Chapter III

ON FOURIER COEFFICIENTS WITH POSITIVE FUNCTIONS

3.1 A function $\phi(x)$ is said to belong to the class $L(p,\alpha)$ (Askey and Wainger [1]) if

$$\int_0^\pi |\phi(x)|^p (\sin x)^{\alpha p} dx < \infty,$$

α is a real number and $p > 0$.

We define the norm of a function $\phi(x) \in L(p,\alpha)$ as :

$$\|\phi(x)\|_{p,\alpha} = \left\{ \int_0^\pi |\phi(x)|^p (\sin x)^{\alpha p} dx \right\}^{1/p}.$$

It is evident that $L(p,\alpha) \Rightarrow L^p$ for $\alpha < 0$, $L^p \Rightarrow L(p,\alpha)$ for $\alpha > 0$ and $L(p,\alpha) \equiv L^p$ where $\alpha = 0$.

Throughout this Chapter, K with or without suffixes, denotes a positive constant, not necessarily the same at each occurrence.

3.2 Concerning the Fourier series of positive functions Askey and Boas [1] proved the following theorems.

__Theorem__ A. Let $G(x)\downarrow$ on $(0,\pi)$, G bounded below and

$$\int_0^\pi x\, dG(x) ,$$

finite, so that dG has generalized sine coefficients

$$b_n = \frac{2}{\pi}\int_0^\pi \sin nx\, dG(x)$$

If $1 < p < \infty$ and $\frac{1}{p} < \gamma < 1 + \frac{1}{p}$, then $\{n^{-\gamma} b_n\} \in \ell^p$ if, and only if

$$t^{\gamma-1-2/p} \int_0^t x\, dG(x) \in L^p .$$

__Theorem__ B. Let $F(x)\downarrow$ on $(0,\pi)$, F bounded below and

$$\int_0^\pi x^2\, dF(x)$$

finite. Let

$$a_n = -\frac{2}{\pi}\int_0^\pi (1-\cos nx)\, dF(x)$$

be the generalized cosine coefficients of dF. If $1 < p < \infty$ and $\frac{1}{p} < \gamma < 2+ 1/p$, then $\{n^{-\gamma} a_n\} \in \ell^p$ if, and only if

$$t^{\gamma-2-2/p} \left(\int_0^t u^2\, dF(u)\right) \in L^p .$$

Theorem C. If $-\frac{1}{p'} < \gamma < \frac{1}{p}$ and $\{n^{-\gamma} a_n\} \in \ell^p$, where a_n are the Fourier coefficients of dF with F monotonic, then

$$t^{\gamma - 2/p} \left[F(t) - F(0)\right] \in L^p, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem D. If $-\frac{1}{p'} < \gamma < \frac{1}{p}$, a_n are the Fourier cosine coefficients of f and $t^{\gamma-2/p} \left(\int_0^t x \,|\, df(x)| \right) \in L^p$, then $\{n^{-\gamma} a_n\} \in \ell^p$.

Recently, Mazhar and Khan [1] have generalised the above theorems in the following form.

Theorem E. Let $G(x)$ satisfy the conditions of Theorem A. If $1 < p < \infty$, $\lambda(x)$ is a positive function such that

(3.2.1) $x^{1+\delta} \lambda(x)$ is decreasing for some small $\delta > 0$,

(3.2.2) $x^{p+1-\delta} \lambda(x) \uparrow +\infty$ for some small $\delta > 0$, as $x \to \infty$, then

$$\left\{ \lambda^{1/p}(n)\, b_n \right\} \in \ell^p$$

if, and only if,

$$\lambda^{1/p} (x/t)\; t^{-1-2/p} \int_0^t x \, dG(x) \in L^p.$$

Theorem F. Let $F(x)$ satisfy the conditions of Theorem B and let

$$a_n = - \frac{2}{\pi} \int_0^{\pi} (1 - \cos nx)\, dF(x)$$

be the generalized cosine coefficients of dF. If $1 < p < \infty$ and $\lambda(x)$ is a positive function such that

(3.2.3) $x^{1+\delta} \lambda(x)$ is decreasing for some small $\delta > 0$,

(3.2.4) $x^{2p+1-\delta} \lambda(x) \uparrow +\infty$ for some small $\delta > 0$, then

$$\{ \lambda^{1/p}(n)\, a_n \} \in \ell^p$$

if, and only if

$$\lambda^{1/p}(\pi/t)\, t^{-2-2/p} \int_0^t x^2\, dF(x) \in L^p.$$

Theorem G. If a_n are the Fourier coefficients of dF with F monotonic and $\mu(x)$ is a positive function such that

(3.2.5) $x^{1+\delta-p} \mu(x) \downarrow 0$ for some small $\delta > 0$, $x \to \infty$,

(3.2.6) $x^{1-\delta} \mu(x) \uparrow +\infty$ for some small $\delta > 0$, $x \to \infty$, and

$$\{ \mu^{1/p}(n)\, a_n \} \in \ell^p,$$

then

$$\mu^{1/p}(\pi/t)\, t^{-2/p}\,[F(t) - F(0)] \in L^p.$$

<u>Theorem H.</u> If a_n are the Fourier cosine coefficients of f and if

$$\mu^{1/p}(\pi/t)\, t^{-2/p} \int_0^t x\,|df(x)| \in L^p,$$

then

$$\{\mu^{1/p}(n)\, a_n\} \in \ell^p,$$

where $\mu(x)$ satisfies the same conditions as in Theorem G.

The object of this chapter is to prove the following theorems in which the class L^p has been replaced by a wider class $L(p,\alpha)$.

3·3. We prove the following Theorems.

<u>Theorem 1.</u> <u>Let</u> $G(x)$ <u>satisfy the conditions of Theorem</u> A. <u>If</u> $1 < p < \infty$, $-1 < \alpha p < p-1$, <u>then</u>

$$\{n^{-\alpha}\, \lambda^{1/p}(n)\, b_n\} \in \ell^p$$

<u>if, and only if</u>

$$\lambda^{1/p}(\pi/t)\, t^{-1-2/p} \left(\int_0^t x\,d\,G(x)\right) \in L(p,\alpha).$$

where $\lambda(x)$ *is a positive function such that*

(3.3.1) $x^{1+\delta-\alpha p}\lambda(x)\downarrow$ *for some small* $\delta > 0$,

(3.3.2) $x^{p+1-\delta}\lambda(x)\uparrow +\infty$ *for some small* $\delta > 0$, $x \to \infty$.

Theorem 2. Let $F(x)$ *satisfy the conditions of Theorem* B, *and let*

$$a_n = -\frac{2}{\pi}\int_0^\pi (1-\cos nx)\, dF(x)$$

be the generalized cosine coefficients of dF. If $1 < p < \infty$, *and* $-1 < \alpha p < p-1$, *then*

$$\{n^{-\alpha}\lambda^{1/p}(n)\, a_n\} \in \ell^p,$$

if, and only if

$$\lambda^{1/p}(\pi/t)\, t^{-2-2/p}\left(\int_0^t x^2\, dF(x)\right) \in L(p,\alpha),$$

where $\lambda(x)$ *is a positive function such that*

(3.3.3) $x^{1+\delta-\alpha p}\lambda(x)\downarrow$ *for some small* $\delta > 0$,

(3.3.4) $x^{2p+1-\delta}\lambda(x)\uparrow +\infty$ *for some small* $\delta > 0$.

<u>Theorem</u> 3. <u>If</u> a_n <u>are the Fourier coefficients of</u> dF <u>with</u> F <u>monotonic and</u> $\mu(x)$ <u>is a positive function such that</u>

(3.3.5) $x^{1+\delta-p-\alpha p}\mu(x)\downarrow 0$ <u>for some small</u> $\delta > 0$, $x\to\infty$, <u>and</u>

(3.3.6) $x^{1-\eta}\mu(x)\uparrow +\infty$ <u>for some small</u> η, $x\to\infty$.

<u>If</u> $\{n^{-\alpha}\mu^{1/p}(n)\, a_n\}\in \ell^p$, <u>then</u>

$$\mu^{1/p}(\pi/t)\; t^{-2/p}\,[F(t)-F(0)] \in L(p,\alpha),$$

<u>where</u> $1 < p < \infty$ <u>and</u> α <u>is a real number such that</u> $-1<\alpha p<\eta$.

<u>Theorem</u> 4. <u>If</u> a_n <u>are the Fourier cosine coefficients of</u> f <u>and if</u>

$$\mu^{1/p}(\pi/t)\,t^{-2/p}\Big(\int_0^t x\,|df(x)|\Big)\in L(p,\alpha),\ 1<p<\infty,\ -1<\alpha p< p-1,$$

<u>then</u> $\{n^{-\alpha}\mu^{1/p}(n)\, a_n\}\in \ell^p$, <u>where</u> $\mu(x)$ <u>satisfies the same conditions as in Theorem</u> 3.

3.4. The following lemmas are pertinent for the proof of our theorems.

<u>Lemma</u> 1. (Askey and Boas [1]). <u>If</u> $G(x)\downarrow$, $\int_0^\pi x|dG(x)|<\infty$,

and b_n are the generalised sine coefficients of dG, then

$$\left| \sum_{\nu=1}^{n}{}' b_\nu \right| \geq K n^2 \int_0^{\pi/n} x \, |dG(x)| ,$$

where Σ' has the last term of the sum halved.

Lemma 2. (Khan [3]). Let $\lambda(n)$ be a positive monotonic decreasing sequence such that

$$\sum_{k=n}^{\infty} \lambda(k) = O(n \lambda(n)) , \; n \to \infty .$$

Let $s_n = \sum_{k=1}^{n} a_k$, $a_k \geq 0$ and $\sum_{n=1}^{\infty} \lambda(n) \; (n \, a_n)^p < \infty$, $p>1$,

then $\sum_{n=1}^{\infty} \lambda(n) \, s_n^p < \infty$ and

$$\sum_{n=1}^{\infty} \lambda(n) \, s_n^p \leq K \sum_{n=1}^{\infty} \lambda(n) \; (n \, a_n)^p .$$

Lemma 3. (Khan [3]). Suppose that $\phi(x)$ increases and is bounded with $\phi(+0) = 0$. Let $\psi(x)$ be a function such that

$$\int_0^{\pi} \psi(u) \, \phi^p(u) \, du < \infty ,$$

then

$$\int_0^{\pi} \psi(u) \, u^{sp} \left(\int_u^{\pi} x^{-s} \, d\phi(x) \right)^p du < \infty ,$$

where $s > 0$, $p > 1$, and

(3.4.1) $x^{1-\delta+\alpha p}\,\psi(x)$ *is a positive increasing function for some small* $\delta > 0$.

3.5 *Proof of Theorem* 1. *Necessity*. We have by Lemma 1 and 2

$$\int_0^\pi u^{-p-2}\,\lambda(\pi/u)\,\left(\int_0^u x\,|dG(x)|\right)^p \sin^{\alpha p} u\,du$$

$$= \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} u^{-p-2}\,\lambda(\pi/u)\,\sin^{\alpha p} u\,\left(\int_0^u x\,|dG(x)|\right)^p du + \int_{\pi/2}^{\pi} u^{-p-2}\,\lambda(\pi/u)\sin^{\alpha p} u\left(\int_0^u x\,|dG(x)|\right)^p du$$

$$\leq K \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} u^{\alpha p-p-2}\,\lambda(\pi/u)\left(\int_0^u x\,|dG(x)|\right)^p du + \left(\tfrac{\pi}{2}\right)^{-p-2}\lambda(2)\left(\int_0^\pi x\,|dG(x)|\right)^p \int_{\pi/2}^{\pi} \sin^{\alpha p} u\,du$$

$$\leq K \sum_{n=2}^{\infty} n^{p-\alpha p}\lambda(n)\,\left(\int_0^{\pi/n} x\,|dG(x)|\right)^p + K \int_0^{\pi/2} \sin^{\alpha p} u\,du$$

$$\leq K \sum_{n=2}^{\infty} n^{-p-\alpha p}\,\lambda(n)\,\left(\sum_{k=1}^{n}{}' \,|b_k|\right)^p + K \int_0^{\pi/2} u^{\alpha p}\,du$$

$$\leq K \sum_{n=2}^{\infty} n^{-p-\alpha p}\,\lambda(n)\,\left(n\,|b_n|\right)^p + K$$

$$= K \sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \, |b_n|^p + K$$

$$< \infty .$$

<u>Sufficiency</u>. Let $\|a_n\|$ denote the norm of a_n in ℓ^p space, that is to say

$$\|a_n\| = \Big(\sum_{n=1}^{\infty} |a_n|^p \Big)^{1/p}.$$

We have

$$\|\lambda^{1/p}(n)\, n^{-\alpha}\, b_n\| = K \| \lambda^{1/p}(n)\, n^{-\alpha} \int_0^{\pi} \sin nx \, dG(x) \|$$

$$= K \| \lambda^{1/p}(n) . n^{-\alpha} \Big(\int_0^{1/n} + \int_{1/n}^{\pi} \Big) \sin nx \, dG(x) \|$$

$$\leq K \| \lambda^{1/p}(n) n^{-\alpha} \int_0^{1/n} \sin nx \, dG(x) \| +$$

$$+ K \| n^{-\alpha} \lambda^{1/p}(n) \int_{1/n}^{\pi} \sin nx \, dG(x) \|$$

$$\leq K \| \lambda^{1/p}(n) n^{1-\alpha} \int_0^{1/n} x \, |dG(x)| \| +$$

$$+ K \| \lambda^{1/p}(n) n^{-\alpha} \int_{1/n}^{\pi} |dG(x)| \|$$

$$\leq K \Big\{ \sum_{n=1}^{\infty} \lambda(n) n^{p-\alpha p} \Big(\int_0^{\pi/n} x \, |dG(x)| \Big)^p \Big\}^{1/p} +$$

$$+ K\left\{ \sum_{n=1}^{\infty} \lambda(n) n^{-\alpha p} \left(\int_{1/n}^{\pi} |dG(x)| \right)^{p} \right\}^{1/p}$$

$$\leq K\left(\sum_{n=3}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t).\ t^{-p-2+\alpha p} \left(\int_{0}^{t} x\,|dG(x)|\right)^{p} dt \right)^{1/p} +$$

$$+ K\left(\sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{1/n}^{\pi/n} + \int_{\pi/n}^{\pi}\right)^{p} \right)^{1/p} + K$$

$$\leq K\left(\int_{0}^{\pi/2} t^{-p-2+\alpha p} \lambda(\pi/t) \left(\int_{0}^{t} x\,|dG(x)|\right)^{p} dt \right)^{1/p} +$$

$$+ K\left(\sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{1/n}^{\pi/n} |dG(x)| \right)^{p} \right)^{1/p} +$$

$$+ K\left(\sum_{n=2}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{\pi/n}^{\pi} |dG(x)| \right)^{p} \right)^{1/p} + K$$

$$= I_1^{1/p} + I_2^{1/p} + I_3^{1/p} + K \text{, say .}$$

Since

$$\int_{0}^{\pi} \lambda(\pi/t)\ t^{-2-p} \sin^{\alpha p} t \left(\int_{0}^{t} x\,|dG(x)| \right)^{p} dt < \infty,$$

We have

$$I_1 = K \int_{0}^{\pi/2} \lambda(\pi/t)\ t^{-p-2+\alpha p} \left(\int_{0}^{t} x\,|dG(x)|\right)^{p} dt$$

$$\leq K \int_0^{\pi/2} \lambda\, (\pi/t)\, t^{-p-2} \sin^{\alpha p} t \, \left(\int_0^t x\, |dG(x)|\right)^p dt < \infty.$$

Also we have

$$I_2 = \sum_{n=2}^{\infty} \lambda(n)\, n^{-\alpha p} \left(\int_{1/n}^{\pi/n} |dG(x)|\right)^p$$

$$\leq K + K \sum_{n=3}^{\infty} \lambda(n)\, n^{p-\alpha p} \left(\int_{1/n}^{\pi/n} x\, |dG(x)|\right)^p$$

$$\leq K + K \sum_{n=3}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t)\, t^{-p-2+\alpha p} \left(\int_{1/n}^{t} x\, |dG(x)|\right)^p dt$$

$$\leq K + K \int_0^{\pi/2} \lambda\, (\pi/t)\, t^{-p-2+\alpha p} \left(\int_0^t x\, |dG(x)|\right)^p dt$$

$$< \infty .$$

Now

$$I_3 \leq K \sum_{n=1}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{\pi/n}^{\pi} |dG(x)|\right)^p$$

$$\leq K \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dG(x)|\right)^p dt$$

$$= K \int_0^{\pi} t^{\alpha p-2} \lambda\, (\pi/t) \left(\int_t^{\pi} |dG(x)|\right)^p dt.$$

By taking $\phi(t) = \int_0^t x\,|dG(x)|$, $s=1$, $\psi(t) = t^{-p-2+\alpha p}\,\lambda(\pi/t)$

and applying Lemma 3, we find, in view of I_1 , that

$$\int_0^\pi \lambda(\pi/t).t^{-p-2+\alpha p}.t^p\;\left(\int_t^\pi x^{-1}\,x\,|dG(x)|\right)^p dt < \infty ,$$

that is to say

$$I_3 < \infty .$$

Hence

$$\| \lambda^{1/p}(n)\; n^{-\alpha}\; b_n \| < \infty .$$

This proves Theorem 1.

3.6 Proof of Theorem 2. The proof of this theorem is similar to that of Theorem 1. However, for the sake of completeness we include a proof here.

Necessity. We have in view of (3.3.3) and (3.3.4)

$$\int_0^\pi \lambda(\pi/t)\; t^{-2p-2}\;\sin^{\alpha p} t\;\left(\int_0^t u^2\;|dF(u)|\right)^p dt$$

$$= \sum_{n=2}^{\infty} \int_{\pi/_{n+1}}^{\pi/n} \lambda(\pi/t)\,t^{-2p-2}\;\sin^{\alpha p} t\left(\int_0^t u^2\;|dF(u)|\right)^p dt +$$

$$+ \int_{\pi/2}^{\pi} \lambda(\pi/t)\, t^{-2p-2} \sin^{\alpha p} t \left(\int_0^t u^2 \, |dF(u)|\right)^p dt$$

$$\leq K \sum_{n=2}^{\infty} \int_{\pi/n+1}^{\pi/n} \lambda(\pi/t)\, t^{\alpha p-2p-2} \left(\int_0^t u^2 \, |dF(u)|\right)^p dt \, +$$

$$+ \lambda(2)(\pi/2)^{-2p-2} \left(\int_0^{\pi} u^2 \, |dF(u)|\right)^p \int_{\pi/2}^{\pi} \sin^{\alpha p} t \, dt$$

$$\leq K \sum_{n=2}^{\infty} \lambda(n) n^{2p-\alpha p} \left(\int_0^{\pi/n} u^2 \, |dF(u)|\right)^p + K \int_0^{\pi/2} \sin^{\alpha p} t \, dt$$

$$\leq K \sum_{n=2}^{\infty} \lambda(n)\, n^{-\alpha p} \, |a_n|^p + K \int_0^{\pi/2} t^{\alpha p} \, dt$$

$$< \infty ,$$

by virtue of the hypothesis and the fact that

$$|a_n| = \frac{2}{\pi} \int_0^{\pi} (1 - \cos nx) \, |dF(x)|$$

$$= \frac{2}{\pi} \cdot 2 \int_0^{\pi} \sin^2 \frac{nx}{2} \, |dF(x)|$$

$$\geq K \int_0^{\pi/n} \sin^2 \frac{nx}{2} \, |dF(x)|$$

$$\geq K \int_0^{\pi/n} n^2 x^2 \, |dF(x)|$$

$$= K n^2 \int_0^{\pi/n} x^2 \, |dF(x)| \, .$$

Sufficiency.

$$\| \lambda^{1/p}(n).n^{-\alpha} a_n \| = K \| \lambda^{1/p}(n) n^{-\alpha} \int_0^{\pi} (1-\cos nx)\, dF(x) \|$$

$$\leq K \| \lambda^{1/p}(n) n^{2-\alpha} \int_0^{1/n} x^2 \, |dF(x)| \| +$$

$$+ K \| \lambda^{1/p}(n) n^{-\alpha} \int_{1/n}^{\pi} |dF(x)| \|$$

$$\leq K \Big(\sum_{n=1}^{\infty} \lambda(n) n^{2p-\alpha p} \big(\int_0^{\pi/n} x^2 \, |dF(x)| \big)^p \Big)^{1/p} +$$

$$+ K \Big(\sum_{n=1}^{\infty} \lambda(n) n^{-\alpha p} \big(\int_{1/n}^{\pi} |dF(x)| \big)^p \Big)^{1/p}$$

$$\leq K \Big(\sum_{n=3}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t)\, t^{-2p-2+\alpha p} \big(\int_0^{t} x^2 \, |dF(x)| \big)^p dt \Big)^{\frac{1}{p}}$$

$$+ K \Big(\sum_{n=2}^{\infty} \lambda(n) n^{-\alpha p} \big(\int_{1/n}^{\pi/n} |dF(x)| \big)^p \Big)^{1/p}$$

$$+ K \Big(\sum_{n=2}^{\infty} \lambda(n) n^{-\alpha p} \big(\int_{\pi/n}^{\pi} |dF(x)| \big)^p \Big)^{1/p} + K.$$

$$\leq J_1^{1/p} + J_2^{1/p} + J_3^{1/p} + K \text{ , say.}$$

Since by virtue of the hypothesis

$$\int_0^{\pi} \lambda(\pi/t)\, t^{-2p-2} \sin^{\alpha p} t \left(\int_0^t x^2 |dF(x)|\right)^p dt < \infty,$$

We have

$$J_1 = K \int_0^{\pi/2} \lambda(\pi/t)\, t^{-2p-2+\alpha p} \left(\int_0^t x^2\, |dF(x)|\right)^p dt$$

$$\leq K \int_0^{\pi/2} \lambda(\pi/t)\, t^{-2p-2} \sin^{\alpha p} t \left(\int_0^t x^2\, |dF(x)|\right)^p dt$$

$$< \infty .$$

Also we have

$$J_2 = K \sum_{n=2}^{\infty} \lambda(n)\, n^{-\alpha p} \left(\int_{1/n}^{\pi/n} |dF(x)|\right)^p$$

$$\leq K \sum_{n=2}^{\infty} \lambda(n)\, n^{2p-\alpha p} \left(\int_{1/n}^{\pi/n} x^2\, |dF(x)|\right)^p$$

$$\leq K \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/n-1} \lambda(\pi/t)\, t^{-2p-2+\alpha p} \left(\int_{1/n}^{t} x^2\, |dF(x)|\right)^p dt$$

$$\leq K \int_0^{\pi} \lambda(\pi/t)\, t^{-2p-2+\alpha p} \left(\int_0^t x^2\, |dF(x)|\right)^p dt$$

$$< \infty .$$

Now

$$J_3 \leq K \sum_{n=1}^{\infty} n^{-\alpha p} \lambda(n) \left(\int_{\pi/n}^{\pi} |dF(x)|\right)^p$$

$$\leq K \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dF(x)| \right)^p dt$$

$$\leq K \int_0^{\pi} t^{\alpha p-2} \lambda(\pi/t) \left(\int_t^{\pi} |dF(x)| \right)^p dt.$$

By taking $\phi(t) = \int_0^t x^2 |dF(x)|$, $s = 2$, $\psi(t) = \lambda(\pi/t)\, t^{-2p-2+\alpha p}$,

and applying Lemma 3, we observe that

$$\int_0^{\pi} \lambda(\pi/t)\, t^{-2p-2+\alpha p} . t^{2p} \left(\int_t^{\pi} x^{-2} x^2 |dF(x)| \right)^p dt < \infty ,$$

that is to say, $J_3 < \infty$.

Hence, $\left\{ n^{-\alpha} \lambda^{1/p}(n)\, a_n \right\} < \infty$.

This completes the proof of Theorem 2.

3.7 <u>Proof of Theorem 3</u>. Since $\left\{ \mu^{1/p}(n) n^{-\alpha} a_n \right\} \in \ell^p$,

we have

$$\sum_{n=1}^{\infty} n^{-1} | a_n| = \sum_{n=1}^{\infty} n^{-1+\alpha} \mu^{-1/p}(n) . n^{-\alpha} \mu^{1/p}(n) | a_n|$$

$$\leq \left(\sum_{n=1}^{\infty} n^{-p'(1-\alpha)} \mu^{-p'/p}(n) \right)^{1/p'} \left(\sum_{n=1}^{\infty} n^{-\alpha p} \mu(n) |a_n|^p \right)^{1/p}$$

$$< \infty ,$$

by the condition (3.3.6) and $\alpha p < \eta$. Hence $\Sigma n^{-1} a_n \sin nx$ is the Fourier series of a function F so that $\Sigma a_n \cos nx$ is the Fourier-Stieltjes series of $F(x)$. Suppose F is an increasing function.

Now

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k = \frac{2}{\pi} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})x}{2\sin x/2}\, dF(x)$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin nx \cot \frac{x}{2}\, dF(x) + \frac{1}{\pi} \int_0^{\pi} \cos nx\, dF(x)$$

$$= \alpha_n + \beta_n \text{, say.}$$

Using (3.3.5), Lemma 2 and the hypothesis, we have

$$\sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} |\alpha_n + \beta_n|^p = \sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} \left| \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \right|^p$$

$$\leq K + K \sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} \left(\sum_{k=1}^{n} |a_k| \right)^p$$

$$< K + K \sum_{n=1}^{\infty} \mu(n) n^{-\alpha p - p} (n|a_n|)^p$$

$$= K + K \sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p} |a_n|^p$$

$$< \infty .$$

Also $\beta_n = O(1)$ and hence ,

$$\sum_{n=1}^{\infty} \mu(n)\, n^{-p-\alpha p} \,|\beta_n|^p \leq K \sum_{n=1}^{\infty} \mu(n)\, n^{-p-\alpha p} < \infty .$$

From this it follows that $\sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p-p} |a_n|^p < \infty$.

Let

$$dG(x) = -\cot {}^{x}/_{2}\, dF(x) , \qquad 0 \leq x \leq {}^{\pi}/_{2} \text{ and}$$

$$dG(x) = 0, \quad {}^{\pi}/_{2} < x \leq \pi .$$

$$a_n = -\frac{1}{\pi} \int_0^{\pi} \sin nx \, dG(x) + O(1).$$

We have already proved that $\sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p-p} |a_n|^p < \infty$ and $\sum_{n=1}^{\infty} \mu(n)\, n^{-\alpha p-p} < \infty$.

Also

$$\int_0^{\pi} x\, |dG(x)| \leq \int_0^{\pi} x \cot {}^{x}/_{2}\, dF(x)$$

$$\leq K \int_0^{\pi} dF(x)$$

$$< \infty .$$

Thus, if we put $\lambda(n) = n^{-p} \mu(n)$ in Theorem 1, we find that all the conditions are satisfied. Hence, by Theorem 1

$$\mu^{1/p}({}^{\pi}/_{t})\ t.\ t^{-2/p-1} \left(\int_0^{t} x \, dG(x)\right) \in L\ (p,\alpha).$$

Now

$$\int_0^t dF(x) = - \int_0^t \tan {}^x/_2 \, dG(x)$$

$$\leq \int_0^t x \, |dG(x)| ,$$

that is to say

$$(F(t) - F(0)) \leq \int_0^t x \, |dG(x)| .$$

Hence $\mu^{1/p} ({}^\pi/t) \, t^{-2/p} \, [F(t) - F(0)] \in L (p,\alpha)$.

This proves Theorem 3.

3.8 <u>Proof of Theorem</u> 4. We are given that

$$\mu^{1/p} ({}^\pi/t) \, t^{-2/p} \, (\int_0^t x \, |df(x)|) \in L (p,\alpha).$$

Also

$n \, a_n = - \frac{2}{\pi} \int_0^\pi \sin nt \, df(t)$, hence by virtue of Theorem 1 $\{ \mu^{1/p}(n) n^{-\alpha} a_n \} \in \ell^p$, since sufficiency part of Theorem 1 is true even when $dG(x)$ is replaced by $|dG(x)|$, where $G(x)$ is not necessarily monotonic.

This proves Theorem 4 .