Chapter VI

ON SOME INEQUALITIES FOR FOURIER SERIES

6.1 A non-decreasing continuous real-valued function Φ defined on the non-negative half line and vanishing only at the origin will be called an Orlicz Function (OF). Function $\Phi \in$ OF is said to satisfy Δ_α ($\alpha > 0$) condition for large u if there are constants $C > 0$ and $u_o \geq 0$ such that $\Phi(\alpha u) \leq C\ \Phi(u)$, $u \geq u_o$ for every $\alpha > 1$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u \to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{\Phi(u)}{u} = \infty$$

Young function (YF). Function Φ belongs to YF iff it admits a representation

$$\Phi(u) = \int_0^u \phi(x)\, dx ,$$

where $\phi(x)$ $(x \geq 0)$ is positive, $\phi(0) = 0$, continuous on the right, non-decreasing and $\lim_{x \to \infty} \phi(x) = \infty$. We have for such functions the relation

$$(6.1.1) \qquad \frac{\Phi(u)}{u} \leq \phi(u) \leq \frac{\Phi(2u)}{u}.$$

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [1] .

" There exist a convex function Λ , $\lambda > 1$ and $0<k<1$, such that the inequality

$$\Lambda(u) \leq \Phi^{k}(u) \leq \lambda \Lambda(u) \text{ holds for all } u ."$$

We write

$$G(x) = \int_{x/2}^{x} \frac{f(t)}{t}\, dt.$$

6.2 M. Izumi and S.Izumi [1] in 1968, proved among others, the following theorems.

__Theorem A.__ Let $p > 1$ and $s > -1$ and let f be a non-negative, non-increasing and integrable function on $(0,\pi)$. If $x^s f^p(x)$ is integrable, then we have

$$\int_0^\pi x^s G^p(x)\, dx \leq A^{*}\int_0^\pi x^s(f(\tfrac{x}{2})-f(x))^p dx + A(\int_{\pi/2}^{\pi} f(x)\,dx)^p.$$

__Theorem B.__ Let $p > 1$ and $s < -1$ and let f be a non-negative and integrable on $(0,\pi)$. If $x^s f^p(x)$ is integrable , then

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*where A denotes a constant, not necessarily the same at each occurrence.

$$\int_0^{\pi} x^s G^p(x)\, dx \le \left(\frac{p}{-s-1}\right)^p \int_0^{\pi} x^s \left| f\left(\frac{x}{2}\right) - f(x) \right|^p dx.$$

Theorem C. Let $p > 1$ and $s \neq 1$ and let f be a non-negative, non-constant and integrable on $(0, \infty)$. If

$$x^s f^p(x) \in L(0,\pi) ,$$

then

$$\int_0^{\infty} x^s G^p(x)\, dx \le \left|\frac{p}{-s-1}\right|^p \int_0^{\infty} x^s \left| f\left(\frac{x}{2}\right) - f(x) \right|^p dx.$$

Theorem D. Let $p > 1$ and $\{a_n\}$ be a monotonic non-increasing sequence tending to zero and

$$\frac{1}{n} \sum_{m=0}^{n-1} a_m \le 2 \sum_{m=n}^{2n}{}^{*} a_m \text{ for all } n \ge 1 ,$$

where Σ^* denotes the sum whose first and last terms are halved. Suppose that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$$

and that $f(x)$ is non-negative.

(I) If f is L^p integrable and decreases monotonically, then

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p \le A \int_0^{\pi} \left(f\left(\frac{x}{2}\right) - f(x)\right)^p dx + A\left(\int_{\pi/2}^{\pi} f(x)\, dx\right)^p.$$

(II) More generally, if $-1 < s < p-1$ and $x^s f^p(x)$ is integrable on $(0,\pi)$, then

$$\sum_{n=1}^{\infty} n^{p-s-2} a_n^p \leq A \int_0^{\pi} x^s \left| f(\tfrac{x}{2}) - f(x) \right|^p dx + A \frac{\left(\int_{\pi/2}^{\pi} f(x)\,dx\right)^{p^2}}{\left\{\int_0^{\pi/2} f(x)\,dx\right\}^{p^2-p}}$$

(III) If $s < -1$ and $x^s f^p(x)$ is integrable on, $(0,\pi)$, then

$$\sum_{n=1}^{\infty} a_n^p n^{p-s-2} \leq A \int_0^{\pi} x^s \left| f(x) - f(\tfrac{x}{2}) \right|^p dx.$$

The object of this chapter is to obtain certain generalizations of the above theorems. In our theorems we replace the special class L^p of functions by a more general class $L(\Phi)$ where Φ satisfies certain properties. In Theorem 1 we have shown that condition of monotonicity in Theorem A is redundant . A similar remark is also applicable to Theorem D (part I).

6.3 We prove the following theorems.

<u>Theorem 1.</u> <u>Let</u> $\Phi \in \Delta_{\alpha} \cap YF$ <u>and</u> f <u>be a non-negative and integrable on</u> $(0,\pi)$. <u>If</u> $x^s \Phi(f(x))$ <u>is integrable and</u> $s > -1$, <u>then we have</u>

$$\int_0^{\pi} x^s \, \Phi(G(x)) \, dx$$

$$\leq A \int_0^{\pi} x^s \, \Phi\left(\left| f(x) - f\left(\frac{x}{2} \right) \right| \right) dx + A \, \Phi\left(\int_{\pi/2}^{\pi} f(x) \, dx \right).$$

Theorem 2. Let $\Phi \in \Delta_\alpha \cap YF$ and f be non-negative. If $x^s \, \Phi(f(x))$ is integrable and $s < -1$, then

$$\int_0^{\pi} x^s \, \Phi(G(x)) \, dx \leq A \int_0^{\pi} x^s \, \Phi\left(\left| f(x) - f\left(\frac{x}{2} \right) \right| \right) dx.$$

Theorem 3. Let $\Phi \in \Delta_\alpha \cap YF$, $s \neq -1$ and f be a non-negative, non-constant and integrable on $(0, \infty)$. If $x^s \, \Phi(f(x)) \in L(0,\infty)$, then

$$\int_0^{\infty} x^s \, \Phi(G(x)) \, dx \leq A \int_0^{\infty} x^s \, \Phi\left(\left| f\left(\frac{x}{2} \right) - f(x) \right| \right) dx.$$

Theorem 4. Let $\Phi \in \Delta_\alpha \cap YF \cap M$. Let $\{a_n\}$ satisfy the conditions of Theorem D. Suppose that

$$f(x) \sim \frac{1}{2} a_0 + \sum_1^{\infty} a_n \cos nx = \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$$

and that $f(x)$ is non-negative.

(i) If $x^s \, \Phi(f(x))$ is integrable and $-1 < s < \beta - 1$, $\beta > 1$, then

$$\sum_{n=1}^{\infty} n^{-s-2} \, \Phi(n a_n)$$

$$\leq A \int_0^{\pi} x^s \, \Phi\left(\left| f(x) - f\left(\frac{x}{2}\right) \right| \right) dx + A \, \Phi\left(\int_{\pi/2}^{\pi} f(x) \, dx \right).$$

(ii) *If* $s < -1$ *and* $x^s \Phi(f(x))$ *is integrable on* $(0,\pi)$, *then*

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n\, a_n) \leq A \int_0^{\pi} x^s \, \Phi(|f(x) - f(\tfrac{x}{2})|)\, dx.$$

6.4 We require the following lemmas for the proof of our theorems.

Lemma 1. (Izumi M. and Izumi S. [1]). *Let* $\{a_n\}$ *be a monotonic decreasing sequence tending to zero and*

$$\frac{1}{n} \sum_{m=0}^{n-1} a_m \leq 2 \sum_{m=n}^{2n}{}^{*} a_m$$

for all $n \geq 1$, *where* Σ^* *denotes the sum whose first and last terms are halved. If* $f(x) \sim \sum_{n=0}^{\infty}{}^{*} a_n \cos nx$ *is non-negative and integrable, then the* n-th *Fourier coefficient* γ_n *of the even function* Q *defined by*

$$Q(x) = \int_{x/2}^{\pi} \frac{f(t)}{2 \tan t/2}\, dt, \qquad 0 < x < \pi,$$

is non-negative and further

$$: \ 0 < \theta < 1, \ \gamma_n \geq \frac{\theta}{n} \sum_{m=n}^{2n}{}^{*} a_m$$

for all $n \geq 1$.

Lemma 2. (Zygmund [1]). *Let* $f(x)$ *be a non-negative function defined for* $x \geq 0$, *and let* $r > 1$, $s < r-1$. *Then, if*

$$x^{s} f^{r}(x) \in L(0,\infty)$$

so is

$$x^{s} \left(\frac{F(x)}{x}\right)^{r},$$

where

$$F(x) = \int_0^x f(t)\, dt.$$

Moreover,

$$\int_0^\infty x^{s} \left(\frac{F(x)}{x}\right)^{r} dx \leq \left(\frac{r}{r-s-1}\right)^{r} \int_0^\infty x^{s} f^{r}(x)\, dx.$$

Lemma 3.† *If* $\Phi \in F$, $s < \alpha-1$, $\alpha > 1$, *and* $x^{s}\, \Phi(f(x)) \in L(0,\infty)$, *then*

$$\int_0^\infty \Phi\left(\frac{F(x)}{x}\right) x^{s}\, dx \leq A \int_0^\infty \Phi(f(x))\, x^{s}\, dx,$$

where

$$F(x) = \int_0^x f(t)\, dt$$

and $f(x) \geq 0$ *for* $x \geq 0$.

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† Woyczynski [1] proved this lemma for $s \leq 0$ only. Following the same technique here we extend the result to the case $s < \alpha-1$, $\alpha > 1$.

Proof of Lemma 3. By virtue of the assumption and the Jensen's inequality, there exist an $\alpha > 1$ and a convex function $\Lambda(x)$ such that

$$\Phi^{1/\alpha}\left(\frac{F(x)}{x}\right) \leq \lambda \Lambda\left(\frac{F(x)}{x}\right)$$

$$\leq \lambda \frac{1}{x} \int_0^x \Lambda(f(t))\, dt$$

$$\leq \lambda \frac{1}{x} \int_0^x \Phi^{1/\alpha}(f(t))\, dt .$$

Now we have

$$\int_0^\infty x^s \Phi\left(\frac{F(x)}{x}\right) dx$$

$$\leq \lambda^\alpha \int_0^\infty x^s \left(\frac{1}{x} \int_0^x \Phi^{1/\alpha}(f(t))\, dt \right)^\alpha dx$$

$$\leq A \int_0^\infty x^s \Phi(f(x))\, dx$$

by Lemma 2.

This proves Lemma 3.

6.3 Proof of Theorem 1. Integrating by parts, we have

$$\int_0^x x^s \Phi(G(x))\, dx$$

$$= \left[x^{s+1} \frac{\Phi(G(x))}{s+1} \right]_0^{\pi} - \frac{1}{s+1} \int_0^{\pi} x^{s+1} \phi(G(x)) \left(\frac{f(x)}{x} - \frac{f(x/2)}{x/2} \cdot \frac{1}{2} \right) dx$$

$$\leq \frac{\pi^{s+1} \Phi(G(\pi))}{s+1} + \frac{1}{s+1} \int_0^{\pi} x^{s} \phi(G(x)) \, | f(x) - f(x/2) | \, dx.$$

Taking T to be arbitrary large positive number, we have, since $\phi(x)$ is non-decreasing

$$\phi\,(G(x)) \, | f(x) - f(\tfrac{x}{2}) |$$

$$= T^{-1} \{ T \phi(G(x)) \, | f(x) - f(\tfrac{x}{2}) | \}$$

$$\leq T^{-1} \operatorname{Max} \{ G(x) \, \phi(G(x)),$$

$$T \, | f(x) - f(\tfrac{x}{2}) | \, \phi\,(T \, | f(x) - f(\tfrac{x}{2}) |) \}$$

$$\leq T^{-1} \Phi(2\, G(x)) + T^{-1} \Phi(2T \, | f(x) - f(\tfrac{x}{2}) |)$$

$$\leq AT^{-1} \Phi(G(x)) + AT^{-1} \Phi(| f(x) - f(\tfrac{x}{2}) |)$$

Thus we have

$$\int_0^{\pi} x^{s} \Phi(G(x)) \, dx$$

$$\leq A \, \Phi(G(\pi)) + \frac{1}{s+1} \int_0^{\pi} x^{s} \phi(G(x)) \, | f(x) - f(\tfrac{x}{2}) | \, dx$$

$$\leq A\,\Phi\left(\int_{\pi/2}^{\pi} \frac{f(t)}{t}\,dt\right) + \frac{A}{T(s+1)} \int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$+ \frac{A}{T(s+1)} \int_0^{\pi} x^s\,\Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right)\,dx$$

or,

$$\left(1 - \frac{A}{T(s+1)}\right) \int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$\leq A\,\Phi\left(\frac{2}{\pi}\int_{\pi/2}^{\pi} f(t)\,dt\right) + \frac{A}{T(s+1)} \int_0^{\pi} x^s\,\Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right)dx$$

or,

$$\int_0^{\pi} x^s\,\Phi(G(x))\,dx$$

$$\leq A\,\Phi\left(\int_{\pi/2}^{\pi} f(t)\,dt\right) + A\int_0^{\pi} x^s\,\Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right)dx$$

by choosing $T > \frac{A}{s+1}$.

This completes the proof of Theorem 1.

6.6 Proof of Theorem 2. First of all we have by Jensen's inequality,

(6.6.1) $\Phi(G(x)) = \left(\int_{\pi/2}^{x} \frac{f(t)}{t}\,dt\right)$

$$\leq \Phi\left(\frac{2}{x} \int_{x/2}^{x} f(t)\, dt\right)$$

$$\leq \frac{2}{x} \int_{x/2}^{x} \Phi(f(t))\, dt$$

$$\leq A\, x^{-s-1} \int_{x/2}^{x} t^{s}\, \Phi(f(t))\, dt$$

$$= o(1)\,, \quad x \to 0\,.$$

Now, integrating by parts, we have

$$\int_0^{\pi} x^{s}\, \Phi(G(x))\, dx$$

$$= \left[\frac{x^{s+1}\, \Phi(G(x))}{s+1}\right]_0^{\pi} - \frac{1}{s+1} \int_0^{\pi} x^{s+1} \phi(G(x)) \left(\frac{f(x)}{x} - \frac{\frac{1}{2} f(\frac{x}{2})}{x/2}\right) dx$$

$$= \frac{\pi^{s+1}\, \Phi(G(\pi))}{s+1} - \frac{1}{s+1} \int_0^{\pi} x^{s}\, \phi(G(x)) (f(x) - f(\tfrac{x}{2}))\, dx.$$

First term on the right side is negative since $s < -1$.
Hence we have

$$\int_0^{\pi} x^{s}\, \Phi(G(x))\, dx$$

$$\leq -\frac{1}{s+1} \int_0^{\pi} x^{s}\, \phi(G(x))\, | f(x) - f(\tfrac{x}{2}) |\, dx.$$

Taking T to be arbitrarily large positive integer, we have, since $\phi(x)$ is non-decreasing

$$\phi\left(G(x)\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\right)$$

$$= T^{-1}\left\{T\,\phi(G(x))\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\right\}$$

$$\leq T^{-1}\,\mathrm{Max}\left\{G(x)\,\phi(G(x)),\right.$$

$$\left.T\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\phi\left(T\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\right)\right\}$$

$$\leq T^{-1}\,\Phi(2(G(x)) + T^{-1}\,\Phi\left(T\,2\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\right)$$

$$\leq AT^{-1}\,\Phi(G(x) + AT^{-1}\,\Phi\left(\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\right).$$

Thus we have

$$\int_0^{\pi} x^{s}\,\Phi(G(x))\,dx$$

$$\leq \frac{-1}{s+1}\int_0^{\pi} x^{s}\,\phi(G(x))\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\,dx$$

$$\leq \frac{-A}{(s+1)T}\int_0^{\pi} x^{s}\,\Phi(G(x))\,dx - \frac{A}{(s+1)T}\int_0^{\pi} x^{s}\,\Phi\left(\left|f(x)-f\left(\tfrac{x}{2}\right)\right|\right)dx$$

or,

$$\left(1 + \frac{A}{(s+1)T}\right) \int_0^x x^s \; \Phi(G(x)) \; dx$$

$$\leq - \frac{A}{(s+1)T} \int_0^x x^s \; \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right| \right) dx$$

or,

$$\int_0^x x^s \, \Phi(G(x)) \; dx \quad < \quad A \int_0^x x^s \, \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) \; dx$$

by choosing $T > - \frac{A}{s+1}$.

This completes the proof of Theorem 2.

6.7 <u>Proof of Theorem</u> 3. It is easy to verify that the relation (6.6.1) holds also when $x \to \infty$ and then following the lines of proof of Theorem 2 we have

$$\int_0^\infty x^s \; \Phi(G(x)) \; dx$$

$$= - \frac{\lambda}{s+1} \int_0^\infty x^s \; \phi\,(G(x)) \; \left(f(x) - f\left(\frac{x}{2}\right) \right) \; dx$$

$$\leq \frac{\lambda}{|s+1|} \int_0^\infty x^s \; \phi\,(G(x)) \; \left|f(x) - f\left(\frac{x}{2}\right)\right| \; dx$$

$$\leq \frac{A}{T\,|s+1|} \int_0^\infty x^s \; \Phi\,(G(x)) \; dx \; +$$

$$+ \; \frac{A}{T\,|s+1|} \int_0^\infty x^s \; \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right| \right) \; dx$$

or,

$$\int_0^\infty x^s \Phi(G(x))\ dx \leq \frac{A}{T.|s+1|}\int_0^\infty x^s\ \Phi(\ |f(x)-f(\frac{x}{2})|\ )dx.$$

This proves Theorem 3.

6.8 <u>Proof of Theorem</u> 4. Let γ_n be the Fourier coefficients of $Q(x)$ of Lemma 1. Then by virtue of this lemma, $\gamma_n \geq 0$ for all $n \geq 1$. We put

$$Q_1(x) = \int_0^x Q(t)\ dt\ ,$$

$$Q_2(x) = \int_0^x Q_1(t)\ dt \qquad \text{for } x \geq 0\ .$$

Then

$$Q_2(x) = \frac{\gamma_0 x^2}{4} + \sum_{j=1}^\infty \gamma_j\ j^{-2}\ (1 - \cos jx)$$

$$\geq \sum_{j=1}^{[\frac{n}{2}]} \gamma_j.\ j^{-2}\ (1-\cos jx)$$

$$= 2\sum_{j=1}^{[\frac{n}{2}]} \gamma_j\ \ j^{-2}\ \sin^2\frac{jx}{2}$$

$$\geq 2\sum_{j=1}^{[\frac{n}{2}]} \gamma_j.\ j^{-2}\ \frac{4}{\pi^2}.\ \frac{j^2 x^2}{4}\ ,\quad \frac{\pi}{n+1} \leq x \leq \frac{\pi}{n}\ .$$

$$= \frac{\varrho}{x^{\varrho}}\, x^{\varrho} \sum_{j=1}^{[\frac{n}{\varrho}]} \Upsilon_j$$

$$\geq \frac{\varrho}{x^{\varrho}}\, \theta\, x^{\varrho} \sum_{j=1}^{[\frac{n}{\varrho}]} \frac{1}{j} \sum_{k=j}^{\varrho j}{}^{*} a_k , \quad 0 < \theta < 1 ,$$

$$\geq A\ x^{\varrho} \sum_{j=1}^{[\frac{n}{\varrho}]} a_{\varrho j}$$

$$\geq A\ x^{\varrho}\, n\, a_n ,$$

or, we have

$$n\, a_n \leq \frac{1}{A}\, \frac{Q_\varrho(x)}{x^{\varrho}}, \quad \frac{\pi}{n+1} \leq x \leq \frac{\pi}{n} , \quad n \geq 1.$$

Now, we have

$$\sum_{n=1}^{\infty} n^{-s-\varrho}\, \bar{\Phi}(\, n\, a_n\,)$$

$$\leq A \sum_{n=1}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} x^{s}\ \bar{\Phi}\left(\frac{Q_\varrho(x)}{A\ x^{\varrho}}\right) dx$$

$$\leq A \int_0^{\pi} x^{s}\ \bar{\Phi}\left(\frac{Q_\varrho(x)}{x^{\varrho}}\right) dx$$

$$= A \int_0^{\pi} x^{s}\ \bar{\Phi}\left(\frac{1}{x^{\varrho}} \int_0^{x} Q_1(t)\ dt\ \right) dx$$

$$\leq A \int_0^{\pi} x^s \, \Phi\left(\frac{G_1(x)}{x}\right) dx$$

$$\leq A \int_0^{\pi} x^s \, \Phi(G(x)) \, dx$$

by Lemma 3.

Now, by Theorem 1, we have

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n a_n)$$

$$\leq A \int_0^{\pi} x^s \, \Phi(G(x)) \, dx$$

$$\leq A \int_0^{\pi} x^s \, \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) dx$$

$$+ A \, \Phi\left(\int_{\pi/2}^{\pi} f(x) \, dx\right).$$

This proves part (i) of Theorem 4.

Now, if we suppose that $s < -1$ and use Theorem 2, then we get

$$\sum_{n=1}^{\infty} n^{-s-2} \Phi(n a_n) \leq A \int_0^{\pi} x^s \, \Phi(G(x)) \, dx$$

$$\leq A \int_0^{\pi} x^s \, \Phi\left(\left|f(x) - f\left(\frac{x}{2}\right)\right|\right) dx.$$

This proves part (ii) of Theorem 4.