Chapter II

ON THE INTEGRABILITY OF POWER SERIES*

2.1 A non-decreasing continuous real-valued function Φ defined on the non-negative half line and vanishing only at the origin will be called an Orlicz function (OF). Function $\Phi \in$ OF is said to satisfy Δ_α $(\alpha > 0)$ condition for large u if there are constants $C > 0$ and $u_0 \geq 0$ such that $\Phi(\alpha u) \leq C\, \Phi(u)$, $u \geq u_0$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty,$$

is called a Young function (YF). Function Φ belongs to YF iff it admits a representation of the form

$$\Phi(u) = \int_0^u \phi(t)\, dt,$$

where $\phi(t)$, $t \geq 0$, is positive, $\phi(0) = 0$, continuous on the right, non-decreasing and $\lim_{t\to\infty} \phi(t) = \infty$

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We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [1] . " There exist a convex function Λ , $\lambda > 1$ and $0 < \alpha < 1$, such that the inequality

$$\Lambda(u) \leq \Phi^{\alpha}(u) \leq \lambda \Lambda(u) \text{ holds for all } u."$$

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone (Shah [1] ; Szasz [1]) if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n}\right).$$

An equivalent definition of quasi-monotone sequence is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$ (Shah [2]).

Let $L_{\Phi}(X, \mu)$, where $\Phi \in \Delta_{\alpha}$, be the Orlicz space, i.e., the set of all complex valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)d\mu$ is finite. In this chapter by Hardy-Orlicz space H_{Φ} we mean simply a closed subspace of $L_{\Phi}((0,2\pi), dx)$ spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int} .$$

This chapter consists of two section. In section A we obtain a generalisation of certain results concerning equivalence of some statements pertaining to integrability of a power series, while in section B, we study integrability problem for a power series under certain Tauberian condition.

SECTION A

2.2 Recently, Woyczyński [1] proved the following theorem.

<u>Theorem</u> A. Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k , \quad 0 \leq x < 1.$$

If $a_n \geq a_{n+1} \geq 0$ ($n = 0,1,2,---$), then the following four statements are equivalent :

(2.2.1) $f(x) \in L_{\Phi}(0, 1)$;

(2.2.2) $g(t) \equiv f(e^{it}) \in H_{\Phi}(0, 2\pi)$;

(2.2.3) $\{n\, a_n\} \in L_{\Phi}(N, \nu)$;

(2.2.4) $\{A_n\} \in L_{\Phi}(N, \nu)$,

where $\Phi \in \Delta_{\alpha} \cap M \cap YF$, $d\mu = dx$, N stands for the set of all positive integers and ν is the measure on N concentrating the mass n^{-2} at the point $n \in N$, and $A_n = a_0+a_1+a_2+ \ldots a_n$.

In this section our object is to obtain a generalisation of the above theorem. We denote throughout this section by

B (with or without suffixes) a positive constant, not necessarily the same at each occurrence.

We prove the following Theorem.

Theorem 1. *Let* $\Phi \in \Delta_\alpha \cap M \cap YF$ *and* $f(x) = \sum_{k=0}^{\infty} a_k x^k$,

$0 \leq x < 1$. *If* $\{a_n\}$ *is a quasi-monotone sequence such that* $0 < B_1 \leq n^\beta a_n \leq B_2$ *with some* $\beta > 0$, $(n = 1,2,\ldots)$, *and* $0 \leq \gamma < 1$, *then the following four statements are equivalent* :

$$(2.2.5) \qquad (1-x)^{-\gamma} \Phi(f(x)) \in L(0,1) ;$$

$$(2.2.6) \qquad x^{-\gamma} \Phi(|f(e^{ix})|) \in L(0,\pi) ;$$

$$(2.2.7) \qquad \sum_{1}^{\infty} n^{\gamma-2} \Phi(n a_n) < \infty ;$$

$$(2.2.8) \qquad \sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n) < \infty ,$$

where $A_n = a_0 + a_1 + \cdots + a_n$.

2.3 We require the following lemmas for the proof of Theorem 1.

Lemma 1. (Woyczynski [1]). *Let* $X = R^+$ *and* $d\mu = x^s\, dx$ ($s \leq 0$). *If* $\Phi \in M$, *then*

$$\int_X \Phi\left(\frac{F(x)}{x}\right) d\mu \leq B \int_X \Phi(f(x))\, d\mu ,$$

<u>where</u> $F(x) = \int_0^x f(t)\,dt$ and $f(t) \geq 0$.

<u>Lemma 2.</u> (Woyczyński [1]). <u>Let</u> $\Phi \in \Delta_\alpha \cap YP$, $X = R^+$ <u>and</u> $d\mu = x^s\,dx$ ($s < -1$). <u>If</u> $f(x)$ <u>is a non-negative function and</u> $x f(x) \in L_\Phi(X, \mu)$, <u>then</u> $F(x) \in L_\Phi$, <u>where</u> $F(x) = \int_0^x f(t)\,dt$.

<u>Lemma 3.</u> (Askey and Wainger [1]). <u>Let</u> $\{a_n\}$ <u>be positive and tend to zero and</u> $\{n^{-k} a_n\}$ <u>be monotonically decreasing for some non-negative</u> k^*, <u>then</u>

$$\sum_{j=n}^{\infty} |\Delta a_j| \leq B \sum_{j=n}^{\infty} a_j/j + a_n ,$$

<u>where B is some positive constant.</u>

2.4 <u>Proof of Theorem 1.</u> We shall prove the following implications :

(i) (2.2.5) $\Longleftrightarrow$ (2.2.8) $\Rightarrow$ (2.2.6)

(ii) (2.2.6) $\Rightarrow$ (2.2.7) $\Rightarrow$ (2.2.8).

<u>Proof of</u> (2.2.5) $\Rightarrow$ (2.2.8). We write $(1-x)=y$, then by virtue of the fact that $(1-\frac{1}{n})^n$ $(n=1,2,\ldots)$ is an increasing sequence, we have for $\frac{1}{n+1} \leq y \leq \frac{1}{n}$, $n \geq 2$,

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* In Lemma 3 the authors have assumed that k is a positive integer but it can be easily verified that the Lemma remains true even for $k \geq 0$.

$$f(1-y) \geq \sum_{k=0}^{n} a_k (1-y)^k \geq (1-y)^n \sum_{k=0}^{n} a_k$$

$$\geq \left(1-\frac{1}{n}\right)^n \sum_{k=0}^{n} a_k \geq \frac{1}{4} A_n \,.$$

Thus we get

$$f(1-y) \geq \frac{1}{4} A_n \text{ for } \frac{1}{n+1} \leq y \leq \frac{1}{n}, \quad (n = 2, 3, \ldots).$$

Now,

$$\sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n) \leq B \sum_{1}^{\infty} \int_{n}^{n+1} t^{\gamma-2} \Phi(A_{[t]})\,dt$$

$$= B \sum_{1}^{\infty} \int_{1/n+1}^{1/n} u^{-\gamma} \Phi(A_{[1/u]})\,du$$

$$= B \int_{1/2}^{1} u^{-\gamma} (A_{[1/u]})\,du + B \sum_{n=2}^{\infty} \int_{1/n+1}^{1/n} u^{-\gamma} \Phi(A_n)\,du$$

$$\leq B + B \int_{0}^{1/2} u^{-\gamma} \Phi(4 f(1-u))\,du$$

$$\leq B + B \int_{0}^{1/2} u^{-\gamma} \Phi(f(1-u))\,du$$

$$\leq B + B \int_{0}^{1} u^{-\gamma} \Phi(f(1-u))\,du$$

$$\leq B + B \int_{0}^{1} (1-x)^{-\gamma} \Phi(f(x))\,dx$$

$$< \infty .$$

<u>Proof of</u> (2.2.8) => (2.2.5).

$$\int_0^1 (1-x)^{-\gamma}\, \Phi(f(x))\, dx = \sum_{n=2}^{\infty} \int_{1-\frac{1}{n-1}}^{1-\frac{1}{n}} (1-x)^{-\gamma}\, \Phi(f(x))\,dx$$

$$= \sum_{n=2}^{\infty} \int_{1/n}^{\frac{1}{n-1}} x^{-\gamma}\, \Phi(f(1-x))\,dx$$

$$= \sum_{n=2}^{\infty} \int_{1/n}^{\frac{1}{n-1}} x^{-\gamma}\, \Phi\Big(\sum_{k=0}^{\infty} a_k (1-x)^k\Big)\, dx$$

$$\leq \sum_{n=2}^{\infty} \int_{1/n}^{\frac{1}{n-1}} x^{-\gamma}\, \Phi\Big(\sum_{k=0}^{\infty} a_k \big(1-\tfrac{1}{n}\big)^k\Big)\, dx$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j \big(1-\tfrac{1}{n}\big)^j\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} \big(1-\tfrac{1}{n}\big)^{nk} \sum_{j=0}^{n(k+1)} a_j\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} e^{-k} \Big(\sum_{j=0}^{n} a_j + \sum_{j=n+1}^{n(k+1)} a_j\Big)\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} e^{-k} \Big(A_n + B_2 \sum_{j=n+1}^{n(k+1)} j^{-\beta}\Big)\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} e^{-k} \big(A_n + B\, n^{1-\beta} k\big)\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi\Big(\sum_{k=0}^{\infty} B_{\cdot}\, (k+1)\, e^{-k}\, A_n\Big)$$

$$\leq B \sum_{n=2}^{\infty} n^{\gamma-2}\, \Phi(A_n) < \infty .$$

<u>Proof of</u> (2.2.8) => (2.2.6). We shall prove that $x^{-\gamma}\Phi(|\mathrm{Re}\, f(e^{ix})|)$ and $x^{-\gamma}\Phi(|\mathrm{Im}\, f(e^{ix})|)$ are both in $L(0,\pi)$.

Writing $D_n(x) = \dfrac{\sin(n+\frac{1}{2})x}{2\sin x/2}$, we have

$$|\mathrm{Re}\, f(e^{ix})| = \left| a_0 + \sum_{k=1}^{\infty} a_k \cos kx \right|$$

$$\leq \sum_{k=0}^{n} a_k + \left| \sum_{k=n+1}^{\infty} a_k \cos kx \right|$$

$$= A_n + \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_n D_n(x) \right|$$

$$\leq A_n + B\frac{1}{x}\sum_{k=n}^{\infty}|\Delta a_k| + B\frac{1}{x}a_n, \quad \pi/n+1 \leq x \leq \pi/n$$

$$\leq A_n + Bn\left(\sum_{k=n}^{\infty}|\Delta a_k| + a_n\right)$$

$$\leq A_n + Bn\left(\sum_{k=n}^{\infty}\frac{a_k}{k} + a_n + a_n\right) \quad \text{(by Lemma 3)}$$

$$\leq A_n + B_2\left(n\sum_{k=n}^{\infty}\frac{1}{k^{1+\beta}} + 2n^{1-\beta}\right)$$

$$\leq A_n + Bn^{1-\beta} \leq B A_n .$$

Now

$$\int_0^{\pi} x^{-\gamma}\Phi(|\mathrm{Re}\, f(e^{ix})|)dx = \sum_{1}^{\infty}\int_{\pi/n+1}^{\pi/n} x^{-\gamma}\Phi(|\mathrm{Re}\, f(e^{ix})|)dx$$

$$\leq B \sum_{1}^{\infty} n^{\gamma-2} \Phi(B A_n)$$

$$\leq B \sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n)$$

$$< \infty .$$

Almost the same proof remains valid for $x^{-\gamma}$ $(|\text{Im } f(e^{ix})|)$ $\in L(0,\pi)$ and so $x^{-\gamma}$ $\Phi(|f(e^{ix})|) \in L(0,\pi)$, whenever $\sum_{1}^{\infty} n^{\gamma-2} \Phi(A_n) < \infty$.

Proof of (2.2.6) => (2.2.7) Let us write

$$r(t) = \text{Re } f(e^{it}) , \quad R(t) = \int_0^t r(x)\, dx,$$

$$R_1(t) = \int_0^t R(x)\, dx.$$

Then

$$R_1(t) = a_0 \frac{t^2}{2} + \sum_{j=1}^{\infty} a_j . \, j^{-2} (1 - \cos j.t)$$

$$\geq \sum_{j=1}^{n} a_j \, j^{-2} (1 - \cos jt)$$

$$= 2 \sum_{j=1}^{n} j^{-2} a_j \sin^2 \frac{jt}{2}$$

$$\geq 2 \sum_{j=1}^{n} j^{-2} a_j . \frac{4}{\pi^2} \frac{j^2 t^2}{4}, \quad \pi/n+1 \leq t \leq \pi/n$$

$$\geq B t^2 \sum_{j=1}^{n} a_j$$

$$\geq B t^2 n a_n .$$

Thus

$$\sum_{n=1}^{\infty} n^{\gamma-2}\, \Phi(n\, a_n) \le \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{-\gamma}\, \Phi\left(\frac{1}{B}\, \frac{R_1(x)}{x^2}\right) dx$$

$$\le B \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} x^{-\gamma}\, \Phi\left(\frac{R_1(x)}{x^2}\right) dx$$

$$= B \int_0^{\pi} x^{-\gamma}\, \Phi\left(\frac{1}{x^2} \int_0^x R(t)\, dt\right) dx$$

$$\le B \int_0^{\pi} x^{-\gamma}\, \Phi\left(\frac{1}{x^2} \int_0^x |R(t)|\, dt\right) dx$$

$$\le B \int_0^{\pi} x^{-\gamma}\, \Phi\left(\frac{1}{x} \int_0^x \frac{|R(t)|}{t}\, dt\right) dx$$

$$\le B \int_0^{\pi} x^{-\gamma}\, \Phi\left(\frac{|R(x)|}{x}\right) dx$$

$$\le B \int_0^{\pi} x^{-\gamma}\, \Phi(|r(x)|)\, dx$$

$$\le B \int_0^{\pi} x^{-\gamma}\, \Phi(|f(e^{ix})|)\, dx$$

$$< \infty,$$

by Lemma 1 and by virtue of the hypothesis.

<u>Proof of</u> (2.2.7) => (2.2.8). Let $a(x)$ be the function equal to a_n if $n-1 \le x < n$, $n=1,2,\ldots$, and let $A(x)=\int_0^x a(t)\,dt$. The assumption $\Sigma\, n^{\gamma-2}\, \Phi(n\, a_n)$ implies that $t^{\gamma-2}\, \Phi(t\, a(t))$

is integrable on the positive half line, and by virtue of Lemma 2 ($s = \gamma - 2 < -1$) $t^{\gamma-2}\,\Phi(A(t))$ is integrable as well. But this is equivalent to the convergence of the series $\Sigma\, n^{\gamma-2}\,\Phi(A_n)$. Hence (2.2.7) => (2.2.8).

This proves Theorem 1.

Remarks : Our proof shows that the implication : (2.2.5) => (2.2.8) holds true under a less restrictive condition, namely $a_n \geq 0$. It is also known (Askey and Karlin [1]) that (2.2.8) => (2.2.5) under certain lighter conditions, namely $0 \leq \gamma < 2$, Φ increasing, positive and convex.

SECTION B

2.3 Heywood [2] proved the following result :

Theorem B. Suppose that $F(x) = \sum_{0}^{\infty} c_n x^n$ for $0 \leq x < 1$, $\gamma < 1$ and that there is a positive number ε such that $c_n > \frac{-K}{n^{\gamma+\varepsilon}}$, for all sufficiently large values of n, K being some positive constant. Then $(1-x)^{-\gamma} F(x) \in L\,(0,1)$ iff $\Sigma\, n^{\gamma-1}\,|c_n|$ converges.

The object of this section is to obtain a generalisation of Theorem B.

2.6 We prove the following result :

Theorem 2. Let $F(x) = \sum_{0}^{\infty} c_n x^n$, $0 \leq x < 1$ *and* $\gamma < 1$. *Suppose that there is a positive number* ϵ *such that*

$$(2.6.1) \qquad c_n > \frac{-K}{n^{\gamma/p+(1-1/p)+\epsilon}} \qquad (0 < p \leq \infty)$$

for all sufficiently large values of n, *where* K *is some positive constant. Then* $(1-x)^{-\gamma}(|F(x)|)^p \in L(0,1)$ *iff* $\sum n^{\gamma-2}\left(\sum_{k=1}^{n} |c_k|\right)^p$ *converges.*

It may be remarked that in view of Lemma 6 and Abel's transformation, our Theorem includes as a special case for $p \geq 1$ the above theorem of Heywood.

2.7 We require the following lemmas to prove Theorem 2.

Lemma 4. (Titchmarsh [1] , p.58). *If* b *is a constant, then*

$$\frac{\Gamma(x)}{\Gamma(x+b)} \sim x^{-b}, \text{ as } x \to \infty .$$

Lemma 5. (Khan [2]). *Let* $F(x) = \sum_{0}^{\infty} c_n x^n$, $c_n \geq 0$, $0 \leq x < 1$, $s_n = \sum_{k=1}^{n} c_k$ *and* $\gamma < 1$. *Then , for* $0 < p \leq \infty$

$$\left(\int_0^1 (1-x)^{-\gamma}(F(x))^p\, dx\right)^{1/p} < \infty \text{ iff } \left(\sum_{1}^{\infty} n^{\gamma-2} s_n^p\right)^{1/p} < \infty .$$

Lemma 6. (Hardy [1], p.255). *If* $c > 1$, $a_n = \sum_{k=1}^{n} a_k$, $a_k \geq 0$, *then* $\sum_{n=1}^{\infty} n^{-c} a_n^p \leq K \sum_{n=1}^{\infty} n^{-c} (n\, a_n)^p$ $(p \geq 1)$.

Lemma 7. (Konyushkov [1], p.83). *If* $c > 1$, $0 < p < 1$, $a_n \geq 0$ *and* $\{n^{-j} a_n\}$ *is monotonic decreasing for some* $j > 0$, *then*

$$\sum_{n=1}^{\infty} n^{-c} \left(\sum_{1}^{n} a_k\right)^p \leq K \sum_{n=1}^{\infty} n^{-c} (n\, a_n)^p .$$

2.8 *Proof of Theorem* 2. We may suppose without loss of any generality that $\frac{\gamma-1}{p} + \epsilon$ is not an integer. This will ensure the existence of the Gamma function at all relevant points. Let

$$G(x) = 2K\, \Gamma\!\left(\frac{1-\gamma}{p} - \epsilon\right) (1-x)^{\frac{\gamma-1}{p} + \epsilon}$$

$$= \sum_{n=0}^{\infty} a_n x^n, \qquad \text{for } 0 \leq x < 1.$$

Then, since

$$(1-x)^{\frac{\gamma-1}{p} + \epsilon} = \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1-\gamma}{p} - \epsilon\right)}{\Gamma(n+1)\, \Gamma\left(\frac{1-\gamma}{p} - \epsilon\right)} x^n ,$$

We have

$$(2.8.1)\qquad a_n = gK\,\frac{\Gamma(n+\frac{1-\gamma}{p}-\epsilon)}{\Gamma(n+1)} \sim \frac{gK}{n^{\frac{\gamma-1}{p}+1+\epsilon}}\,, \text{ as } n\to\infty,$$

by Lemma 4. It follows from (2.6.1) that $c_n + a_n$ is positive for all sufficiently large values of n. Since

$$F(x) + G(x) = \sum_{n=0}^{\infty} (c_n+a_n)\, x^n$$

for $0 \leq x < 1$, Lemma 5 now shows that

$$(1-x)^{-\gamma}\big(F(x)+G(x)\big)^p \in L(0,1), \text{ iff } \sum n^{\gamma-2}\Big(\sum_{k=1}^{n} (c_k+a_k)\Big)^p$$

Converges.

But, $(1-x)^{-\gamma}\, G^p(x)$ is a multiple of $(1-x)^{\epsilon p-1}$ and therefore integrable L in (0,1). Moreover (2.8.1) shows that $\sum n^{\gamma-2}\big(\sum_{k=1}^{n} |a_k|\big)^p$ is convergent by Lemmas 6 and 7 . Therefore, it follows that

$$(1-x)^{-\gamma}\,(|F(x)|)^p \in L(0,1)$$

iff

$$\sum n^{\gamma-2}\Big(\sum_{k=1}^{n} |c_k|\Big)^p < \infty .$$

Thus the theorem is proved.