Chapter IV

INTEGRABILITY THEOREMS FOR TRIGONOMETRIC SERIES WITH QUASI-MONOTONE COEFFICIENTS

4.1. Let $f(x)$ and $g(x)$ be defined by the following trigonometric series :

(4.1.1) $$f(x) = \sum_{1}^{\infty} a_n \cos nx ,$$

(4.1.2) $$g(x) = \sum_{1}^{\infty} a_n \sin nx .$$

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone (Shah [1] ; Szász [1]) if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n} \right)$$

for all $n > n_0(\alpha)$, where $n_0(\alpha)$ is a positive number depending upon α.

An equivalent definition of quasi-monotone sequence (Shah [2]) is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

It is said to be a quasi-monotone of $\alpha = \alpha_0$ (Yong; [2]) if

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha_0}{n} \right).$$

A positive function $L(x)$ is said to be "slowly increasing" in the sense of Karamata [1] if it is continuous for $x \geq 0$ and

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for every fixed } t > 0.$$

Some of the important properties of such functions are as follows. 1)

P_1 : $\frac{L(tx)}{L(x)} \to 1$ as $x \to \infty$, uniformly for $0 < a \leq t \leq b < \infty$.

P_2 : $x^{\alpha} L(x) \to \infty$, $x^{-\alpha} L(x) \to 0$ as $x \to \infty$ for every $\alpha > 0$.

P_3 : If we write for some $\alpha > 0$,

$$\bar{L}_1(x) = x^{-\alpha} \underset{0\leq t\leq x}{\mathrm{Max}} \{t^{\alpha} L(t)\}, \quad \underline{L}_1(x) = x^{\alpha} \underset{0<t\leq x}{\mathrm{Min}} \{t^{-\alpha} L(t)\},$$

$$\bar{L}_2(x) = x^{\alpha} \underset{x\leq t\leq\infty}{\mathrm{Max}} \{t^{-\alpha} L(t)\}, \quad \underline{L}_2(x) = x^{-\alpha} \underset{x\leq t<\infty}{\mathrm{Min}} \{t^{\alpha} L(t)\},$$

then $\bar{\underline{L}}_k(x) \sim L(x)$ as $x \to \infty$ for $k = 1, 2$.

P_4 : For $\alpha > 0$, we have

$$L(tu) \leq A_1 t^{-\alpha} L(u) \text{ for every } u \geq 0 \text{ and } 0 < t \leq 1,$$

$$L(u/t) \leq A_2 t^{-\alpha} L(u) \text{ for every } u \geq 0 \text{ and } 0 < t \leq 1,$$

---

1) P_1, P_2, P_3 are due to J.Karamata [1] where P_4 is due to Igari [1].

where A_1 and A_2 are positive constants depending on α and L.

A function $\phi(x)$ is said to belong to the class $L(p,\alpha)$ (Askey and Wainger [1]) if

$$\int_0^\pi |\phi(x)|^p (\sin x)^{\alpha p}\, dx < \infty,\ p > 0,\ \alpha \text{ being any real number.}$$

We define the norm of a function $\phi(x) \in L(p,\alpha)$ as :

$$\|\phi(x)\|_{p,\alpha} = \left\{ \int_0^\pi |\phi(x)|^p (\sin x)^{\alpha p}\, dx \right\}^{1/p}.$$

4.2 Concerning the integrability of trigonometric series Igari [1] , in 1960, proved the following theorems.

__Theorem A.__ Suppose that $a_n \downarrow 0$, $p \geq 1$ and $-1 < \lambda < 0$. Then a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} n^{-1+p+p\lambda} L(n)\, a_n^p$$

should converge is that

$$x^{-1-\lambda p} L(1/x)\, f^p(x) \in L(0,\pi).$$

__Theorem B.__ Suppose that $a_n \downarrow 0$, $p \geq 1$ and $-1 < \lambda < 1$. Then a necessary and sufficient condition that

$$\sum_{n=1}^{\infty} n^{-1+p+p\lambda} L(n)\, a_n^p$$

should converge is that

$$x^{-1-\lambda p} L(1/x)\, g^{p}(x) \in L(0, \pi).$$

Theorem A is similar to Theorem B. The only difference between the two is that in the latter case range of λ has been extended to $\lambda < 1$.

These theorems were subsequently extended by Yong [2] to quasi-monotone sequences. His theorems are as follows :

<u>Theorem C</u>. Let $\{a_n\}$ be quasi-monotone of $\alpha < 1$ and such that $M_2 \geq n^{\beta} L_1(n)\, a_n \geq M_1 > 0$ with some $\beta > 0$ $(n=1,2,\ldots)$. If $p \geq 1$ and $1-p < \lambda < 1$, then $x^{-\lambda} L_2(1/x)\, f^{p}(x) \in L(0,\pi)$ iff $\sum_1^{\infty} n^{\lambda+p-2} L_2(n)\, a_n^{p}$ converges.

<u>Theorem D</u>. Let $\{a_n\}$ be quasi-monotone such that $M_2 \geq n^{\beta} L_1(n) a_n \geq M_1 > 0$ with some $\beta > 0$ $(n=1,2,\ldots)$. If $p \geq 1$ and $1-p<\lambda<1+p$, then $x^{-\lambda} L_2(1/x)\, g^{p}(x) \in L(0,\pi)$ iff $\sum_1^{\infty} n^{\lambda+p-2} L_2(n) a_n^{p}$ converges.

Concerning $L(p,\alpha)$ class, Askey and Wainger [1] in 1966 proved the following theorems.

<u>Theorem E</u>. Let $f(x) \in L(p,\alpha)$ with $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Let $f(x) \sim \sum_1^{\infty} a_n \cos nx$ with $a_n \geq 0$ and $A_n = \sum_{j=[n/2]}^{n} a_j$, then

$$\sum_1^{\infty} n^{-2-\alpha p} A_n^{p} < \infty,$$

and

$$\sum_{1}^{\infty} n^{-2-\alpha p} A_n^p \leq B(\alpha,p) \, \|f\|_{p,\alpha}^p .$$

__Theorem F__. Let $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and

$$\left\{ \sum_{n=1}^{\infty} n^{p-\alpha p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p \right\}^{1/p} < \infty ,$$

then $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ is in $L(p,\alpha)$ class and

$$\|f\|_{p,\alpha}^p \leq B(\alpha,p) \sum_{n=1}^{\infty} n^{p-\alpha p-2} \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p ,$$

where $\Delta a_k = a_k - a_{k+1}$.

From Theorems E and F they deduced the following interesting result.

__Theorem G__. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonically decreasing for some non-negative integer k. Let $1 \leq p < \infty$ and $-1 < \alpha p < p-1$, then a necessary and sufficient condition that $f(x) \in L(p,\alpha)$ is that

$$\sum_{1}^{\infty} n^{p-\alpha p-2} a_n^p < \infty ,$$

where $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

Later on Khan [1] in 1968, obtained several results involving $L(p,\alpha)$ class, which generalise all the above results for cosine series. His results are as follows .

<u>Theorem H</u>. Let $L^{1/p}(1/x)\ f(x) \in L(p,\alpha)$ with $1 \leq p < \infty$, $-1 < \alpha p < p-1$, where $f(x) \sim \sum_1^\infty a_n \cos nx$ with $a_n \geq 0$. If $A_n = \sum_{j=[n/2]}^{n} a_j$, then

$$\sum_1^\infty n^{-2-\alpha p}\ L(n)\ A_n^p < \infty ,$$

and

$$\sum_{n=1}^\infty n^{-2-\alpha p}\ L(n)\ A_n^p \leq B(\alpha,p) \| L^{1/p}(1/x)\ f(x) \|_{p,\alpha}^p .$$

<u>Theorem I</u>. Let $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Suppose that $\{ a_n \}$ is a sequence of numbers such that $a_n \to 0$ and

$$\left\{ \sum_1^\infty n^{p-\alpha p-2}\ L(n)\ \left(\sum_{j=n}^\infty |\triangle a_j| \right)^p \right\}^{1/p} < \infty .$$

Then

$$L^{1/p}(1/x)\ f(x) \in L\ (p,\alpha)$$

and

$$\| L^{1/p}(1/x) f(x) \|_{p,\alpha}^p \leq B(\alpha,p) \sum_1^\infty n^{p-\alpha p-2}\ L(n)\ \left(\sum_{j=n}^\infty |\triangle a_j| \right)^p ,$$

where $f(x) \sim \sum_1^\infty a_n \cos nx$ and $\triangle a_j = a_j - a_{j+1}$.

<u>Theorem J</u>. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonically decreasing for some non-negative integer k. If $1 \leq p < \infty$, and $-1 < \alpha p < p-1$, then $L^{1/p}(1/x)\ f(x) \in L\ (p,\alpha)$ iff $\sum_{1}^{\infty} n^{p-\alpha p-2}\ L(n)\ a_n^p < \infty$, where $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

In this chapter we propose to obtain certain generalization of all these results.

4.3 We prove the following theorems.

<u>Theorem 1.Let</u> $\lambda(\pi/x)\ L^{1/p}(1/x) f(x) \in L(p,\alpha)$ <u>with</u> $1 \leq p < \infty$, $-1 < \alpha p < p-1$, <u>where</u> $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ <u>with</u> $a_n \geq 0$. <u>If</u> $A_n = \sum_{j=[n/2]}^{n} a_j$, <u>then</u>

$$\sum_{n=1}^{\infty} n^{-2-\alpha p}\ L(n)\ \lambda^p(n)\ A_n^p < \infty,$$

<u>and</u>

$$\sum_{1}^{\infty} n^{-2-\alpha p}\ L(n)\ \lambda^p(n) A_n^p \leq B(\alpha,p) \| L^{1/p}(1/x)\lambda(\pi/x) f(x) \|_{p,\alpha}^p$$

<u>where</u> $\lambda(x)$ <u>is a positive function such that</u>

(4.3.1) $x^{-\alpha+1-1/p-\delta}\ \lambda(x) \uparrow$ <u>as</u> $x \to \infty$ for some small $\delta > 0$.

Theorem 2. *Let* $1 \leq p < \infty$ and $-1 < \alpha p < p-1$. *Suppose that* $\{a_n\}$ *is a sequence of numbers such that* $a_n \to 0$ *and*

$$\left\{ \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n)\, L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p} \right\}^{1/p} < \infty .$$

Then

$$L^{1/p}(1/x)\, \lambda(\pi/x)\, f(x) \in L(p,\alpha),$$

and

$$\| \lambda(\pi/x)\, L^{1/p}(1/x) f(x) \|_{p,\alpha} \leq B(\alpha,p) \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n) L(n) \left(\sum_{n}^{\infty} |\Delta a_j|\right)^{p}$$

where $\lambda(x)$ *is a positive function such that*

(4.3.p) $\quad x^{-\alpha-1/p+\delta} \lambda(x)\downarrow$ *as* $x \to \infty$ *for some small* $\delta > 0$,

and $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

Theorem 3. *Let* $\{a_n\}$ *be a positive sequence tending to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some non-negative real number* k. *If* $1 \leq p < \infty$, $-1 < \alpha p < p-1$ *and* $0 \leq \gamma < \alpha + 1/p$, *then*

$$x^{-\gamma} L^{1/p}(1/x)\, f(x) \in L(p,\alpha)$$

iff

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^{p} < \infty ,$$

where $f(x) \sim \sum_1^\infty a_n \cos nx$.

The corresponding theorems for sine series are as follows.

Theorem 4. Let $\lambda(\pi/x)\, L^{1/p}(1/x) g(x) \in L(p,\alpha)$ *with* $1 \le p < \infty$, $-1-p < \alpha p < p-1$, *where* $g(x) \sim \sum_1^\infty a_n \sin nx$ *with* $a_n \ge 0$. *If* $A_n = \sum_{j=[\frac{n}{2}]+1}^{n} a_j$, *then*

$$\sum_{n=1}^\infty n^{-2-\alpha p} \lambda^p(n) L(n) A_n^p \le B^*(\alpha,p) \, \| L^{1/p}(1/x) \lambda(\pi/x) g(x) \|_{p,\alpha}^p ,$$

where $\lambda(x)$ *satisfies the condition* (4.3.1).

Theorem 5. Let $1 \le p < \infty$ *and* $-1-p < \alpha p < p-1$. *Suppose that* $\{a_n\}$ *is a sequence of numbers such that* $a_n \to 0$ *and*

$$\left\{ \sum_{n=1}^\infty n^{p-\alpha p-2} \lambda^p(n)\, L(n) \left(\sum_{j=n}^\infty |\triangle a_j| \right)^p \right\}^{1/p} < \infty ,$$

then

$$L^{1/p}(1/x)\, \lambda(\pi/x)\, g(x) \in L(p,\alpha),$$

and

$$\| L^{1/p}(1/x) \lambda(\pi/x) g(x) \|_{p,\alpha}^p \le B(\alpha,p) \sum_1^\infty n^{p-\alpha p-2} \lambda^p(n) L(n) \left(\sum_{j=n}^\infty |\triangle a_j| \right)^p ,$$

---

* where B and $B(\alpha,p)$ are constants, not necessarily the same at each occurrence.

where $\lambda(x)$ *is a positive function such that*

(4.3.3) $\quad x^{-\alpha-1/p-1+\delta}\,\lambda(x)\downarrow$ *as* $x\to\infty$ *for some small* $\delta > 0$, *and* $g(x)\sim\sum_{1}^{\infty} a_n \sin nx.$

Theorem 6. Let $\{a_n\}$ *be a positive sequence tending to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some non-negative real number* k. *If* $1 \leq p \leq \infty$, $-1-p < \alpha p < p-1$ *and* $0 \leq \gamma < \alpha+1+1/p$, *then*

$$x^{-\gamma} L^{1/p}\left(\frac{1}{x}\right) g(x) \in L(p,\alpha)$$

iff

$$\sum_{1}^{\infty} n^{p-\alpha p-2+\gamma p} L(n)\, a_n^p < \infty,$$

where

$$g(x)\sim\sum_{1}^{\infty} a_n \sin nx.$$

4.4 We require the following lemmas for the proof of our Theorems.

Lemma 1. (Askey and Wainger [1]). *Let* $\{a_n\}$ *be positive and tend to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some* *non-negative real number* k, *then*

$$\sum_{j=n}^{\infty} |\Delta a_j| \leq B \sum_{j=n}^{\infty} \frac{a_j}{j} + a_n ,$$

where B *is some positive constant.**

Lemma 2. (Khan [1]). *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of positive terms such that*

$$\sum_{n=1}^{\infty} n^{-c} L(n) (n a_n)^p < \infty , \quad p \geq 1, \quad c < 1.$$

If

$$A_n = \sum_{\nu=n}^{\infty} a_\nu ,$$

then

$$\sum_{1}^{\infty} n^{-c} L(n) A_n^p < \infty ,$$

and

$$\sum_{1}^{\infty} n^{-c} L(n) A_n^p \leq K \sum_{1}^{\infty} n^{-c} L(n) (n a_n)^p ,$$

where K *is some constant depending on* c *and* p *and* L(x) *is a*

---

* In Lemma 1 the authors have assumed that k should be a non-negative integer but it can be easily verified that the Lemma remains true for all $k \geq 0$.

<u>slowly increasing function in the sense of Karamata</u> [1] .

<u>Lemma</u> 3. <u>Let</u> $f(x) \geq 0$ <u>for</u> $x \geq 0$ <u>and let</u> $F(x) = \int_0^x f(u)\,du$. <u>If</u> $q \geq p \geq 1$, <u>then</u>

$$(4.4.1) \quad \left\{ \int_0^\infty t^{-1} \{\mu(t)\, L(1/t) F(t)\}^q \, dt \right\}^{1/q} \leq K \left\{ \int_0^\infty t^{-1} \{t\, \mu(t) L(1/t) f(t)\}^p \, dt \right\}^{1/p},$$

<u>and</u>

$$\left\{ \int_0^\infty t^{-1} \{\mu(t)\, L(t)\, F(t)\}^q \, dt \right\}^{1/q} \leq K \left\{ \int_0^\infty t^{-1} \{t\, \mu(t)\, L(t)\, f(t)\}^p \, dt \right\}^{1/p},$$

<u>where</u> $\mu(t)$ <u>is a positive function such that</u> $t^{\delta} \mu(t)$ <u>is decreasing as</u> $t \to \infty$ <u>for some small</u> $\delta > 0$ <u>and</u> K <u>is a positive constant</u>.

It may be remarked that the special case $\mu(t) = t^{-1-\gamma}$, $\gamma > -1$ of this result is due to Igari [1] .

<u>Proof of Lemma 3</u>. First we prove (4.4.1). Consider the case $q \geq p > 1$. Put

$$J = \left[\int_0^\infty t^{-1} \{t\, \mu(t)\, L(1/t) f(t)\}^p \, dt \right]^{1/p}$$

and let α be a constant such that $\alpha < \frac{1}{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$. Applying Hölder's inequality with indices q, p' and $\frac{pq}{q-p}$, we have

$$F(t) = \int_0^t f(u)\, du$$

$$= \int_0^t u^{\alpha - \frac{(p-1)(q-p)}{pq}} \left(\mu(u)\, L(1/u)\right)^{-\frac{q-p}{q}} f^{p/q}(u)\, u^{-\alpha} \times$$

$$\left\{ u^{p-1} \left(\mu(u) L\left(\tfrac{1}{u}\right) f(u)\right)^p \right\}^{\frac{q-p}{pq}} du$$

$$\leq \left(\int_0^t u^{\alpha q - \frac{(p-1)(q-p)}{p}} \left(\mu(u) L\left(\tfrac{1}{u}\right)\right)^{-(q-p)} f^p(u)\, du \right)^{1/q} \times$$

$$\left(\int_0^t u^{-\alpha p'}\, du\right)^{1/p'} \left(\int_0^t u^{-1}\left(u\, \mu(u) L\left(\tfrac{1}{u}\right) f(u)\right)^p du\right)^{\frac{q-p}{pq}}$$

$$\leq K\, t^{1/p' - \alpha}\, J^{1-p/q} \left\{ \int_0^t u^{\alpha q - \frac{(p-1)(q-p)}{p}} \left(\mu(u) L\left(\tfrac{1}{u}\right)\right)^{-q+p} \cdot f^p(u)\, du \right\}^{1/q}$$

that is

$$F^q(t) \leq K t^{q/p' - \alpha q}\, J^{q-p} \int_0^t u^{\alpha q - \frac{(p-1)(q-p)}{p}} \left(\mu(u) L\left(\tfrac{1}{u}\right)\right)^{-q+p} f^p(u)\, du.$$

Now, we have

$$t^{-1}\left(\mu(t)\,L\left(\tfrac{1}{t}\right)\right)^{q} F^{q}(t)$$

$$\le K\, t^{-1-\alpha q+q/p'}\left(\mu(t)L\left(\tfrac{1}{t}\right)\right)^{q} J^{q-p}\int_{0}^{t} u^{\alpha q-\frac{(p-1)(q-p)}{p}} \left(\mu(u)L\left(\tfrac{1}{u}\right)\right)^{-q+p} f^{p}(u)\,du$$

or

$$\int_{0}^{\infty} t^{-1}\left(\mu(t)\,L\left(\tfrac{1}{t}\right) F(t)\right)^{q} dt$$

$$\le K\, J^{q-p}\int_{0}^{\infty} t^{-1-\alpha q+q/p'}\left(\mu(t)L\left(\tfrac{1}{t}\right)\right)^{q}\left(\int_{0}^{t} u^{\alpha q-\frac{(p-1)(q-p)}{p}} \cdot \left(\mu(u)L\left(\tfrac{1}{u}\right)\right)^{-q+p} f^{p}(u)\,du\right) dt$$

Changing the order of integration the above expression is

$$= K\, J^{q-p}\int_{0}^{\infty} u^{\alpha q-\frac{(p-1)(q-p)}{p}}\left(\mu(u)\ L\left(\tfrac{1}{u}\right)\right)^{-q+p} f^{p}(u) \left(\int_{u}^{\infty} t^{-1-\alpha q+q/p'}\left(\mu(t)L\left(\tfrac{1}{t}\right)\right)^{q} dt\right) du$$

$$= K\, J^{q-p}\int_{0}^{\infty} u^{-1+p}\left(\mu(u)L\left(\tfrac{1}{u}\right)\right)^{-q+p} f^{p}(u)\, K(u)\, du$$

where

$$K(u) = u^{\alpha q - q/p'} \int_u^\infty t^{-\alpha q + q/p' - 1} \left(\mu(t) L\left(\tfrac{1}{t}\right)\right)^q dt$$

$$\leq \mu^q(u)\, u^{\alpha q - q/p' + q\delta} \int_u^\infty t^{-\alpha q + q/p' - 1 - q\delta} L^q\left(\tfrac{1}{t}\right) dt$$

$$= \mu^q(u) \int_0^1 T^{\alpha q + q\delta - q/p' - 1} L^q\left(T/u\right) dT$$

$$\leq K\ \mu^q(u)\, L^q\left(\tfrac{1}{u}\right) \int_0^1 T^{\alpha q + q\delta - q/p' - 1 - \varepsilon q}\, dT$$

$$\leq K\ \mu^q(u)\, L^q\left(\tfrac{1}{u}\right)$$

provided that we first choose $\delta > \frac{1}{p'} - \alpha$ and thus $\varepsilon > 0$ such that $\alpha + \delta - \frac{1}{p'} - \varepsilon > 0$.

Thus we obtain

$$\int_0^\infty t^{-1} \left\{\mu(t)\, L\left(\tfrac{1}{t}\right) F(t)\right\}^q dt$$

$$\leq K\, J^{q-p} \int_0^\infty u^{-1+p} \left(\mu(u)\, L\left(\tfrac{1}{u}\right) f(u)\right)^p du$$

$$\leq K\, J^q$$

that is to say

$$\left\{\int_0^\infty t^{-1} \left(\mu(t) L\left(\tfrac{1}{t}\right) F(t)\right)^q dt\right\}^{1/q} \leq K \left\{\int_0^\infty t^{-1} \left(t\, \mu(t) L\left(\tfrac{1}{t}\right) f(t)\right)^p dt\right\}^{1/p}.$$

Thus, we have proved (4.4.1). In the case $q \geq p = 1$, put $\alpha = 0$ and the inequality may be obtained by similar arguements with indices q and q'. The inequality (4.4.2) can be proved in a similar manner.

This completes the proof of Lemma 3.

4.5 Proof of Theorem 1. Let

$$f_1(x) = \int_0^x f(u)\, du\ , \quad f_2(x) = \int_0^x f_1(u)\, du\ ,$$

then

$$f_2(x) = \sum_{j=1}^{\infty} a_j \cdot j^{-2} (1 - \cos jx)$$

$$\geq \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-2} (1 - \cos jx)$$

$$= 2 \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-2} \sin^2 \frac{jx}{2}$$

$$\geq 2 \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-2} \frac{4}{\pi^2} \cdot \frac{j^2 x^2}{4}\ , \quad \pi/n+1 \leq x \leq \pi/n\ ,$$

$$= B\, x^2 \sum_{j=[\frac{n}{2}]}^{n} a_j\ ,$$

where B is some positive constant.

Now

$$\sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p} (n) A_n^{p}$$

$$\leq B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p-2p} L\left(\frac{1}{x}\right) \lambda^{p} (\pi/x) f_2^{p} (x) \, dx$$

$$= B \int_{0}^{\pi/2} x^{\alpha p-2p} L\left(\frac{1}{x}\right) \lambda^{p} (\pi/x) f_2^{p} (x) \, dx$$

$$\leq B \int_{0}^{\pi/2} x^{\alpha p-p} L\left(\frac{1}{x}\right) \lambda^{p}(\pi/x) \left|f_1(x)\right|^{p} dx$$

$$\leq B \int_{0}^{\pi/2} x^{\alpha p} L\left(\frac{1}{x}\right) \lambda^{p} (\pi/x) \left|f(x)\right|^{p} dx$$

$$\leq B \int_{0}^{\pi/2} (\sin x)^{\alpha p} L\left(\frac{1}{x}\right) \lambda^{p} (\pi/x) \left|f(x)\right|^{p} dx$$

$$\leq B \left\| L^{1/p} \left(\frac{1}{x}\right) \lambda (\pi/x) f(x) \right\|_{p,\alpha}^{p}$$

$$< \infty,$$

by virtue of the result (4.4.1) of Lemma 3 and the hypotheses of Theorem 1.

This completes the proof of Theorem 1.

4.6 <u>Proof of Theorem</u> 2. Since we are given that

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n) L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p} < \infty ,$$

it follows on putting $n = 1$ that

$$\sum_{j=1}^{\infty} |\Delta a_j| < \infty ,$$

and therefore the fourier series of f converges for $x > 0$ (Zygmund [1]), and we have

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \sum_{n=1}^{k} a_n \cos nx + \sum_{n=k+1}^{\infty} a_n \cos nx.$$

Let $D_n(x) = \frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu x$, then

$$\sum_{n=k+1}^{N} a_n \cos nx = \sum_{n=k+1}^{N-1} \Delta a_n \left(\sum_{\nu=0}^{n} \cos \nu x \right) + a_N \sum_{\nu=0}^{N} \cos \nu x$$

$$- a_{k+1} \sum_{\nu=0}^{k} \cos \nu x$$

$$= \sum_{n=k+1}^{N-1} \Delta a_n \left(\frac{1}{2} + D_n(x)\right) + a_N\left(\frac{1}{2} + D_N(x)\right) - a_{k+1}\left(\frac{1}{2} + D_k(x)\right)$$

$$= \sum_{n=k}^{N} \Delta a_n \left(\frac{1}{2} + D_n(x) \right) - \Delta a_N \left(\frac{1}{2} + D_N(x) \right)$$

$$- \Delta a_k\left(\frac{1}{2} + D_k(x)\right) + a_N\left(\frac{1}{2} + D_N(x)\right) - a_{k+1}\left(\frac{1}{2} + D_k(x) \right)$$

$$= \sum_{n=k}^{N} (\triangle a_n)\, D_n(x) + a_{N+1}\, D_N(x) - a_k\, D_k(x).$$

Now making $N \to \infty$, we have

$$\sum_{n=k+1}^{\infty} a_n \cos nx = \sum_{n=k}^{\infty} (\triangle a_n)\, D_n(x) - a_k\, D_k(x).$$

Hence for any n ,

$$|f(x)| \leq \sum_{n=1}^{k} |a_n| + \sum_{n=k}^{\infty} |\triangle a_n|\, |D_n(x)| + |a_k|\, |D_k(x)|$$

Since, $D_n(x) = O(1/x)$, $\pi/n+1 \leq x \leq \pi/n$, we have

$$|f(x)| \leq A_k + O(1/x) \sum_{n=k}^{\infty} |\triangle a_n| + O(1/x)\, |a_k| \, .$$

where

$$A_k = \sum_{n=1}^{k} |a_n| \, .$$

Now,

$$\int_0^{\pi/2} L(\tfrac{1}{x})\, \lambda^p(\pi/x)\, (\sin x)^{\alpha p}\, |f(x)|^p\, dx$$

$$\leq B \int_0^{\pi/2} x^{\alpha p}\, L(\tfrac{1}{x})\, \lambda^p(\pi/x)\, |f(x)|^p\, dx$$

$$= B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p}\, L(\tfrac{1}{x})\, \lambda^p(\pi/x)\, |f(x)|^p\, dx$$

$$\leq B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) \Big(A_n + \tfrac{1}{x} \sum_{k=n}^{\infty} |\Delta a_k| + \tfrac{1}{x} |a_n| \Big)^{p} dx$$

$$\leq B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) A_n^{p}\, dx +$$

$$+ B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) x^{-p} \Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^{p} dx +$$

$$+ B \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{\alpha p} L(\tfrac{1}{x}) \lambda^{p}(\pi/x) x^{-p} |a_n|^{p} dx$$

$$\leq B \sum_{n=1}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) A_n^{p} +$$

$$+ B \sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n) \lambda^{p}(n) \Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^{p} +$$

$$+ B \sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n) \lambda^{p}(n) |a_n|^{p}$$

$$\leq B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) A_{n-2}^{p}$$

$$+ B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^{p}(n) \big(|a_{n-1}| + |a_n|\big)^{p}$$

$$+ B \sum_{n=2}^{\infty} n^{p-\alpha p-2} L(n) \lambda^{p}(n) \Big(\sum_{k=n}^{\infty} |\Delta a_k|\Big)^{p}$$

$= J_1 + J_2 + J_3$, say.

Evidently, in view of the hypothesis $J_3 = O(1)$.

Again,

$$J_2 \leq B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^p(n) |a_{n-1}|^p$$

$$+ B \sum_{n=2}^{\infty} n^{-2-\alpha p} L(n) \lambda^p(n) |a_n|^p$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p-2} L(n) \lambda^p(n) |a_n|^p$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p-2} L(n) \lambda^p(n) \left(\sum_{k=n}^{\infty} |\Delta a_k| \right)^p$$

$$= O(1) .$$

Now put $q(x) = |a_n|$ for $n \leq x < n+1$ with $a_0 = 0$ ($n = 0, 1, 2, \ldots$) and

$$Q(x) = \int_0^x q(t)\, dt ,$$

then we have

$$J_1 \leq B \sum_{n=2}^{\infty} \int_{n-1}^{n} x^{-\alpha p-2} L(x) \lambda^p(x) Q^p(x)\, dx$$

$$= B \int_1^{\infty} x^{-\alpha p-2} L(x) \lambda^p(x) Q^p(x)\, dx .$$

Applying the result (4.4.2) of Lemma 3 (on taking q=p and $\mu(x) = x^{-\alpha-1/p}\lambda(x)$) we observe that the above integral is

$$\leq B \int_1^\infty x^{-2-\alpha p+p} L(x)\, \lambda^p(x)\, q^p(x)\, dx$$

$$= B \sum_{n=1}^\infty \int_n^{n+1} x^{-\alpha p-2+p} L(x)\, \lambda^p(x)\, q^p(x)\, dx$$

$$\leq B \sum_{n=1}^\infty n^{p-\alpha p-2} L(n)\, \lambda^p(n)\, |a_n|^p$$

$$= O(1).$$

Similarly we can prove

$$\int_{\pi/2}^{\pi} L^{1/p}(1/x)\, \lambda^p(\pi/x)\, |f(x)|^p \sin^{\alpha p} x\, dx < \infty .$$

Hence,

$$L^{1/p}(1/x)\, \lambda(\pi/x)\, f(x) \in L(p,\alpha),$$

and

$$\| L^{1/p}(1/x)\lambda(\pi/x) f(x) \|_{p,\alpha} \leq B \sum_{n=1}^\infty n^{p-\alpha p-2} L(n) \lambda^p(n) \Big(\sum_{k=n}^\infty |\Delta a_k| \Big)^p.$$

This completes the proof of Theorem 2.

4.7 <u>Proof of Theorem 5.</u> <u>Necessity</u>: Suppose that $x^{-\gamma} L^{1/p}(1/x) f(x) \in L(p,\alpha)$, then we have to prove that

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p < \infty .$$

Since $\{n^{-k} a_n\}$ is monotonic decreasing, we have

$$a_n = a_n \cdot n^{-k}\, n^{k} \leq B\, n^{k-1} \sum_{j=[\frac{n}{2}]}^{n} a_j \cdot j^{-k}$$

$$\leq B\, n^{-1} \sum_{j=[\frac{n}{2}]}^{n} a_j .$$

We have therefore by Theorem 1,

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p$$

$$\leq B \sum_{n=1}^{\infty} n^{-\alpha p+\gamma p-2} L(n) \left(\sum_{j=[\frac{n}{2}]}^{n} a_j \right)^p$$

$$< \infty .$$

<u>Sufficiency</u> : Now suppose that

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p < \infty .$$

Then

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^p$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n) \left(\sum_{j=n}^{\infty} \frac{a_j}{j} + a_n \right)^p, \quad \text{(by Lemma 1)}$$

$$\leq B \sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n) \left(\sum_{j=n}^{\infty} \frac{a_j}{j} \right)^p + B \sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2} L(n)\, a_n^p$$

$$< \infty, \qquad \text{(by Lemma 2)}$$

and therefore by Theorem 2 ,

$$L^{1/p}(1/x)\, x^{-\gamma} f(x) \in L\,(p, \alpha).$$

Thus Theorem 3 is proved.

4.8 The proofs of Theorems 4, 5 and 6 are similar to those of Theorems 1, 2 and 3 and hence omitted.