Chapter V

INTEGRABILITY THEOREMS FOR CERTAIN TRIGONOMETRIC SERIES WITH δ-QUASI-MONOTONE COEFFICIENTS

5.1 A sequence $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n$, $n = 1, 2, \ldots$. It is said to be a null sequence if $a_n \to 0$.

The idea of decreasing null sequence was generalised in the form of a quasi-monotone sequence by Shah [1] and Szász [1] in the following manner.

A sequence $\{a_n\}$ of positive numbers is said to be quasi-monotonic if, and only if , $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$, or equivalently, if, and only if, $\triangle a_n \geq -\frac{\alpha a_n}{n}$ for some $\alpha > 0$, where $\triangle a_n = a_n - a_{n+1}$.

It is clear that if $\{a_n\}$ is a positive monotonic decreasing sequence then it is also quasi-monotonic. However, the converse need not be true.

The quasi-monotonic sequences are known to share many of the important properties of decreasing sequences. For example , Oliver's theorem, Cauchy's condensation test for convergence and a number of results about trigonometric series have been found to be true for quasi-monotonic sequences.

In 1965, Boas [2] considered a more general definition of quasi-monotonic sequence. According to him a sequence $\{a_n\}$ is said to be δ-quasi-monotonic if $a_n \to 0$; $a_n > 0$ ultimately and $\triangle a_n \geq -\delta_n$ where $\{\delta_n\}$ is a sequence of positive numbers. By taking $\delta_n = \frac{\alpha\, a_n}{n}$ we find that δ-quasi-monotone sequence becomes simply a quasi-monotone sequence.

A number of theorems concerning δ-quasi-monotone sequences was obtained by Boas [2], who restricted δ_n, his various results, by the condition

$$\sum_{n=1}^{\infty} \delta_n n^{\gamma} < \infty \quad (\gamma > 0) \quad \text{or} \quad \sum_{n=1}^{\infty} \delta_n \log n < \infty .$$

A positive function L(x) is said to be " slowly increasing" in the sense of Karamata [1] if it is continuous for $x \geq 0$ and

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for every fixed } t > 0 .$$

Some of the important properties of such a function are as follows*.

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*P_1, P_2, P_3 and P_4 are due to Karamata [1] and P_5 due to Igari [1].

P_1 : $\frac{L(tx)}{L(x)} \to 1$ as $x \to \infty$, uniformly for $0 < a \leq t \leq b < \infty$.

P_2 : If $f(x) \sim L(x)$, $x \to \infty$, then $f(x)$ is also a slowly increasing function.

P_3 : $x^{\alpha} L(x) \to \infty$, $x^{-\alpha} L(x) \to 0$, as $x \to \infty$ for every $\alpha > 0$.

P_4 : If we write for some $\alpha > 0$,

$$\bar{L}_1(x) = x^{-\alpha} \underset{0 \leq t \leq x}{\text{Max}} \{ t^{\alpha} L(t) \}, \quad \underline{L}_1(x) = x^{\alpha} \underset{0 < t \leq x}{\text{Min}} \{ t^{-\alpha} L(t) \},$$

$$\bar{L}_2(x) = x^{\alpha} \underset{x \leq t < \infty}{\text{Max}} \{ t^{-\alpha} L(t) \}, \quad \underline{L}_2(x) = x^{-\alpha} \underset{x \leq t < \infty}{\text{Min}} \{ t^{\alpha} L(t) \},$$

then $\overline{\underline{L}}_k(x) \sim L(x)$ as $x \to \infty$, for $k = 1, 2$.

P_5 : For any $\alpha > 0$, we have

$$L(tu) \leq C_1 \, t^{-\alpha} L(u) \quad \text{for every } u \geq 0 \text{ and } 0 < t \leq 1,$$

$$L\left(\frac{u}{t}\right) \leq C_2 \, t^{-\alpha} L(u) \quad \text{for every } u \geq 0, \ 0 < t \leq 1,$$

where C_1 and C_2 are positive constants depending on α and L only.

5.2 Concerning integrability of trigonometric series $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$ and $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$, Boas [1] proved the following results, in which he assumed the sequence of coefficients to be monotonic decreasing and tending to zero .

Theorem A. If $a_n \downarrow 0$ and $0 < \gamma < 1$, then $x^{-\gamma} f(x) \in L(0,\pi)$ if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Theorem B. If $a_n \downarrow 0$ and $0 \leq \gamma \leq 1$, then $x^{-\gamma} g(x) \in L(0,\pi)$ if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Later on, G. Sunouchi [1] proved Theorems A and B by a different method.

Shah [2] obtained the following results concerning the integrability of trigonometric series for quasi-monotone sequences which generalize the above theorems.

Theorem C. Let $\{a_n\}$ be a quasi-monotone sequence.

(i) If $0 < \gamma < 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if $x^{-\gamma} f(x) \in L(0,\pi)$.

(ii) $0 < \gamma \leq 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if $x^{-\gamma} g(x) \in L(0,\pi)$.

This theorem was subsequently extended by Boas [2] for

δ-quasi-monotone sequences. His results are as follows :

<u>Theorem</u> D. Let $0 < \gamma < 1$, and let $\{a_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n^{\gamma} \delta_n < \infty$. Then $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if $x^{-\gamma} f(x) \in L(0,\pi)$.

<u>Theorem</u> E. Let $0 < \gamma \leq 1$, and let $\{a_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n^{\gamma} \delta_n < \infty$. Then $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if $x^{-\gamma} g(x) \in L(0,\pi)$.

<u>Remark</u> :- Suppose that $\{a_n\}$ is a positive sequence tending to zero and $\delta_n = \frac{\alpha\, a_n}{n}$. Let $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$, and $0 < \gamma < 1$. Then Theorem D asserts that $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges iff $x^{-\gamma} f(x) \in L(0,\pi)$. Considering the case $\int \to \Sigma$ we observe that Boas had already assumed what he wished to prove. Thus his results (Theorem D and E) suffer from this serious defect. Of course, he could have avoided this by

(say in Theorem D) assuming the convergence of $\frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$ instead of that of $\sum_{n=1}^{\infty} n^{\gamma} \delta_n$ in the part $\int \to \Sigma$.

Aljančić , Bojanić and Tomić [2] , in 1955, generalised Theorems A and B in a different direction. They proved, among others, the following results.

<u>Theorem</u> F. If $0 < \gamma < 1$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x) f(x) \in L(0,\pi)$, if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n < \infty$.

<u>Theorem</u> G. If $0 < \gamma < 2$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x) g(x) \in L(0,\pi)$ if, and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n < \infty$.

These results were extended by Yong [1] in 1965 to quasi-monotone sequences in the following form.

<u>Theorem</u> H. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$ as $n \to \infty$ and $0 < \gamma < 1$. Then $\sum_{1}^{\infty} n^{\gamma-1} L(n) a_n$ converges if, and only if, $\frac{1}{2} a_0 + \sum_{1}^{\infty} a_n \cos nx$ converges everywhere to $f(x)$, save possibly $x = 0$, and $x^{-\gamma} L(1/x) f(x) \in L(0,\pi)$.

<u>Theorem</u> I. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$, as $n \to \infty$.

(i) For $0 < \gamma < 2$, if $\sum_{1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges, then $\sum_{1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$.

(ii) For $0 < \gamma < 1$, if $\sum_{1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$, then $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges.

It may be remarked that if we examine the proof of $\int \to \Sigma$ in Theorems H and I, we find that it is sufficient to assume that $\{a_n\}$ is only a positive sequence.

In the present chapter our object is to generalise all the results stated above. In what follows, we prove the following theorems.

<u>Theorem 1. Let</u> $\{a_n\}$ <u>be a</u> δ-<u>quasi-monotone sequence and</u> $0 < \gamma < 1$. <u>If</u>

(5.2.1) $$\sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n < \infty ,$$

<u>and</u>

(5.2.2) $$\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n \quad \text{converges, then,}$$

(5.2.3) $$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{converges everywhere}$$

to $f(x)$ *except possibly at* $x=0$ *and*

(5.2.4) $$x^{-\gamma} L(1/x)\, f(x) \in L(0,\pi).$$

Conversely, if $\{a_n\}$ *is any sequence which is ultimately positive such that* (5.2.3) *holds and if* (5.2.4) *holds, then* (5.2.2) *holds.*

Theorem 2. (i) *Let* $\{a_n\}$ *be a* δ*-quasi-monotone sequence and* $0 < \gamma < 2$. If the series $\sum_1^\infty n^\gamma L(n)\, \delta_n$ *and* $\sum_1^\infty n^{\gamma-1} L(n) a_n$ *are convergent, then* $\sum_1^\infty a_n \sin nx$ *converges everywhere to* $g(x)$ *and* $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$.

(ii) *For* $0 < \gamma < 1$, *if* $\sum_1^\infty a_n \sin nx$ *converges everywhere to* $g(x)$ *and* $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$, *then* $\sum_1^\infty n^{\gamma-1} L(n)\, a_n$ *converges, where* $\{a_n\}$ *is any sequence which is ultimately positive.*

We follow Yong [1] in the proof of our theorems.

5.3 We require the following lemmas for the proof of our theorems.

Lemma 1. (Aljančić, Bojanić and Tomić [2]). *For* $\gamma > 0$,

$$0 < A_1 n^\gamma L(n) \le \sum_{k=1}^{n} k^{\gamma-1} L(k) \le A_2 n^\gamma L(n) \;,$$

where A_1 *and* A_2 *are positive constants.*

Lemma 2. (Aljančić, Bojanić and Tomić [1] *). *Let* α *and* β *be positive. If the integral*

$$\int_{+0}^{\infty} y^{k} \, |f(y)| \, dy \quad < \infty ,$$

for $-\alpha < k < \beta$, *then*

$$\int_{+0}^{\infty} f(y) \, L(\lambda y) \, dy \sim L(\lambda) \int_{+0}^{\infty} f(y) \, dy \, , \quad \lambda \to \infty .$$

Lemma 3. If $\{a_n\}$ *is a* δ*-quasi-monotone sequence with* $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, \delta_n < \infty$, $\gamma > 0$ *and* $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) \, a_n$ *converges, then* $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, |\triangle a_n| < \infty$. *Conversely, if* $a_n \to 0$ *and* $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, |\triangle a_n| < \infty$, *then* $\sum_{1}^{\infty} n^{\gamma-1} L(n) \, a_n$ *is convergent.*

Taking $L(x) = 1$ we get a result of Boas [2] while for $\delta_n = \frac{c \, a_n}{n}$ we have the corresponding result of Yong [1] .

Proof of Lemma 3. Since $\triangle a_n \geq -\delta_n$, we have

$$|\triangle a_n| \leq \triangle a_n + 2 \, \delta_n \, .$$

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* The authors use in their paper the asymptotic relation

$$\int_{c/\lambda}^{\infty} f(x) L(\lambda x) \, dx \sim L(\lambda) \int_{+0}^{\infty} f(x) \, dx \quad \text{for a fixed } c > 0.$$

First suppose that $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges. Then, by Lemma 1, we have

$$\sum_{k=1}^{\infty} k^{\gamma} L(k)\, |\Delta a_k| \leq A^{*} \sum_{k=1}^{\infty} |\Delta a_k| \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$\leq A \sum_{k=1}^{\infty} (\Delta a_k + 2\delta_k) \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$= A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu) \sum_{k=\nu}^{\infty} (\Delta a_k + 2\delta_k)$$

$$= A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu) \left(a_{\nu} + 2 \sum_{k=}^{\infty} \delta_k \right)$$

$$\leq A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu)\, |a_{\nu}| + 2A \sum_{k=1}^{\infty} \delta_k \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$\leq A \sum_{\nu=1}^{\infty} \nu^{\gamma-1} L(\nu)\, |a_{\nu}| + A \sum_{k=1}^{\infty} k^{\gamma} L(k)\, \delta_k$$

Now suppose that $\sum_{k=1}^{\infty} k^{\gamma} L(k)\, |\Delta a_k|$ converges. Then, by Abel's transformation, we have

$$\sum_{k=1}^{n+1} k^{\gamma-1} L(k)\, a_k = \sum_{k=1}^{n} \Delta a_k \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu) + a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu)$$

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* where A is a constant not necessarily the same at occurrence.

$$\leq \sum_{k=1}^{n} |\Delta a_k| \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu) + a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu)$$

$$\leq A \sum_{k=1}^{n} |\Delta a_k| \, k^{\gamma} L(k) + a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu).$$

Now

$$\left| a_{n+1} \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu) \right| = \left| \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu) \sum_{k=n+1}^{\infty} \Delta a_k \right|$$

$$\leq \sum_{\nu=1}^{n+1} \nu^{\gamma-1} L(\nu) \sum_{k=n+1}^{\infty} |\Delta a_k|$$

$$\leq \sum_{k=n+1}^{\infty} |\Delta a_k| \sum_{\nu=1}^{k} \nu^{\gamma-1} L(\nu)$$

$$\leq A \sum_{k=n+1}^{\infty} k^{\gamma} L(k) \, |\Delta a_k|$$

$\to 0$, $n \to \infty$, by virtue of the hypotheses.

Hence, $\sum_{k=1}^{\infty} k^{\gamma-1} L(k) \, a_k$ converges.

This proves Lemma 3.

5.4 <u>Proof of Theorem 1</u>. Proof of $\Sigma \to \int$: By virtue of the hypothesis $\Delta a_n \geq -\delta_n$, we have

$$|\Delta a_n| \leq \Delta a_n + 2\, \delta_n .$$

Also the convergence of the series $\sum_{n=1}^{\infty} n^{\gamma} L(n) \, \delta_n$ implies that $\sum_{n=1}^{\infty} \delta_n < \infty$. Therefore, using the condition that $a_n \to 0$, we have

$$\sum_{n=1}^{\infty} |\triangle a_n| \leq \sum_{n=1}^{\infty} \triangle a_n + 2 \sum_{n=1}^{\infty} \delta_n < \infty .$$

Thus $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges to $f(x)$ for all x except possibly $x = 0$.

By Abel's transformation, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n - a_{n+1}) \left(\sum_{k=1}^{n} \cos kx \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n - a_{n+1}) \left(\frac{\sin(n+\frac{1}{2})x - \sin {}^{x}/_{2}}{2 \sin {}^{x}/_{2}} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n - a_{n+1}) \frac{\cos(n+1)^{x}/_{2} \sin \frac{nx}{2}}{\sin {}^{x}/_{2}}$$

Since $\left(\frac{1}{x} \right)^{-\epsilon} L\left({}^{1}/_{x} \right) \to 0$, as $x \to 0$, it is easy to see that $x^{-\gamma} L\left({}^{1}/_{x} \right) \in L(0,\pi)$ for $0 < \gamma < 1$.

Therefore, it is sufficient to prove that

$$\int_0^{\pi} x^{-\gamma-1} L({}^{1}/_{x}) \left| \sum_{n=1}^{\infty} (a_n - a_{n+1}) \cos \left(\frac{n+1}{2}\right) x \sin \frac{nx}{2} \right| dx < \infty.$$

Now

$$\int_0^{\pi} x^{-\gamma-1} L({}^{1}/_{x}) \left| \sum_{n=1}^{\infty} (a_n - a_{n+1}) \cos\left(\frac{n+1}{2}\right) x \sin \frac{nx}{2} \right| dx$$

$$\leq A \int_0^\pi x^{-\gamma-1} L(1/x) \sum_{n=1}^\infty |\Delta a_n| \; |\sin \frac{nx}{2} \cos(\frac{n+1}{2}) \; x| \, dx$$

$$\leq A \sum_{n=1}^\infty |\Delta a_n| \int_0^\pi x^{-\gamma-1} L(1/x) \, |\sin \frac{nx}{2}| \, dx$$

$$(5.4.1) = A \sum_{n=1}^\infty |\Delta a_n| \, n^\gamma \, L(n) \, K_n ,$$

where

$$K_n = \frac{1}{n^\gamma L(n)} \int_0^\pi x^{-\gamma-1} L(1/x) \; |\sin \frac{nx}{2}| \, dx.$$

Putting $y = \frac{1}{nx}$, we have

$$K_n = \frac{1}{L(n)} \int_{1/n\pi}^\infty y^{\gamma-1} L(n\,y) \; |\sin \frac{1}{2y}| \, dy$$

$$\sim \frac{L(n)}{L(n)} \int_{+0}^\infty y^{\gamma-1} \; |\sin \frac{1}{2y}| \, dy \, , \; n \to \infty , \quad \text{by Lemma 2,}$$

$$\leq \int_{+0}^1 y^{\gamma-1} \, dy + A \int_1^\infty y^{\gamma-2} \, dy$$

$$< \infty .$$

Hence the expression in (5.4.1) is

$$\leq A \sum_{n=1}^\infty n^\gamma L(n) \; |\Delta a_n| < \infty ,$$

by virtue of Lemma 3 and the hypotheses.

Therefore,

$$x^{-\gamma} L(1/x)\, f(x) \in L(0,\pi).$$

<u>Proof of</u> $\int \to \Sigma$: By assumption $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges for $x > 0$. The condition

$$x^{-\gamma} L(1/x) f(x) \in L(0,\pi) \Rightarrow f(x) \in L(0,\pi),$$

and hence a_n is the Fourier coefficient of $f(x)$. Writing $p = [\frac{1}{\delta}]$ and $q = [\frac{1}{x}]$, where $0 < \delta < \pi$, we have

$$p \leq q \quad \text{for} \quad 0 \leq x \leq \delta ;$$

and

$$q \leq p \quad \text{for} \quad \delta \leq x \leq \pi .$$

Supposing, $a_n > 0$ for $n > p_0$, then

$$\sum_{n=p_0}^{p} n^{\gamma-1} L(n) a_n \leq \sum_{n=p_0}^{p} \max_{n \leq k < \infty} \{ k^{\gamma-1} L(k) \} a_n$$

$$= \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n)\, a_n$$

$$= \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \cos nx \, dx$$

$$\leq \frac{2}{\pi} \int_0^\pi |f(x)| \left| \sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \cos nx \right| dx$$

$$= A \left(\int_0^\delta + \int_\delta^\pi \right) = I_1 + I_2 \text{, say.}$$

$$I_1 \leq A \int_0^\delta |f(x)| \left(\sum_{n=p_0}^{p} n^{\gamma-1} \bar{L}_2(n) \right) dx$$

$$\leq A \int_0^\delta |f(x)| \; q^\gamma \; \bar{L}_2(q) \; dx$$

$$\leq A \int_0^\delta x^{-\gamma} L(1/x) \; |f(x)| \; dx$$

$$< \infty .$$

Also

$$I_2 \leq A \int_\delta^\pi |f(x)| \left(\sum_{n=p_0}^{q} n^{\gamma-1} \bar{L}_2(n) \; dx + A \int_\delta^\pi |f(x)| \; . \right.$$

$$\left| \sum_{n=q+1}^{p} n^{\gamma-1} \bar{L}_2(n) \cos nx \right| dx$$

$$\leq A \int_\delta^\pi x^{-\gamma} L(1/x) \; |f(x)| \; dx + A \int_\delta^\pi |f(x)| q^{\gamma-1} \bar{L}_2(q) \; .$$

$$\text{Max} \left| \sum_{q}^{p+1} \cos nx \right| dx$$

$$\leq A \int_\delta^\pi x^{-\gamma} L(1/x) \; |f(x)| \; dx + A \int_\delta^\pi |f(x)| \; q^{\gamma-1} \; \bar{L}_2(q) \frac{1}{\sin^x/2} \; dx$$

$$\leq A \int_\delta^\pi x^{-\gamma} L(1/x) \; |f(x)| \, dx$$

$$< \infty .$$

This proves $\int \to \Sigma$ part of Theorem 1.

Thus the Theorem 1 is proved.

5.5 <u>Proof of Theorem</u> 2. (i) <u>Proof of</u> $\Sigma \to \int$:

Since $\Delta a_n \geq -\delta_n$, we have $|\Delta a_n| \leq \Delta a_n + 2\delta_n$.
The convergence of the series $\sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n$ implies that
$\sum_{n=1}^{\infty} \delta_n < \infty$. Therefore, by using the condition that $a_n \to 0$,
we have

$$\sum_{n=1}^{\infty} |\Delta a_n| \leq \sum_{n=1}^{\infty} \Delta a_n + 2 \sum_{n=1}^{\infty} \delta_n < \infty .$$

Thus, $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ for every x.

Using Abel's transformation, we have

$$g(x) = \frac{1}{2 \sin \frac{x}{2}} \sum_{n=1}^{\infty} \left(\cos \frac{x}{2} - \cos \left(n + \tfrac{1}{2}\right) x \right) (a_n - a_{n+1})$$

$$= -\frac{1}{2} \tan \frac{x}{4} \sum_{n=1}^{\infty} \Delta a_n + \frac{1}{2 \sin \frac{x}{2}} \sum_{n=1}^{\infty} \Delta a_n \left(1 - \cos\left(n + \tfrac{1}{2}\right) x \right)$$

and hence

$$|g(x)| \leq \frac{1}{2} \tan \frac{x}{4} \sum_{n=1}^{\infty} |\Delta a_n| + \frac{x}{2 \sin \frac{x}{2}} G(x) ,$$

where $G(x) = \frac{1}{x} \sum_{1}^{\infty} |\triangle a_n| \; (1 - \cos (n + \frac{1}{2}) x)$.

Since $(\frac{1}{x})^{-\varepsilon} L(\frac{1}{x}) \to 0$, as $x \to 0$, it is easy to see that $x^{-\gamma} L (\frac{1}{x}) \tan \frac{x}{4} \in L(0,\pi)$. for $0 < \gamma < 2$.

Now

$$(5.5.1) \int_0^{\pi} x^{-\gamma} L(\frac{1}{x}) \; G(x) \; dx$$

$$= \int_0^{\pi} x^{-\gamma-1} L(\frac{1}{x}) \sum_{n=1}^{\infty} |\triangle a_n| \; (1-\cos(n+ \frac{1}{2}) \; x) \; dx$$

$$= \sum_{n=1}^{\infty} n^{\gamma} L(n) \; |\triangle a_n| \; I_n \; ,$$

where

$$I_n = \frac{1}{n^{\gamma} \; L(n)} \int_0^{\pi} x^{-\gamma-1} L(\frac{1}{x}) \; \left(1-\cos \; (n + \frac{1}{2})x \right) dx$$

$$= \frac{2}{n^{\gamma} \; L(n)} \int_0^{\pi} x^{-\gamma-1} L(\frac{1}{x}) \; \sin^2 \; (n+ \frac{1}{2}) \frac{x}{2} \; dx.$$

Putting $y = \frac{1}{(n+\frac{1}{2}) x}$, we have

$$I_n = \frac{2(n+ \frac{1}{2})^{\gamma}}{n^{\gamma} L(n)} \int_{1/(n+ \frac{1}{2})\pi}^{\infty} y^{\gamma-1} \sin^2 \frac{1}{2y} \; L(n+ \frac{1}{2}) \; y \;) \; dy$$

$$\sim \frac{2}{L(n)} \int_{1/(n+\frac{1}{2})x}^{\infty} y^{\gamma-1} L((n+\tfrac{1}{2})y)\ \sin^2 \frac{1}{2y}\, dy$$

$$\sim 2 \int_{+0}^{\infty} y^{\gamma-1} \sin^2 \frac{1}{2y}\, dy , \quad \text{as } n \to \infty, \text{ by Lemma 2.}$$

Then, as $n \to \infty$,

$$I_n \to 2 \int_{+0}^{\infty} y^{\gamma-1} \sin^2 \frac{1}{2y}\, dy < \infty, \quad 0 < \gamma < 2.$$

Hence the expression in (5.5.1) is

$$\leq A \sum_{n=1}^{\infty} n^{\gamma} L(n)\ |\Delta a_n| < \infty ,$$

by Lemma 3 and the hypotheses of Theorem 2.

Thus we have

$$\int_0^{\pi} x^{-\gamma} L(\tfrac{1}{x})\ |g(x)|\ dx \leq A + A \int_0^{\pi} x^{-\gamma} L(\tfrac{1}{x})\, G(x)\, dx$$
$$< \infty .$$

This proves $\Sigma \to \int$ part of Theorem 2.

(ii) <u>Proof of</u> $\int \to \Sigma$. The proof of this part is similar to that of Theorem 1 and hence omitted.