Chapter I

INTRODUCTION

1.1 The Theory of Integrability of trigonometric series and transforms is of recent origin. It was in 1949 that the celebrated mathematician Professor B. Sz-Nagy, a distinguished researcher in the domain of functional analysis, initiated the discussion of problems concerning this topic. Within a short period of two decades, it has now grown into a fully developed discipline of mathematical analysis. The present thesis is based on certain investigations of the author into the theory of "Integrability of Trigonometric Series and Transforms."

1.2 Before giving a résumé of the earlier researches, in the light of which various new and interesting results have been obtained by the author, it seems desirable to state here the definitions and notations which will be required in the sequel.

ORLICZ AND YOUNG FUNCTIONS

A non-decreasing, continuous, real-valued function Φ defined on the non-negative half-line and vanishing only at the origin will be called an Orlicz function (OF). Function $\Phi \in$ OF is said to satisfy Δ_α $(\alpha > 0)$ condition for large u if there are constants $c > 0$ and $u_0 \geq 0$ such that $\Phi(\alpha u) \leq C\,\Phi(u)$, $u \geq u_0$ for every $\alpha > 1$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$$

is called a Young function (YF). Function Φ belongs to YF if, and only if, it admits a representation

$$\Phi(u) = \int_0^u \phi(x)\,dx ,$$

where $\phi(x)$, $x \geq 0$, is positive, $\phi(0) = 0$, continuous on the right, non-decreasing and $\lim_{x\to\infty} \phi(x) = \infty$.

We have for such functions the relation

$$\frac{\Phi(u)}{u} \leq \phi(u) \leq \frac{\Phi(2u)}{u} .$$

We denote by M the class of Orlicz functions Φ which satisfy the following condition of Mulholland [1].

" There exist a convex function Λ, $\lambda > 1$ and $0 < k < 1$, such that the inequality $\Lambda(u) \leq \Phi^{k}(u) \leq \lambda\Lambda(u)$ holds for all u."

Let $L_\Phi(X,\mu)$, where $\Phi \in \Delta_\alpha$, be the Orlicz space, i.e., the set of all complex valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)\, d\mu$ is finite. By Hardy-Orlicz space H_Φ we mean simply closed subspace of $L_\Phi((0, 2\pi), dx)$ spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int}.$$

Functions of $L(p,\alpha)$ class.

A function $\phi(x)$ is said to belong to the class $L(p,\alpha)$ (Askey and Wainger [1]) if

$$\int_0^\pi |\phi(x)|^p \sin^{\alpha p} x \, dx < \infty,$$

where α is any real number and $p > 0$.

We define the norm of a function $\phi(x) \in L(p,\alpha)$ as :

$$\| \phi(x) \|_{p,\alpha} = \left\{ \int_0^\pi |\phi(x)|^p \sin^{\alpha p} x \, dx \right\}^{1/p}.$$

It is evident that $L(p,\alpha) \Rightarrow L^p$ for $\alpha < 0$, $L^p \Rightarrow L(p,\alpha)$ for $\alpha > 0$ and $L^p \equiv L(p,\alpha)$ when $\alpha = 0$.

Slowly increasing function.

A positive function $L(x)$ is said to be "Slowly increasing in the sense of Karamata [1] if it is continuous for $x \geq 0$ and

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \qquad \text{for every } t > 0.$$

Monotonic sequences.

A sequence $\{a_n\}$ of non-negative numbers is said to be quasi-monotone (Shah [1] ; Szász [1]) if for some $\alpha > 0$,

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n} \right)$$

for all $n > n_o(\alpha)$, where $n_o(\alpha)$ is a positive number depending upon α.

An equivalent definition of quasi-monotone sequence (Shah [2]) is that $n^{-\beta} a_n \downarrow 0$ for some $\beta > 0$.

It is said to be quasi-monotone of $\alpha = \alpha_o$ (Yong [2]) if

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha_o}{n} \right).$$

A sequence $\{a_n\}$ is said to be δ-quasi-monotone (Boas [2]) if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta a_n \geq -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers.

Fourier series and transforms.

Let $f(x)$ and $g(x)$ be defined by the following trigonometric series.

(1.2.1) $$f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx ,$$

(1.2.2) $$g(x) = \sum_{n=1}^{\infty} a_n \sin nx .$$

Let $F(x)$ be the Fourier sine transforms of $f(t)$, that is to say

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt \, dt.$$

1.3 Integrability of trigonometric series of L-class with monotonic coefficients.

Concerning the integrability of trigonometric series for monotonic coefficients, Boas [1], in 1952, proved the following theorems†

Theorem A. If $a_n \downarrow 0$ and $0 < \gamma < 1$, then $x^{-\gamma} f(x) \in L(0,\pi)$ if, and only if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Theorem B. If $a_n \downarrow 0$ and $0 \leq \gamma \leq 1$, then $x^{-\gamma} g(x) \in L(0,\pi)$ if, and only, if $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

Later on Sunouchi [1] proved these theorems by using a different method.

In 1962, Shah [2] extended above theorems in the following form.

Theorem C. Let $\{a_n\}$ be a quasi-monotone.

(i) If $0 < \gamma < 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if, $x^{-\gamma} f(x) \in L(0,\pi)$.

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† For detailed literature concerning L-class reference may be made to the recent monograph of Boas [3].

(ii) If $0 < \gamma \leq 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if, $x^{-\gamma} g(x) \in L(0,\pi)$.

Theorem C was extended by Boas [2] for δ-quasi-monotonic sequences in the following manner.

<u>Theorem D</u>. Let $0 < \gamma < 1$, and let $\{a_n\}$ be δ-quasi-monotonic sequence with $\sum_{n=1}^{\infty} n^{\gamma} \delta_n < \infty$, then $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$, and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if, $x^{-\gamma} f(x) \in L(0,\pi)$.

<u>Theorem E</u>. Let $0 < \gamma \leq 1$, and let $\{a_n\}$ be δ-quasi-monotonic sequence with $\sum_{k=1}^{\infty} n^{\gamma} \delta_n < \infty$, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if, and only if, $x^{-\gamma} g(x) \in L(0,\pi)$.*

Note: Suppose that $\{a_n\}$ is a positive sequence tending to zero and $\Delta a_n > -\frac{\alpha a_n}{n}$. Let $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$ and $0 < \gamma < 1$. Then Theorem D asserts that $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges (except perhaps at integral multiples of 2π) to $f(x)$ and $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ converges if and only, $x^{-\gamma} f(x) \in L(0,\pi)$. Considering

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*Theorem E ($\Sigma \to f$) part has been generalized in a different direction by Hasegawa [2] .

the case $\int - \Sigma$ we observe that Boas had already assumed what he wished to prove. Thus his results (Theorem D and E) suffer from such a defect. Of course, he could have avoided this by (say in Theorem D) assuming the convergence of $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ instead of that of $\sum_1^{\infty} n^{\gamma} \delta_n$ in part $\int - \Sigma$.

Aljenčić, Bojanić and Tomić [2], in 1955, generalized Theorem A and B in a different direction. They proved, among others, the following results.

<u>Theorem F</u>. If $0 < \gamma < 1$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x)\ f(x) \in L(0,\pi)$, if and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n < \infty$.

<u>Theorem G</u>. If $0 < \gamma < 2$, $a_n \downarrow 0$, then $x^{-\gamma} L(1/x)\ g(x) \in L(0,\pi)$, if and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\ a_n < \infty$.

These results were subsequently extended by Yong [1], in 1965, to quasi-monotonic sequences in the following form.

<u>Theorem H</u>. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$ as $n \to \infty$ and $0 < \gamma < 1$. Then $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\ a_n$ converges if, and only if, $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges everywhere to $f(x)$, save possibly $x=0$, and $x^{-\gamma} L(1/x)\ f(x) \in L(0,\pi)$.

<u>Theorem I</u>. Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$, as $n \to \infty$.

(i) For $0 < \gamma < 2$, if $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$.

(ii) For $0 < \gamma < 1$, if $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x)\, g(x) \in L(0,\pi)$, then $\sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$ converges.

It may be remarked that if we examine the proof of $\int \to \Sigma$ in Theorem H and I we find that it is sufficient to assume that $\{a_n\}$ is only a positive sequence.

The Chapter V of this thesis is concerned with the generalization of all the results stated above. We prove the following theorems.

<u>Theorem 1.</u> <u>Let</u> $\{a_n\}$ <u>be a</u> δ-<u>quasi-monotone sequence and</u> $0 < \gamma < 1$. <u>If</u>

$$(1.3.1) \qquad \sum_{n=1}^{\infty} n^{\gamma} L(n)\, \delta_n < \infty ,$$

<u>and</u>

$$(1.3.2) \qquad \sum_{n=1}^{\infty} n^{\gamma-1} L(n)\, a_n$$

<u>converges, then</u>

(1.3.3) $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

converges, everywhere to $f(x)$ except possibly at $x = 0$ and

(1.3.4) $x^{-\gamma} L(1/x) f(x) \in L(0,\pi)$.

Conversely, if $\{a_n\}$ is any sequence which is ultimately positive such that (1.3.3) holds and if (1.3.4) holds, then (1.3.2) holds.

Theorem 2. (i) Let $\{a_n\}$ be a δ-quasi-monotone sequence and $0 < \gamma < 2$. If $\sum_{n=1}^{\infty} n^{\gamma} L(n) \delta_n$ and $\sum_{n=1}^{\infty} n^{\gamma-1} L(n) a_n$ are convergent, then $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x) g(x) \in L(0,\pi)$.

(ii) For $0 < \gamma < 1$, if $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and $x^{-\gamma} L(1/x) g(x) \in L(0,\pi)$, then $\sum_{1}^{\infty} n^{\gamma-1} L(n) a_n$ converges where $\{a_n\}$ is any sequence which is ultimately positive.

1.4 Recently, Hasegawa [1] has proved the following Theorems.

Theorem J. Let $f(x)$ be an even function, continuous on $(0,\pi)$, and let its Fourier series be

$$(1.4.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} a_n \cos nx .$$

Suppose that $\alpha(x)$ is an even function, positive and non-increasing on $(0,\pi)$, that $x\,\alpha(x) \in L(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-2} \int_0^t x\,\alpha(x)\,dx \leq C\,\alpha(t) \qquad \text{for all } t, \quad 0 < t \leq \eta.$$

If the series $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\,dx$ converges absolutely, then the Fourier series (1.4.1) converges uniformly to $f(x)$ and $(\, f(x) - f(x)\,)\,\alpha\,(x-s) \in L(0,\pi)$ for each s, $0 < s < \pi$.

<u>Theorem K</u>. Let $g(x)$ be an odd function, continuous on $(0,\pi)$, and let its Fourier series be

$$(1.4.2) \qquad g(x) \sim \sum_{1}^{\infty} b_n \sin nx .$$

Suppose that $\alpha(x)$ is an odd function, positive on $(0,\pi)$, that $x\alpha(x)$ is Lebesgue integrable and non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-1} \int_t^{\eta} \alpha(x)\,dx \leq C\,\alpha(t)$$

for all t, $0 < t \leq \eta$. If the series $\sum_{n=1}^{\infty} n\,b_n \int_0^{1/n} x\,\alpha(x)\,dx$

converges absolutely, then the Fourier series (1.4.2) converges uniformly to $g(x)$ and $(\, g(x) - g(s)\,)\,\alpha(x-s) \in L(0,\pi)$ for each s , $0 \leq s \leq \pi$. In particular, in case $s=0$, Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} g(x)\ \alpha(x)\ dx = \sum_{n=1}^{\infty} b_n\, q_n$$

holds, where q_n's are defined by

$$q_n = \frac{2}{\pi}\int_0^{\pi} \alpha(x)\ \sin nx\ dx\ ,$$

and the series $\sum_{n=1}^{\infty} b_n\, q_n$ converges absolutely.

In Chapter VII , we have obtained certain generalizations of Theorems J and K. For example Theorem J has been generalized in the following form.

In what follows we assume that $\alpha(x)$, $\phi(x)$ and $\psi(x)$ are positive functions defined on $(0,\pi)$, such that $\alpha(x)\ \phi(x) \in L(0,\pi)$ and $\alpha(x)\ \psi(x)$ is either even or odd.

<u>Theorem</u> 3. <u>Let</u> $f(x)$ <u>be an even function and continuous on</u> $(0,\pi)$ <u>and let its Fourier series be</u> (1.4.1). <u>Suppose that</u> $\alpha(x)\ \frac{\psi(x)}{\phi(x)}$ <u>is non-increasing on</u> $(0,\pi)$ <u>and that there is a positive number</u> $\eta \leq \pi$ <u>such that</u>

$$\int_0^t \phi(x)\ \alpha(x)\ dx \leq C\ \Phi_1(t)\ \alpha(t)$$

<u>for all</u> t , $0 < t \leq \eta$, <u>where</u> $\Phi_1(t) = \int_0^t \phi(x)\ dx$. <u>If the</u>

series $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx$ *converges absolutely, then the Fourier series* (1.4.1) *converges uniformly to* $f(x)$ *and* $(f(x) - f(s))\,\alpha(x-s)\,\psi(x-s) \in L(0,\pi)$ *for each* s, $0 < s < \pi$, *where* $x\,\dfrac{\psi(x)}{\phi(x)} < M$, $0 < x \leq \pi$ *and*

$$\frac{\phi(x)\ \Phi_1(x)}{\psi(x)(\Phi_1(x) - \Phi_1(\frac{x}{2}))} = O(x), \quad x \to 0.$$

By taking $\psi(x) = 1$ and $\phi(x) = x$ we get theorem J. On the other hand if we take $\phi(x) = x$ and $\psi(x)\,\alpha(x) = \beta(x)$ in Theorem 3, we get Theorem J in which one of our conditions namely, " $\frac{\beta(x)}{x} \downarrow$ " is lighter than the corresponding condition " $\beta(x) \downarrow$ " in Theorem J. Also we extend the scope of our theorem by assuming that $\beta(x)$ is either even or odd instead of restricting it to be even in Theorem J. Thus our Theorem is a definite generalization of Theorem J.

1.5 *Integrability of trigonometric series of L^p and other associated classes with monotone coefficients.*

Concerning the integrability of trigonometric series for L^p class, Hardy and Littlewood (see Zygmund [1]) proved the following theorems.

Theorem L. If $a_n \downarrow 0$ and $1 < p < \infty$, then $f \in L^p(0,\pi)$

if and only if, $\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty$, where $f(x) \sim \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx$.
Similar theorem holds for sine series.

They also obtained the following dual theorem in which decreasing coefficients were replaced by a decreasing function.

<u>Theorem [illegible]</u>. If $f \geq 0$ and decreases, $1 < p < \infty$, and a_n are the Fourier cosine coefficents of f, then $\sum_1^{\infty} |a_n|^p < \infty$, if, and only if, $x^{p-2} f^p \in L(0,\pi)$.

A similar result is true for sine series.

In 1956, Theorem [illegible] was extended by Chen [1] in the following form.

<u>Theorem [illegible]</u>. Suppose that $a_n \downarrow 0$, $f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx$. Then for $p > 1$, $0 < \gamma < 1$, $x^{-\gamma} f^p(x) \in L(0,\pi)$ if and only , $\sum_1^{\infty} n^{\gamma+p-2} a_n^p < \infty$. The result is also true for sine series.

He has also shown that if a_n is ultimately positive and decreases steadily to zero as $n \to \infty$, then Theorem [illegible] remains true.

Later on he [3] extended his results to other values of γ. His results were subsequently extended by Robertson [1] for the case when $\{a_n\}$ can be partitioned into k monotonic sequences.

A different kind of generalization has been discussed by Chen([2] ; [3]) in great details in a series of papers which generalize not only the power function multipliers but also the L^p classes.

Igari [1] , in 1960, obtained a theorem as an extension of Theorem A in a different direction. This theorem was subsequently generalized by Yong [2] for quasi-monotone sequences. His theorem is as follows :

<u>Theorem C</u>. Let $\{a_n\}$ be quasi-monotone of $\alpha < 1$ and such that $C_2 \geq n^{\beta} L_1(n)\, a_n \geq C_1 > 0$ with some $\beta > 0$ $(n=1,2,\dots)$. If $p \geq 1$ and $1 > \gamma > 1-p$, then $x^{-\gamma} L_2(1/x)\, f^p(x)$ is integrable $L(0,\pi)$ if, and only if, $\sum_1^{\infty} n^{\gamma+p-2} L_2(n)\, a_n^p$ converges, where $L_1(x)$ $L_2(x)$ are " slowly increasing " functions in the sense of Karamata and $f(x) = \sum_1^{\infty} a_n \cos nx$.

Similar result holds for sine series for $1+p > \gamma > 1-p$.

Mikarev and Necaev [1] have obtained a theorem for Lorenz space $L(q,p)$*. Concerning the space $L(p,\alpha)$ Askey and Wainger [1

* We say that a function $\phi \in L(q,p)$ $(1<p<\infty,\ 1<q<\infty)$ if $(t^{1/q-1/p}\phi^*) \in L^p$, where ϕ^* is ϕ rearranged in decreasing order, i.e. ϕ^* is the decreasing function equimeasurable with ϕ. Similarly $\{a_n\} \in \ell(q,p)$ if $\{n^{\frac{1}{q}-\frac{1}{p}} a_n\} \in \ell^p$.

in 1966, have established the following theorems.

<u>Theorem P</u>. Let $f(x) \in L(p,\alpha)$ with $1 \leq p < \infty$, $-1 < \alpha p \leq p-1$. Let $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ with $a_n \geq 0$ and $A_n = \sum_{j=[\frac{n}{2}]}^{n} a_j$, then $\sum_{n=1}^{\infty} n^{-2-\alpha p} A_n^p < \infty$, and $\sum_{n=1}^{\infty} n^{-2-\alpha p} A_n^p \leq C(\alpha,p) \, \| f \|^p_{p,\alpha}$.

<u>Theorem Q</u>. Let $1 \leq p < \infty$, $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and

$$\Big\{ \sum_{n=1}^{\infty} n^{p-\alpha p-2} \Big(\sum_{k=n}^{\infty} |\triangle a_k| \Big)^p \Big\}^{1/p} < \infty,$$

then $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$ is in $L(p,\alpha)$ class and $\| f \|^p_{p,\alpha} \leq B(\alpha,p) \sum_{n=1}^{\infty} n^{p-\alpha p-2} \Big(\sum_{k=n}^{\infty} |\triangle a_k| \Big)^p$,

where $\triangle a_k = a_k - a_{k+1}$.

From theorems P and Q they deduced the following interesting result.

<u>Theorem R</u>. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonic decreasing for some non-negative integer k. Let $1 \leq p < \infty$ and $-1 < \alpha p < p-1$, then a necessary and sufficient condition that $f(x) \in L(p,\alpha)$ is that

$$\sum_{1}^{\infty} n^{p-\alpha p-2} a_n^p < \infty, \text{ where } f(x) \sim \sum_{1}^{\infty} a_n \cos nx.$$

Later on Khan [1] in 1968, obtained several results involving $L(p,\alpha)$ class, which generalize all the above results for cosine series. One of his typical result is as follows.

Theorem B. Let $\{a_n\}$ be a positive sequence tending to zero and $\{n^{-k} a_n\}$ be monotonic decreasing for some non-negative integer k. If $1 \leq p < \infty$, and $-1 < \alpha p < p-1$, then $L^{1/p}(\frac{1}{x})\ f(x) \in L(p,\alpha)$ if, and only if, $\sum_1^\infty n^{p-\alpha p-2} L(n)\ a_n^p < \infty$, where $f(x) \sim \sum_1^\infty a_n \cos nx$.

In Chapter IV of the present thesis we have obtained certain generalization of all these theorems. A number of results have been obtained. Some of the interesting results are as follows.

Theorem 4. Let $\lambda(1/x)\ L^{1/p}(\frac{1}{x})\ f(x) \in L(p,\alpha)$, with $1 \leq p < \infty$, $-1 < \alpha p < p-1$, where $f(x) \sim \sum_1^\infty a_n \cos nx$, with $a_n \geq 0$. If $A_n = \sum_{j=[\frac{n}{\rho}]}^{n} a_j$, then $\sum_1^\infty n^{-\rho-\alpha p} L(n)\ \lambda^p(n)\ A_n^p < \infty$, and

$$\sum_1^\infty n^{-2-\alpha p} L(n) \lambda^p(n) A_n^p \leq B(\alpha,p) \| L^{1/p}(1/x) \lambda(\pi/x)\ f(x) \|^p_{p,\alpha},$$

where $\lambda(x)$ is a positive function such that $(i)\ x^{-\alpha+1-1/p+\delta} \lambda(x) \uparrow$ as $x \to \infty$ for some small $\delta > 0$.

Theorem 5. Let $1 \leq p < \infty$ and $-1 < \alpha p < p-1$. Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n \to 0$ and

$$\left\{ \sum_{1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n)\, L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p} \right\}^{1/p} < \infty . \quad \text{Then}$$

$$L^{1/p}(1/x)\ \lambda(1/x)\ f(x) \in L(p,\alpha),$$

and

$$\| \lambda(1/x) L^{1/p}(1/x) f(x) \|_{p,\alpha}^{p} \le B(\alpha,p) \sum_{n=1}^{\infty} n^{p-\alpha p-2} \lambda^{p}(n) L(n) \left(\sum_{j=n}^{\infty} |\Delta a_j| \right)^{p}$$

where $\lambda(x)$ *is a positive function such that* (i) $x^{-\alpha-1/p+\delta}\lambda(x) \downarrow$ as $x \to \infty$ for some small $\delta > 0$, and $f(x) \sim \sum_{1}^{\infty} a_n \cos nx$.

Combining Theorems 4 and 5 we deduce the following.

Theorem 6. *Let* $\{a_n\}$ *be a positive sequence tending to zero and* $\{n^{-k} a_n\}$ *be monotonic decreasing for some real number* k. *If* $1 \le p < \infty$, $-1 < \alpha p < p-1$ *and* $0 \le \gamma < \alpha + 1/p$, *then* $x^{-\gamma} L^{1/p}(1/x)\ f(x) \in L(p,\alpha)$ *if, and only if,*

$$\sum_{n=1}^{\infty} n^{p-\alpha p+\gamma p-2}\, L(n)\, a_n^{p} < \infty, \quad \textit{where} \quad f(x) \sim \sum_{1}^{\infty} a_n \cos nx .$$

1.6 *Integrability of trigonometric series of L^p class with positive coefficients.*

In the previous section we studied integrability problems of L^p and other associated classes with monotonic coefficients. In this connection R.P. Boas raised the question as to what would happen if the condition of monotonicity on the coefficients a_n is replaced by the condition of its non-negativeness. R. Askey gave an example to demonstrate that the condition $\sum_{1}^{\infty} n^{p\gamma+p-2} a_n^{p} < \infty$

for $p > \frac{2}{1+\gamma}$ is atleast not necessary and sufficient condition for the function $x^{-\gamma} f(x) \in L^p$. However, Boas [4] found a necessary and sufficient condition for such a case in a different form. He proved the following.

Theorem T. If $a_n \geq 0$ and a_n are the Fourier sine or cosine coefficients of a function $\phi(x)$, and $1<p<\infty$, $\frac{1}{p}<\gamma<\frac{1+p}{p}$, then

$$|x-a|^{-\gamma} \, |\phi(x)-\phi(a)| \in L^p, \quad 0 \leq a < \pi,$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{p\gamma-p-2} \left(\sum_{k=1}^{n} k\, a_k\right)^p < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{p\gamma-2} \left(\sum_{k=n}^{\infty} a_k\right)^p < \infty.$$

1.7 Integrability of trigonometric series of L^p and other associated classes with monotonic function

Concerning the function instead of its Fourier coefficients a_n, Askey and Boas [1] proved a number of theorems. One of their results is as follows.

Theorem U. Let $G(x) \downarrow$ on $(0,\pi)$, G bounded below and $\int_0^\pi x\, dG(x)$ finite so that dG has generalized sine coefficients $b_n = \frac{2}{\pi}\int_0^\pi \sin nx \, dG(x)$. If $1 < p < \infty$ and $\frac{1}{p} < \gamma < 1+\frac{1}{p}$, then $\{n^{-\gamma} b_n\} \in \ell^p$ if, and only if, $t^{\gamma-1-2/p} \int_0^t x\, dG(x) \in L^p$.

From their results, Askey and Boas deduced a number of interesting corollaries.

Recently Mazhar and Khan [1] have generalized all the main results of Askey and Boas [1]. Their generalization of Theorem U is as follows.

Theorem V. Let $G(x)$ satisfy the conditions of Theorem U. If $1 < p < \infty$ and $\lambda(x)$ is a positive function such that

(1.7.1) $x^{1+\delta}\lambda(x)$ is decreasing for some small $\delta > 0$,

(1.7.2) $x^{p+1-\delta}\lambda(x) \uparrow +\infty$ for some small $\delta > 0$, as $x \to \infty$,

then $\{\lambda^{1/p}(n)\, b_n\} \in \ell^p$, if, and only if, $\lambda^{1/p}(\frac{\pi}{t})\, t^{-1-2/p}\int_0^t x\,dG(x) \in L^p$.

In Chapter III of this thesis we have established four theorems which generalise various results of Mazhar and Khan [1]. Thus for example Theorem V has been generalized in the following form.

Theorem 7. Let $G(x)$ satisfy the conditions of Theorem U. If $1 < p < \infty$, $-1 < \alpha p < p-1$, then

$\{n^{-\alpha}\lambda^{1/p}(n)b_n\} \in \ell^p$, if, and only if, $\lambda^{1/p}(\frac{\pi}{t})\,t^{-1-2/p}(\int_0^t x\,dG(x))$ $\in L(p,\alpha)$, where $\lambda(x)$ is a positive function such that

(i) $x^{1+\delta-\alpha p}\,\lambda(x)\downarrow$ <u>for some small</u> $\delta > 0$,

(ii) $x^{p+1-\delta}\,\lambda(x)\uparrow+\infty$ <u>for some small</u> $\delta > 0$, <u>as</u> $x \to \infty$.

Recently, M. Izumi and S. Izumi [1] have proved some theorems involving certain inequalities for Fourier series. With the help of some of their results they have obtained a theorem connecting monotonic decreasing Fourier coefficients with the corresponding generating function. We quote here one typical result. We write

$$G(x) = \int_{x/2}^{x} \frac{f(t)}{t}\,dt.$$

<u>Theorem A</u>. Let $p > 1$ and $a > -1$ and let f be a non-negative, non-increasing and integrable function on $(0,\pi)$. If $x^{a} f^{p}(x)$ is integrable, then we have

$$\int_0^{\pi} x^{a}\, G^{p}(x)\,dx \le A \int_0^{\pi} x^{a}\left(f(\tfrac{x}{2}) - f(x) \right)^{p} dx + A\left(\int_{\pi/2}^{\pi} f(x)\,dx\right)^{p}$$

where A is some positive constant.

In Chapter VI we have replaced the special class L^{p} of functions by a more general class L_{Φ} satisfying certain properties . Our results generalize all the theorems of M. Izumi and S. Izumi. Thus for example Theorem A has been generalized in the following manner.

Theorem 8. Let $\Phi \in \Delta_{\sigma} \cap YF$ *and* f *be a non-negative and integrable on* $(0,\pi)$. *If* $x^{\theta}\ \Phi(f(x))$ *is integrable and* $\theta > -1$, *then we have*

$$\int_0^{\pi} x^{\theta}\ \Phi(G(x))\ dx \leq A \int_0^{\pi} x^{\theta}\ \Phi(|f(x)-f(\tfrac{x}{2})|)\,dx + A\ \Phi(\int_{\pi/2}^{\pi} f(x)\,dx).$$

where A *is some positive constant.*

1.8 *Integrability of power series.*

In 1958, Heywood [1] studied the integrability of power series and proved several theorems. His main theorem is as follows .

Theorem X. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \geq 0$, $0 \leq x < 1$.
Then for $\gamma < 1$,

$$(1-x)^{-\gamma}\ f(x) \in L(0,1)$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma-1}\ a_n\ < \infty .$$

Later on he [2] weakened the hypothesis of positiveness of the coefficients a_n by replacing it with $a_n \geq -\frac{K}{n^{1+\epsilon}}$,

$C > 0$. He proved the following theorem.

<u>Theorem X</u>. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $0 \leq x < 1$, $\gamma < 1$ and that there is a positive number C such that $a_n > -\frac{A}{n^{1+C}}$, for all sufficiently large values of n, A being some positive constant. Then $(1-x)^{-\gamma} f(x) \in L(0,1)$ if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma-1} |a_n| < \infty .$$

In the same year, Kennedy [1] replaced $(1-x)^{-\gamma}$ by a general integrable function which contains Theorem X for $0 \leq \gamma < 1$. He also proved a general theorem which contains Theorem X for $\gamma < 0$. Some of the theorems of Heywood were subsequently generalised by Boas and Gonzaléz-Fernádez [1] . Chen [4] made certain generalization in another direction.

Later on in 1964 Mercer [1] proved some theorems on integrability of power series involving Cauchy integrals .

Askey [1] and Askey and Boas [2] have proved several theorems on the integrability of power series with positive coefficients for L^p class. Recently, Khan [2] and Mazhar and Khan [2] generalized the results of Askey [1] and Askey and Boas [2] The result of Khan, which deals with the case $a_n \geq 0$, includes as a special case for $p=1$ the theorem of Heywood (Theorem X). Almost simultaneously, Leindler [1] generalized the result of Khan.

The question arises whether it is possible to relax the condition of positiveness of a_n in the theorems of Khan. In Section B of Chapter II we answer this question in the affirmative. We prove the following theorem which includes Theorem Y of Heywood for p=1.

<u>Theorem</u> 9. <u>Let</u> $f(x) = \sum_{0}^{\infty} a_n x^n$, $0 \leq x < 1$ <u>and</u> $\gamma < 1$. Suppose that there is a positive number ϵ such that $a_n > -\frac{K}{n^{\gamma/p+1-1/p+\epsilon}}$ $(0 < p \leq \infty)$ <u>for all sufficiently large values of</u> n, <u>where</u> K <u>is some positive constant</u>. <u>Then</u> $(1-x)^{-\gamma} (f(x))^p \in L(0,1)$ <u>if, and only if</u> $\sum n^{\gamma-2}(\sum_{k=1}^{n} |a_k|)^p$ <u>converges</u>.

Recently, Woyczyński [1] proved the following theorem which deals with the equivalence of a number of statements pertaining to integrability of power series.

<u>Theorem Z</u>. Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $0 \leq x < 1$. If $a_n \geq a_{n+1} \geq 0$ $(n = 0,1,2,\ldots)$ then the following four statements are equivalent.

(i) $f(x) \in L_{\Phi}(0, 1)$;

(ii) $g(t) \equiv f(e^{it}) \in H_{\Phi}(0,2\pi)$;

(iii) $\{n a_n\} \in L_{\Phi}(N, \nu)$;

(iv) $\{A_n\} \in L_{\Phi}(N, \nu)$,

where $\Phi \in \Delta_\alpha \cap E \cap YP$, $d\mu = dx$, N stands for the set of all positive integers and ν is the measure on N concentrating the mass n^{-2} at the point $n \in N$, and $A_n = a_0 + a_1 + \dots + a_n$.

Section A of Chapter II deals with a generalization of Theorem 2. We prove the following theorem.

<u>Theorem</u> 10. <u>Let</u> $\Phi \in \Delta_\alpha \cap E \cap YP$ <u>and</u>

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad 0 \leq x < 1.$$

<u>If</u> $\{a_n\}$ <u>is a quasi-monotone sequence such that</u>

$0 < B_1 \leq n^{\beta} a_n \leq B_2$ <u>with some</u> $\beta > 0$, $(n=1,2,\dots)$, <u>and</u>
$0 \leq \gamma < 1$, <u>then the following four statements are equivalent</u>.

(i) $(1-x)^{-\gamma} \Phi(f(x)) \in L(0,1)$;

(ii) $x^{-\gamma} \Phi(|f(e^{ix})|) \in L(0,\pi)$;

(iii) $\sum_{n=1}^{\infty} n^{\gamma-2} \Phi(n a_n) < \infty$;

(iv) $\sum_{n=1}^{\infty} n^{\gamma-2} \Phi(A_n) < \infty$,

<u>where</u> $A_n = a_0 + a_1 + \dots + a_n$.

1.9 <u>Integrability of trigonometric transforms.</u>

Recently Boas [6] has proved the following theorem for

Fourier transforms by a method which is rather more direct than those that have been used for similar problems about Fourier series.

Theorem A'. If $f(x) \downarrow 0$, $x^{1/p} f(x) \in L^p (0,1)$, and $F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt \, dt$, then $x^{\gamma+1-2/p} F(x) \in L^p (0,\infty)$ provided that $x^{-\gamma} f(x) \in L^p (0,\infty)$, where $p > 1$ and $-\frac{1}{p} < \gamma < \frac{1}{p}$.

It may be remarked that in Theorem A', the condition $x^{1/p} f(x) \in L^p (0,1)$ is redundant, in view of the condition $x^{-\gamma} f(x) \in L^p (0,\infty)$.

In Chapter VIII of this thesis we obtain a theorem in which we have replaced L^p class by a more general class L_Φ. Our theorem is as follows.

Theorem 11. Let $F(x)$ be a sine transform of $f(x)$. If $f(x) \downarrow 0$, $x^{-\alpha} \Phi(f(x)) \in L(0,\infty)$ and $-1 < \alpha < 1$, then $x^{\alpha-2} \Phi(x F(x)) \in L (0, \infty)$, where $\Phi(x)$ is a convex Orlicz function satisfying Δ_2 condition.

It may be observed that for $\Phi(t) = t^p$, $p > 1$, we get Theorem A'.