## Chapter VIII

## THE INTEGRABILITY CLASS OF THE SINE TRANSFORM OF A MONOTONIC FUNCTION

8.1 A non-decreasing, continuous and real-valued function $\Phi$ defined on the non-negative half-line and vanishing only at the origin is called an Orlicz Function (OF). Function $\Phi \in$ OF is said to satisfy $\Delta_2$ - condition for large u if there are constants $c > 0$ and $u_0 \geq 0$ such that

$$\Phi(2u) \leq c\, \Phi(u), \quad u \geq u_0.$$

8.2 Recently, Boas [6] has established the following theorem for Fourier transform by a method which is rather more direct than those that have been used for similar problems about Fourier series. His method depends on the Steffensen's version of Jensen's inequality ( see Mitrinović [1] , p.109 ) and a theorem of Edmonds [1] on Parseval's theorem for monotonic function.

<u>Theorem A</u>. If $f(x) \downarrow 0$, $x^{1/p} f(x) \in L^p (0,1)$ , and

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt \, dt ,$$

then $x^{\gamma+1-\frac{2}{p}} F(x) \in L^p (0,\infty)$ provided that

$$x^{-\gamma} f(x) \in L^p (0, \infty) ,$$

where $p > 1$ and $-\frac{1}{p} < \gamma < \frac{1}{p}$.

It may be remarked that in Theorem A, the condition $x^{1/p} f(x) \in L^p (0, 1)$ need not be mentioned because it is already implied by the condition

$$x^{-\gamma} f(x) \in L^p ( 0, \infty ), -\frac{1}{p} < \gamma < \frac{1}{p} .$$

In this chapter it is proposed to obtain a generalization of the above theorem. Instead of considering $L^p$ class we would employ a more general class, namely $L_\Phi$ .

8.3 We prove the following theorem.

__Theorem__. __Let__ $F(x)$ __be the sine transform of__ $f(x)$ . __If__ $f(x) \downarrow 0$ , $x^{-\alpha} \Phi ( f(x) ) \in L (0, \infty)$ __and__ $-1 < \alpha < 1$, __then__ $(x^{\alpha-2}) \Phi (x F(x) ) \in L (0, \infty)$ , __where__ $\Phi (x)$ __is a__ *convex Orlicz function satisfying* $\Delta_2$ *condition.*

It may be observed that for $\Phi(t) = t^p$, $p > 1$ we get theorem A.

8.4 We require the following lemmas for the proof of our theorem.

*Lemma 1.* ( Boas [8] ). *Let* $\lambda$ *be a function of bounded variation on every finite sub-interval of* $(0, \infty)$ ; $\lambda(o) \leq \lambda(x)$ *for all* $x > 0$ ; *and* $\lambda(o) < \Lambda = \sup \lambda(x)$ . *Let* $f(x)$ *decrease and* $f(x) \geq 0$. *If* $\Psi$ *is continuous and convex over* $(0, f(o))$, $\Psi(o) \leq 0$ *and*

$$\int_0^\infty d\,\mu(x) \geq \Lambda - \lambda(o) ,$$

*then*

$$\Psi\left\{\frac{\int_0^\infty f(x)\; d\lambda(x)}{\int_0^\infty d\,u(x)}\right\} \leq \frac{\int_0^\infty \Psi(\, f(x)\,)\; d\lambda(x)}{\int_0^\infty d\,\mu(x)}$$

*Lemma 2.* ( Boas [3] , p.58). *If* $g$ *and* $\beta$ *decrease to* 0 *on* $(0, \infty)$ *and* $x\,g(x)$ , $x\,\beta(x) \in L(0,1)$ , *then* $\beta(y)\; b(y) \in L\,(0, \infty)$ *if, and only if* , $g(u)\; G(u) \in L\,(0, \infty)$ *and Parseval's formula*

$$\int_0^\infty B(y)\ b(y)\ dy = \int_0^\infty G(u)\ g(u)\ du$$

holds, G and b being the sine transforms of B and g respectively.

8.5 Proof of the Theorem. Taking $\lambda(x) = 1-\cos x$, $\Lambda = 2$ and using Lemma 1, we have

(8.5.1)
$$\Phi\left( \frac{1}{2} \int_0^\infty f(x) \sin x\ dx \right) \leq \frac{1}{2} \int_0^\infty \Phi( f(x)) \sin x\ dx .$$

Since sine transform of a positive decreasing function is positive, it follows that reight-hand side is positive. Also in view of the hypotheses, it is finite. Now replacing $f(x)$ by $f(xt)$, multiplying (8.5.1) by $t^{-\alpha}$ and integrating over $(0, \infty)$ we have

(8.5.2)
$$\int_0^\infty t^{-\alpha}\ \Phi\left( \frac{1}{2} \int_0^\infty f(xt) \sin x\ dx \right) dt \leq \frac{1}{2} \int_0^\infty t^{-\alpha}\ dt \int_0^\infty \Phi( f(xt)) \sin x\ dx.$$

Putting $t = \frac{1}{y}$ and $x = yu$ in (8.5.2) we have

$$\int_0^\infty y^{\alpha-2}\ \Phi\left( \frac{1}{2} f(u) \sin yu\ ydu \right) dy$$

$$\leq \frac{1}{2} \int_0^\infty y^{\alpha-2}\, dy \int_0^\infty \Phi(\ f(u)\ ) \sin yu\ y du.$$

That is to say,

$$\int_0^\infty y^{\alpha-2}\ \Phi\left( \frac{1}{2} \sqrt{\frac{\pi}{2}}\ y\ F(y)\ \right) dy$$

$$\leq \frac{1}{2} \int_0^\infty y^{\alpha-1}\, dy \int_0^\infty \Phi(\ f(u)\ ) \sin yu\ du\ .$$

Thus, it follows that,

$$\int_0^\infty y^{\alpha-2}\ \Phi(\ y\ F(y)\ )\ dy$$

$$\leq C \int_0^\infty y^{\alpha-2}\ \Phi\left( \frac{1}{2} \sqrt{\frac{\pi}{2}}\ y\ F(y)\ \right) dy$$

$$\leq C \int_0^\infty y^{\alpha-1}\, dy \int_0^\infty \Phi(\ f(u)\ ) \sin yu\ du$$

$$= C \int_0^\infty B(y)\ b(y)\ dy\ ,$$

where $B(y) = y^{\alpha-1}$, $-1 < \alpha < 1$, $g(u) = \Phi(\ f(u)\ )$ and G and b are the sine transforms of B and g respectively.

Now

$$\int_0^\infty g(u)\ G(u)\ du$$

$$= \int_0^\infty \Phi(f(u))\, du \int_0^\infty y^{\alpha-1} \sin yu\, du$$

$$= \int_0^\infty u^{-\alpha} \Phi(f(u))\, du \int_0^\infty t^{\alpha-1} \sin t\, dt \; {}^{*}$$

$$= \Gamma(\alpha) \sin \frac{\pi\alpha}{2} \int_0^\infty u^{-\alpha} \Phi(f(u))\, du$$

$$< \infty ,$$

by virtue of the hypotheses. Thus $Gg \in L(0, \infty)$. In view of Lemma 2, Parseval's formula holds and therefore,

$$\int_0^\infty y^{\alpha-2} \Phi(y F(y))\, dy < \infty .$$

Thus our theorem is proved.

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* when $\alpha = 0$, the integral $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.