \) the \( n \)-th Cesàro sum of order \( k \). The series \( \Sigma a_n \) is said to be summable \((c,k)\) to the sum \( s \), if \( \lim_{n \to \infty} \frac{S_n^k}{A_n^k} = s \), and harmonic summable to \( s \), if \( t_n \to s \), as \( n \to \infty \), where \( t_n = \frac{T_n}{\log(n+1)} \).

\[ A_n = \left( \frac{n+k}{n} \right), \quad k > -1. \]

The series \( \Sigma a_n \) is said to be summable \((K,1)\) to the sum \( s \), if the series

\[ \sum_{n=1}^{\infty} a_n \frac{\sin nu}{\pi} \int_0^\pi \frac{1}{\tan \frac{u}{2}} \, du, \]

converges in some interval \( 0 < t < t_0 \), to a function \( f(t) \) and \( \lim_{t \to +0} f(t) = s \).

Hirokawa has recently extended the above definition of Zygmund, by introducing a parameter \( \alpha \). According to him, a series \( \Sigma a_n \) is said to be summable \((K,1,\alpha)\) to \( s \), if the series

\[ F(\alpha,t) = B_{\alpha}^{-1} t^{\alpha+1} \sum_{n=1}^{\infty} s_n \frac{\sin \pi x}{x} \int_0^\pi \frac{1}{\tan \frac{x}{2}} \, dx \]

1) Zygmund, A. [67]
2) Hirokawa, H. [25]
converges in some interval \( 0 < t < t_0 \) and \( \lim_{t \to +0} F(\alpha, t) = s \),

where

\[
\tilde{b}_x = \begin{cases} 
\pi/2 & \alpha = -1 \\
(\alpha+1)^{-1} \sin (\alpha+1)\pi/2 & -1 < \alpha < 0 \\
1 & \alpha = 0
\end{cases}
\]

The above method \((K, l, \alpha)\) reduces to \((K, l)\) for \( \alpha = -1 \).

It has been proved by Izumi that for Fourier series summability \((K, l)\) is equivalent to the summability \((R_1)\). Since it is known\(^2\) that for Fourier series summability \((R, l)\) and \((R_1)\) are mutually exclusive, it follows that in general, summability \((K, l)\) and \((R, l)\) are also independent of each other.

It may be remarked that Hirokawa\(^3\) has obtained a number of properties which are similar to that of summability \((R, l, \alpha)\) i.e. Riemann-Cesaro summability.

**Section A**

6.2 Concerning summability \((R, l, \alpha)\) we proved the following theorem in Chapter III.

**Theorem A**: If \( \Sigma a_n \) is harmonic summable and if

\[
(6.2.1) \quad W_n = \sum_{v=1}^{n} |T_v - T_{v-1}| = O(\log n), \quad n \to \infty,
\]

\[\]
then $\sum a_n$ is summable $(R, 1, \alpha)$ for $-1 \leq \alpha \leq 0$.

In theorem 1 of this section we shall show that this result remains valid, also for summability $(K, 1, \alpha)$.

Concerning the summability $(R, 1)$ Sunouchi\(^1\) proved the following theorem.

**Theorem B.** If

$$\sum_{v=1}^{n} S_v = O(n^\alpha)$$

and

$$\sum_{v=n}^{\infty} \frac{|a_v|}{v} = O(n^{-\alpha}), \quad 0 < \alpha < 1,$$

then the series $\sum a_n$ is summable $(R, 1)$ to zero.

This result was subsequently generalized by Hirokawa and Sunouchi\(^2\). Their result includes Theorem B for $\beta = 1$ and $\gamma = \alpha$. The result proved by them is as follows:

**Theorem C.** If

\begin{equation}
(6.2.2) \quad s_n = O(n^\gamma), \quad 0 < \gamma < \beta,
\end{equation}

and

\begin{equation}
(6.2.3) \quad \sum_{v=n}^{\infty} \frac{|a_v|}{v} = O(n^{-1+\delta}),
\end{equation}

where $0 < \delta < 1$, $\beta > 0$, and $\delta = 1 - \frac{\gamma}{\beta}$ then the series $\sum a_n$ is summable $(R, 1)$ to zero.

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1) Sunouchi, G. [53]
2) Hirokawa, H. and Sunouchi, G. [27]
Later on in 1954 Kanno extended Theorem C to the summability \((R,p)\) for integral values of \(p\) by establishing the following result.

**Theorem D:** Let \(p\) be a positive integer. Suppose that (6.2.2) holds and

\[
(6.2.4) \quad \frac{1}{\Sigma v=n} \frac{|a_v|}{v} = O(n^{-1} + \delta)
\]

where \(0 < \delta < 1\) and 

\[
\delta = \frac{p(\beta - \gamma)}{\beta + 1 - p}
\]

Then \(\Sigma a_n\) is summable \((R,p)\) to zero.

It is clear that Theorem D includes, as a special case, for \(p = 1\) the above theorem of Hirokawa and Sunouchi.

The above result of Kanno was subsequently partially generalized by Hirokawa who consider the summability \((R,p,\alpha)\), 

\(-1 \leq \alpha < 0\).

His result is given below.

**Theorem E:** Let \(p\) be a positive odd integer. If the condition (6.2.2) and (6.2.4) are satisfied then the series \(\Sigma a_n\) is summable \((R,p,\alpha)\) to zero where 

\(-1 \leq \alpha < 0\).

In Theorem 2 we have obtained the corresponding result for the summability \((K,1,\alpha)\), 

\(-1 \leq \alpha < 0\).

1) Kanno, K. [23]

2) Hirokawa, M. [23]
Our theorems are as follows:

**THEOREM 1.** Under the hypotheses of Theorem A, \( \sum a_n \) is summable \((K, l, \alpha)\), \(-1 \leq \alpha \leq 0\).

**THEOREM 2.** If \((6.2.2)\) and \((6.2.3)\) hold, then \( \sum a_n \) is summable \((K, l, \alpha)\) to zero for \(-1 \leq \alpha \leq 0\).

6.3 As in Chapter III we write

\[
\left( \sum_{n=0}^{\infty} p_n x^n \right)^{-1} = \left( \sum_{n=0}^{\infty} c_n x^n \right),
\]

then we have

\[
(6.3.2) \quad s_n = \sum_{v=1}^{n} c_{n-v} T_v
\]

and

\[
(6.3.3) \quad a_n = \sum_{v=1}^{n} c_{n-v} (T_v - T_{v-1}).
\]

The following estimates are well known\(^1\). \n
\[
(6.3.4) \quad \begin{cases} 
(1) \quad c_n = O \left( \frac{1}{n \log^2 n} \right) \\
(II) \quad d_n = \sum_{v=0}^{n} c_v = O \left( \frac{1}{\log n} \right).
\end{cases}
\]

We need the following lemmas for the proof of our theorems.

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1) Riesz, M. [48], Iyengar, K.S.K. [30]; Varshney, O.P.[59].
LEMMA 1. Let \[ J_n = \sum_{v=n}^{\infty} \frac{b_v}{v} \], where \( b_v = T_v - T_{v-1} \),

\( T_v = O(\log v) \) and

\[ \sum_{v=1}^{n} |T_v - T_{v-1}| = O(\log n), \quad n \to \infty. \]

Then

(6.3.5) \[ J_n = O(\log n/n), \quad n \to \infty. \]

and

(6.3.6) \[ J_n = \sum_{l=1}^{n} J_l = o(\log n), \quad n \to \infty. \]

LEMMA 2. Let \( \phi(n,t) = \int_{t}^{\pi} \frac{\sin nu}{\tan \frac{u}{2}} \, du \). Then

(6.3.7) \[ \phi(n,t) = O\left( \frac{1}{nt} \right), \]

and

(6.3.8) \[ \Delta^m \phi(n,t) = O\left( \frac{t^{m-1}}{n} \right), \quad \text{where } \Delta^m \phi(n,t) \]

denotes the \( m \)-th difference of \( \phi(n,t) \) with respect to \( n \)

and \( m \) is a non-negative integer.

PROOF. \[ \phi(n,t) = \int_{t}^{\pi} \frac{\sin nu}{\tan \frac{u}{2}} \, du \]

\[ = (\tan \frac{t}{2}) \int_{t}^{\xi} \sin nu \, du, \quad t < \xi < \pi \]

\[ = (\tan \frac{t}{2}) \left[ -\frac{\cos nu}{n} \right]_{t}^{\xi} \]

\[ = O\left( \frac{1}{nt} \right). \]

1) Varshney, O.P. [60 I].
Again

\[ \Delta \varphi(n,t) = \Delta \left[ \int \frac{\sin nu}{\tan u/2} \, du \right] \]

\[ = \int \frac{\sin nu - \sin (n+1)u}{\tan u/2} \, du \]

\[ = -\int \frac{2 \cos(n + \frac{1}{2})u \sin u/2}{\tan u/2} \, du \]

\[ = -\int \frac{(\cos (n+1)u + \cos nu)}{\tan u/2} \, du \]

\[ = t \left[ \frac{\sin (n+1)t}{(n+1)t} + \frac{\sin nt}{nt} \right] \]

Hence

\[ \Delta^m \varphi(n,t) = \Delta^{m-1} \Delta \varphi(n,t) = t \Delta^{m-1} \frac{\sin(n+1)t}{(n+1)t} \frac{\sin(nt)}{nt} \]

\[ = O \left( \frac{t^{m-1}}{n} \right), \text{ by virtue of the fact that} \]

\[ \Delta^m \left( \frac{\sin nt}{nt} \right)^p = O(n^{-p} t^{m-p-1}) \]

**LEMMA 3.** Let \( G_v(t) = t^{\alpha+1} \sum_{n=v}^{\alpha+1} A_{n-v} \varphi(n,t), -1 \leq \alpha \leq 0 \)

Then

(6.3.9) \( G_v(t) = O \left( v^{-1} \right), \)

and

(6.3.10) \( \Delta^k G_v(t) = O \left( \frac{t^k}{v^k} \right), \)

1) Obreschkoff, N. [43].
PROOF. The proof is analogous to that of a lemma of Hirokawa\(^1\).

Let \( G_v(t) = t^{\alpha+1} \left( \sum_{n=v}^{\infty} \frac{\alpha-1}{v+\rho+1} \right) = U_1 + U_2 \), say

\[
\lim_{n \to \infty} \frac{U_2}{t^{\alpha}} = \frac{\alpha-1}{n-v+\rho+1} \sum_{n=v+\rho+1}^{\infty} \frac{\alpha-1}{n-v+\rho+1} = O(t^{\alpha} \sum_{n=v+\rho+1}^{\infty} \frac{\alpha-1}{n-v+\rho+1})
\]

Now by (6.3.8) we have for \(-1 < \alpha < 0\),

\[
U_2 = t^{\alpha+1} \sum_{n=v+\rho+1}^{\infty} A_n \phi(n, t)
\]

and on applying Abel's transformation to \( U_1 \) we have

\[
U_1 = t^{\alpha+1} \sum_{n=0}^{\rho-1} A_n \phi(n+v, t)
\]

\[
= t^{\alpha+1} \sum_{n=0}^{\rho-1} A_n (n+v) + O(t^{\alpha+1} \sum_{n=0}^{\rho-1} A_n (n+v)^{-1} + O(\sqrt{v})
\]

Hence \( G_v(t) = O(v^{-1}) \) for \(-1 < \alpha < 0\). When \( \alpha = 0 \)

\( G_v(t) = t \phi(v, t) = O(1) \). Similarly we have the result when \( \alpha = -1 \).

1) Hirokawa, H. [43].
Now

\[ G_v(t) = t^{\alpha+1} \sum_{n=v}^{\infty} \frac{A^{-1}}{n-v} \Phi(n,t) \]

\[ = t^{\alpha+1} \sum_{n=0}^{\infty} \frac{A^{-1}}{n} \Phi(n+v,t) \]

hence

\[ \Delta^k G_v(t) = t^{\alpha+1} \sum_{n=0}^{\infty} \frac{A^{-1}}{n} \Delta^k \Phi(n+v,t) = O\left( \frac{t^k}{v^k} \right), \]

by using the method of proof of (6.3.9).

**LEMMA 4.** Let \( K_v(t) \equiv \sum_{n=v}^{\infty} G_n(t) \), then

\[ (6.3.11) \quad K_v(t) = O(v^{-1} t^{-1}). \]

**PROOF.** We have

\[ K_v(t) = \sum_{n=v}^{\infty} G_n(t) = t^{\alpha+1} \sum_{n=v}^{\infty} \sum_{k=n}^{\infty} \frac{A^{-1}}{k-n} \Phi(K,t) \]

\[ = t^{\alpha+1} \sum_{n=v}^{\infty} \sum_{r=0}^{\infty} \frac{A^{-1}}{r} \Phi(n+r,t) \]

\[ = t^{\alpha+1} \sum_{r=0}^{\infty} A^{-1} \sum_{n=v}^{\infty} \Phi(n+r,t) \]

The change of order of summation can be easily justified.

To prove the lemma, we first prove that

\[ \gamma_{v+r}(t) = \sum_{n=v}^{\infty} \Phi(n+r,t) = O((v+r)^{-1} t^{-2}) \]
we have
\[ \sum_{n=v}^{\infty} \phi(n+r,t) = \sum_{n=v}^{\infty} \int_{t}^{\tan \frac{x}{2}} \frac{\sin(n+r,x)}{\tan x} \, dx \]
\[ = \sum_{n=v}^{\infty} \frac{1}{\tan t/2} \int_{t}^{\tan t/2} \sin(n+r,x) \, dx, \quad t < t < \pi, \]
\[ = (\tan t/2)^{-1} \sum_{n=v}^{\infty} \left[ \frac{\cos(n+r,x)}{n+r} \right]_t^\xi \]
\[ = O\left( \frac{t^{-2}}{(v+r)} \right), \]
since
\[ \sum_{n=v}^{\infty} \frac{\cos nt}{n} = O\left( \frac{1}{nt} \right). \]

Now we write for \(-1 < \alpha < 0\)
\[ t^{\alpha+1} \sum_{r=0}^{\infty} A^r \gamma_{v+r}(t) = t^\alpha \sum_{r=0}^{\infty} A^{r+1} \gamma_{v+r}(t) = V_1 + V_2, \text{ say.} \]

We have
\[ V_2 = O\left( t^{\alpha+1} \sum_{r=0}^{\infty} \frac{1}{r+1} (v+r)^{-1} t^{-2} \right) \]
\[ = O\left( (v+\rho+1)^{-1} t^{\alpha-1} \rho^\alpha \right) \]
\[ = O\left( \frac{1}{vt} \right) \]
and
\[ V_1 = t^{\alpha+1} \sum_{r=0}^{\infty} A^r \Delta_r \gamma_{r+v}(t) = t^{\alpha+1} A^\alpha \gamma_{v+t}(t) \]
\[ = O\left( v^{-1} t^{-1} \right). \]

Hence (6.3,11) follows for \(-1 < \alpha < 0\). The result for \(\alpha = 0, -1\) is quite obvious.
LEMMA 5. If \( S_n = O(n) \), then we have

\[
\sum_{n=1}^{\infty} s_n O(n,t) = \sum_{n=1}^{\infty} s_n G_n(t),
\]

where

\[
G_n(t) = t^{\alpha+1} \sum_{v=n}^{\infty} A_{v-n} O(n,t) \quad (-1 \leq \alpha \leq 0).
\]

PROOF.

\[
\sum_{n=1}^{\infty} s_n O(n,t) = \sum_{n=1}^{\infty} \sum_{v=n}^{\infty} A_{v-n} O(n,t) = \sum_{k=1}^{\infty} s_k \sum_{n=k}^{\infty} A_{n-k} O(n,t) = \sum_{k=1}^{\infty} s_k G_k(t).
\]

Here we shall prove that the change of order of summation is justified. For this purpose it is sufficient to prove that for fixed \( t > 0 \)

\[
I_N = \sum_{k=1}^{N} \sum_{n=N+1}^{\infty} A_{n-k} O(n,t) = 0(1), \quad \text{as} \quad N \to \infty.
\]

Using Abel's transformation we have

\[
I_N = \sum_{k=1}^{N-1} \sum_{n=N+1}^{\infty} A_{n-k} O(n,t) + \sum_{n=N+1}^{\infty} A_{n-N} O(n,t)
\]

\[
= \mathcal{O}\left(\sum_{k=1}^{N-1} \left| s_k \right|^2 N^{-1} (N-k)^{\alpha-1}\right) + O(N^{-1})
\]

\[
= 0(1) \quad \text{as} \quad N \to \infty.
\]
LEMMA 6. If \( s_n k_{n+1}(t) \to 0 \) as \( n \to \infty \), then the convergence of \( \sum_{n=1}^{\infty} a_n k_n(t) \) implies the convergence of \( \sum_{n=1}^{\infty} s_n G_n(t) \) and
\[
\sum_{n=1}^{\infty} a_n k_n(t) = \sum_{n=1}^{\infty} s_n G_n(t)
\]

PROOF. Since, \( \sum_{v=1}^{m} s_v G_v(t) = \sum_{v=1}^{m} a_v k_v(t) - s_m k_{m+1}(t) \)
we have
\[
\sum_{v=1}^{\infty} s_v G_v(t) = \sum_{v=1}^{\infty} a_v k_v(t).
\]

LEMMA 7. Let \( \gamma_v(t) = \sum_{n=0}^{\infty} c_n k_{n+v}(t) \), then
\[
(6.3.12) \quad \gamma_v(t) = O\left(\frac{1}{vt \log \tau}\right), \quad \tau = \left[\frac{1}{t}\right], \quad 0 < t < 1,
\]
and
\[
(6.3.13) \quad \Delta^m \gamma_v(t) = O\left(\frac{t^{m-1}}{v \log \tau}\right),
\]
where \( m \) is a non-negative integer and \( \Delta^m \gamma_v(t) \) denotes the \( m \)-th difference of \( \gamma_v(t) \) with respect to \( v \).

PROOF. Proof of (6.3.13)
\[
\Delta^m \gamma_v(t) = \Delta^m \sum_{n=0}^{\infty} c_n k_{n+v}(t)
\]
\[
= \Delta^m \sum_{n=0}^{\infty} c_n \Delta k_{n+v}(t)
\]
\[
= \Delta^m \sum_{n=0}^{\infty} c_n G_{n+v}(t)
\]
\[
= \sum_{n=0}^{\infty} c_n \Delta^m G_{n+v}(t)
\]
\[ \sum_{n=0}^{\infty} \frac{1}{n \log^{2} n} = N_{1} + N_{2} \text{, say.} \]

Now using (6.3.4) (i)

\[ N_{2} = \sum_{\tau+1}^{\infty} c_{n} \Delta_{n}^{m-l}(G_{n+v}(t)) = O\left( \sum_{\tau+1}^{\infty} \frac{l}{n \log^{2} n} \frac{t^{m-l}}{(n+v)} \right) \]

\[ = O\left( \frac{t^{m-l}}{(v+\tau+1) \log \tau} \right) = O\left( \frac{t^{m-l}}{v \log \tau} \right) \]

Again applying Abel's transformation and using (6.3.4)(ii) we have

\[ = \sum_{n=0}^{\tau-1} d_{n} \Delta_{n}^{m} G_{n+v}(t) + d_{\tau} \Delta_{\tau}^{m-l} G_{\tau+v}(t) \]

\[ = O\left( \sum_{n=0}^{\tau-1} \frac{l}{\log n} \frac{t^{m}}{(n+v)} \right) + O\left( \frac{t^{m-l}}{(v+\tau) \log \tau} \right) \]

\[ = O\left( \frac{t^{m-l}}{v \log \tau} \right). \]

The proof of (6.3.12) is similar and hence omitted.

**Lemma 8.1**. Let \( V_{w} \) and \( W(n) \) be non-decreasing functions of \( n \). If

(i) \( S_{n} = O\left( V(n) \right) \)

(ii) \( S_{n}^{r} = o(W(n)) \), then for \( 0 < \alpha \leq r \),

\[ S_{n}^{\alpha} = o\left\{ \left( V(n) \cdot W(n) \right)^{\alpha/r} \right\}. \]

---

1) Dixon, A.L. and Ferrar, W.L. [12]
6.4 PROOF OF THEOREM 1. We suppose, as we may without loss of any generality, that $T_n = o(\log n)$ as $n \to \infty$. By virtue of Lemma 5, we have,

$$
\sum_{n=1}^{\infty} S_n \phi(n,t) = \sum_{n=1}^{\infty} s_n G_n(t)
$$

Again,

$$
s_n \cdot k_{n+1}(t) = k_{n+1}(t) \sum_{v=1}^{n} c_{n-v} T_v
$$

$$
= \mathcal{O}\left( \frac{1}{(n+1)t} \sum_{v=1}^{n-1} \frac{\log v}{(n-v) \log^2(n-v)} \right) + \mathcal{O}\left( -\frac{\log n}{nt} \right)
$$

$$
= \mathcal{O}\left( \frac{\log n}{nt} \sum_{v=1}^{\infty} \frac{1}{v \log^2(v+1)} \right)
$$

$$
= \mathcal{O}\left( \frac{\log n}{nt} \right) = o(1), \text{ as } n \to \infty, \text{ for fixed } t > 0.
$$

Therefore by virtue of Lemma 6 it is sufficient to prove that $\sum a_n k_n(t)$ converges in $0 < t < t_0$ and tends to zero as $t \to +0$. Now

$$
(6.4.1) \quad \sum_{n=1}^{\infty} a_n k_n(t) = \sum_{n=1}^{\infty} k_n(t) \sum_{v=1}^{n} c_{n-v}(T_v - T_{v-1})
$$

$$
= \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} k_n(t)
$$

The change of order of summation is justified for

$$
\sum_{v=1}^{\infty} |T_v - T_{v-1}| \sum_{n=0}^{\infty} |c_n k_{n+v}(t)|
$$
\[
= \mathcal{O}\left( \sum_{v=1}^{\infty} \left| T_v - T_{v-1} \right| \sum_{n=0}^{\infty} \left| c_n \right| \frac{1}{(n+v)} \right)
\]

\[
= \mathcal{O}\left( \sum_{v=1}^{\infty} \frac{1}{v} \left| T_v - T_{v-1} \right| \right)
\]

Now \( \sum_{v=1}^{n} \frac{1}{v} \left| T_v - T_{v-1} \right| = \mathcal{O}(1) \), \( n \to \infty \), as shown in chapter III.

Thus the series \( (6.4.1) \) converges absolutely.

Let \( F(t) = \sum_{v=1}^{\infty} (T_v - T_{v-1}) \sum_{n=v}^{\infty} c_{n-v} k_n(t) \)

\[
= \sum_{v=1}^{\infty} (T_v - T_{v-1}) \gamma_v(t).
\]

Now we choose a positive number \( \lambda \), put \( n = \lfloor \lambda \frac{1}{t} \rfloor \) and write

\[
F(t) = \left( \sum_{v=1}^{n} + \sum_{v=n+1}^{\infty} \right) (T_v - T_{v-1}) \gamma(t) = X_1 + X_2, \text{ say}.
\]

Applying Lemma 7, and Abel's transformation we get

\[
X_2 = \mathcal{O}\left\{ \frac{1}{t} \log \tau \sum_{v=n+1}^{\infty} \frac{1}{v} \left| T_v - T_{v-1} \right| \right\}
\]

\[
= \mathcal{O}\left\{ \frac{1}{t} \log \tau \sum_{n+1}^{\infty} \frac{\log v}{v(v+1)} \right\} + \mathcal{O}\left( \frac{\log n}{nt \log \tau} \right)
\]

\[
= \mathcal{O}\left( \frac{\log \lambda}{nt \log \tau} \right) = \frac{\log \lambda}{\lambda} \mathcal{O}(1)
\]
Also by Lemma 1,

\[ X_1 = \sum_{v=1}^{n} (J_v - J_{v+1}) \psi_v(t) \]

\[ = \sum_{v=1}^{n} J_v \left\{ \psi_v(t) - (v-1) \psi_{v-1}(t) \right\} - n J_{n+1} \psi_n(t) \]

\[ = \sum_{v=1}^{n} J_v \psi_{v-1}(t) - \sum_{v=1}^{n} J_v \psi_v(t) + O\left(\frac{\log n}{nt \log T}\right) \]

\[ = O\left(\frac{nt \log n}{\log T}\right) + O\left(\frac{\log n}{\log T}\right) + O\left(\frac{\log n}{nt \log T}\right). \]

Hence \( X_1 = \lambda \log \lambda \cdot 0(1) + \log \lambda \cdot 0(1) + \frac{\log \lambda}{\lambda} \cdot 0(1) \)

as \( t \to +0 \). Therefore,

\[ \lim_{t \to +0} \sup \left| F(t) \right| \leq k \frac{\log \lambda}{\lambda}, \]

being arbitrarily large, we get finally

\[ \lim_{t \to +0} F(t) = 0. \]

This proves Theorem 1.

6.5 PROOF OF THEOREM 2. By the condition (6.2.3) we have \( s_n = O(n^\delta) \) and then using (6.2.2) and Dixon Ferrar's convexity theorem (Lemma 8), we have
Therefore for the proof of the theorem it is sufficient to prove that \( \sum S_n G_n(t) \) converges in \( 0 < t < t_0 \) and its limit is zero as \( t \to +0 \). Let \( m = [\beta]+1 \).

Now
\[
\sum_{n=1}^{\infty} S_n G_n(t) = \sum_{n=1}^{N+m} \sum_{v=1}^{\infty} G_v(t) + \sum_{v=N+m+1}^{\infty} G_v(t)
\]
where \( N = \lceil (\epsilon t)^{-\ell} \rceil \) and \( \epsilon \) being an arbitrarily fixed positive number and \( \ell = \frac{1}{1-\delta} \).

Using Abel's transformation we have
\[
W_2 = \sum_{n=N+m+1}^{\infty} a_{n+1} \sum_{v=n+1}^{\infty} G_v(t) + \sum_{v=N+m+1}^{\infty} G_v(t)
\]
\[
= O\left( t^{-1} \sum_{n=N}^{\infty} \frac{|a_n|}{n} \right) + O\left( t^{-1} N^{-\delta-1} \right)
\]
\[
= O\left( t^{-1} N^{-\delta-1} \right) = O\left( t^{-1} (\epsilon t)^{\ell(1-\delta)} \right) = O(\epsilon) .
\]

Now by the repeated use of Abel's transformation \( m \)-times we have
\[
W_1 = \sum_{n=1}^{N} S_n \Delta^m G_n(t) + \sum_{v=1}^{m} \sum_{v=1}^{v} S_n \Delta^{v-1} G_{N+m-v-1}(t)
\]
\[
= W_{1,1} + W_{1,2} , \text{ say.}
\]

By (6.5.2) and the lemma 3 we get for \( 0 < v \leq m - 1 \)
\[ S_{N+m-v+1} \Delta^{m-1} G_{N+m-v+1} = o\left(N^{-\left(\delta(\beta-v)+v\gamma\right)} \right) \]

\[ = o\left(t^{v-1} - \frac{1}{\beta(1-\delta)} \left\{ \delta(\beta-v)+v\gamma-\beta \right\} \right) \]

\[ = o\left(t^{v(\beta-\gamma)/\beta} \right) = o(1) \text{ as } t \to +0, \]

and for \( v = m \) we have \( S_{n}^{m} = o(n^{m+\gamma-\beta}) \)

\[ S_{N+1}^{m} \Delta^{m-1} G_{N+1}(t) = o\left(N^{m+\gamma-\beta} \right) \]

\[ = o\left(t^{m-1} - \frac{1}{1-\delta} (m+\gamma-\beta-1) \right) \]

\[ = o\left(t^{1-\delta} \left\{ -m\delta + \beta - \gamma + \delta \right\} \right) \]

\[ = o\left(t^{\delta}/(\beta+1-{m}) \right) = o(1) \text{ as } t \to +0. \]

Hence \( W_{1,2} = o(1) \text{ as } t \to +0. \) If \( \beta \) is a positive integer we use Abel's transformation \( \beta \)-times only and get the required result.

Using the identity \( S_{n}^{m} = \sum_{p=1}^{n} A^{m-\beta-1} \frac{\beta}{p} \) we have

\[ W_{1,1} = \sum_{p=1}^{N} S_{p}^{\beta} \sum_{n=p}^{N} A^{m-\beta-1} \Delta^{m} G_{n}(t) \]

\[ = \sum_{p=1}^{N} \sum_{n=p}^{p+T} A^{m-\beta-1-\beta} - \sum_{p=1}^{N-T+1} \sum_{n=p+T+1}^{N} A^{m-\beta-1-\beta} \]

\[ = P_{1} + P_{2} - P_{3}, \text{ say.} \]
Now applying Lemma 3

\[ P_1 = O \left( \sum_{p=1}^{N} \left| \beta_p \right| \sum_{n=p}^{p+\tau} A_{n-p}^{\beta-2} A_{n-p-1}^{\beta} t_{n}^{m} \right) \]

\[ = O \left( t^{\beta} \sum_{p=1}^{N} \left| \beta_p \right| \right) \]

\[ = O \left( t^{\beta} \sum_{p=1}^{N} O(p^{\gamma-1}) = O(t^{\beta} N^{\gamma}) \right) \]

\[ = O(t^{\beta} : t^{-\gamma}) = O(1), \quad t \to + 0. \]

Also

\[ P_2 = \sum_{p=1}^{N-T-1} s_p^\beta \left\{ \sum_{n=p+T+1}^{N-1} A_{n-p+1}^{\beta-2} A_{n-p}^{\beta} \Delta^{m-1} G_{n+1}(t) \right\} \]

\[ + A_{T+1}^{\beta-1} \Delta^{m-1} G_{T+1}(t) - A_{N-p}^{\beta-1} \Delta^{m-1} G_{N+1}(t) \]

\[ = O(t^{m-1} \sum_{p=1}^{N-T-1} \left| \beta_p \right| \sum_{n=p+T+1}^{N-1} A_{n-p+1}^{\beta-2} \right) \]

\[ + O(\sum_{p=1}^{N} \left| \beta_p \right| t^{\beta-1} \sum_{p=1}^{T-m+1} t_{m-1}) \]

\[ + O(\sum_{p=1}^{N} \left| \beta_p \right| t^{m-1} N^{\beta-1}) \]
\[ = \mathcal{O}(t^{m-1} N^\beta + 1 - m) + \mathcal{O}(t^{N-1} N |S_p^\beta|) \]
\[ = \mathcal{O}(t^N) = o(1), \quad t \to + 0. \]

Finally we have

\[ P_3 = \sum_{p=N-T+1}^N S_p^\alpha \sum_{n=N+1}^{m-\beta-1} A_n^{\alpha-1} \Delta G_n(t) \]
\[ = \mathcal{O}(t^N) \sum_{p=N-T+1}^N |S_p^\beta| \sum_{n=N+1}^{m-\beta-1} A_n^{\alpha-1} t^{m-n} \]
\[ = \mathcal{O}(t^N) \sum_{p=N-T+1}^N |S_p^\beta| = \mathcal{O}(t^N) = o(1), \quad t \to + 0. \]

Thus \( W_{1,1} = o(1) \) and consequently \( W_1 = o(1) \).

We have already shown that \( W_2 = \mathcal{O}(\varepsilon) \).

Therefore,

\[ \lim_{t \to + 0} \sup_{n=1}^{\infty} |\sum_{n=1}^{\infty} S_n G_n(t)| = \mathcal{O}(\varepsilon). \]

Since \( \varepsilon \) is an arbitrary positive number, we have

\[ \lim_{t \to + 0} \sum_{n=1}^{\infty} S_n G_n(t) = 0. \]

This establishes the proof of Theorem 2.
6.6 In this section we shall generalize the following theorem due to Hirokawa.

Theorem F. If

\[ S_n - s = O\left( \frac{1}{(\log n)^{1+\delta}} \right), \delta > 0. \]

Then \( \sum a_n \) is summable \((K,l,\alpha)\) to \( s \), when \(-1 < \alpha < 0\).

It is clear that (6.6.1) implies that

\[ \sum_{v=1}^{n} |S_v - s| = O\left( \frac{n}{(\log n)^{1+\delta}} \right), \delta > 0. \]

However the converse need not be true. Therefore the question arises whether Theorem F remains true even when we take the lighter condition, namely (6.6.2). In this section we shall show that answer to this problem is in the affirmative. As a matter of fact we shall prove a still more general theorem.

**Theorem 3.** If

\[ \sum_{v=1}^{n} |S_v - s| = O(n \lambda_n), \quad \text{and} \]

\[ \sum_{n=1}^{\infty} |\lambda_n| < \infty, \quad (\text{i}) \lambda_n \to \infty \text{ as } n \to \infty \quad \text{and} \]

\[ \sum_{n=1}^{\infty} |\lambda_n| = +\infty, \quad \text{(iii)} \]

then \( \sum a_n \) is summable \((K,l,\alpha)\) to \( s \), when \(-1 \leq \alpha \leq 0\).
PROOF OF THE THEOREM: We assume as we may, without any loss of generality that \( s = 0 \). Since \( S_n^1 = O(n) \), we have by virtue of Lemma 5

\[
\sum_{n=1}^{\infty} S_n^1 \phi(n,t) = \sum_{v=1}^{\infty} S_v G_v(t).
\]

Therefore for the proof of our theorem it is sufficient to prove that \( \sum S_v G_v(t) \) converges in \( 0 < t < t_0 \) and its limit as \( t \to +0 \) is zero.

We write

\[
\sum_{v=1}^{\infty} S_v G_v(t) = \sum_{v=1}^{\rho} S_v G_v(t) + \sum_{v=\rho+1}^{\infty} S_v G_v(t).
\]

where \( \rho = \lfloor \frac{1}{t} \rfloor \).

By virtue of Lemma 3 we have

\[
M_2 = \sum_{v=\rho+1}^{\infty} S_v G_v(t) = O \left( \sum_{v=\rho+1}^{\infty} \frac{|S_v|}{v} \right)
\]

and

\[
\sum_{v=\rho+1}^{N} \frac{|S_v|}{v} = \sum_{k=1}^{N-1} \sum_{v=\rho+1}^{\rho+1} \frac{|S_k|}{v} \Delta \left( \frac{1}{v} \right) + \sum_{v=1}^{N} \frac{|S_v|}{N} \frac{1}{N}
\]

\[
= \frac{1}{\rho+1} \sum_{v=1}^{\rho} \frac{|S_v|}{v} + O \left( \sum_{v=\rho+1}^{N} \frac{|S_v|}{v} \right)
\]

\[
= O \left( \sum_{v=\rho+1}^{N} \frac{\lambda \nu}{v^2} \right) + O \left( \frac{\lambda N}{N} \right) +
\]

\[
+ O \left( \frac{\lambda \rho}{\rho} \right)
\]
Making \( N \to \infty \)

\[
\sum_{v=\beta+1}^{\infty} \frac{|S_v|}{v} = O \left( \sum_{v=\beta+1}^{\infty} \frac{1}{v} \right) + o(1)
\]

Applying Abel's transformation

\[
M_1 = \sum_{v=1}^{\rho} S_v g_v(t) = \sum_{v=1}^{\rho-1} S_v \Delta g_v(t) + S_\rho \Delta g_\rho(t)
\]

\[
= o \left( \rho t \sum_{v=1}^{\rho} \frac{|\lambda_v|}{v} \right) + o(1)
\]

Making \( t \to +\infty \) we have the required result.

This proves Theorem 3.