Chapter VII

INTEGRABILITY THEOREMS FOR FOURIER SERIES AND PARSEVALS FORMULAE

7.1 Recently, Hasegawa [1] has proved, among others, the following theorems.

Theorem A. Let $f(x)$ be an even function, continuous on $(0,\pi)$, and let its Fourier series be

$$(7.1.1) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

Suppose that $\alpha(x)$ is an even function, positive and non-increasing on $(0,\pi)$, that $x\alpha(x) \in L(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-2} \int_0^t x\,\alpha(x)\,dx \leq C^{\dagger}\alpha(t) \text{ for all } t, \quad 0 < t \leq \eta.$$

If the series $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)dx$ converges absolutely, then the Fourier series (7.1.1) converges uniformly to $f(x)$ and

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† where C, with or without suffixes, denotes a positive constant, not necessarily the same at each occurrence.

$(f(x) - f(s))\ \alpha\ (x-s)\ \in\ L(0,\pi)$ for each s, $0 < s < \pi$.

<u>Theorem B</u>. Let $g(x)$ be an odd function, continuous on $(0,\pi)$, and let its Fourier series be

(7.1.2) $$g(x) \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

Suppose that $\alpha(x)$ is an odd function, positive on $(0,\pi)$, that $x\,\alpha(x)$ is Lebesgue integrable and non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$t^{-1} \int_t^{\eta} \alpha(x)\ dx \leq C\ \alpha(t) \qquad \text{for all } t,\ 0 < t \leq \eta.$$

If the series $\sum_{n=1}^{\infty} n\, b_n \int_0^{1/n} x\,\alpha(x)\ dx$ converges absolutely, then the Fourier series (7.1.2) converges uniformly to $g(x)$ and $(g(x)-g(s))\ \alpha\ (x-s) \in L(0,\pi)$ for each s, $0 \leq s \leq \pi$. In particular, in case $s=0$, Parseval's formulae

$$\frac{2}{\pi} \int_0^{\pi} g(x)\ \alpha(x)\ dx = \sum_{n=1}^{\infty} b_n\, q_n$$

holds, where q_n's are defined by

$$q_n = \frac{2}{\pi} \int_0^{\pi} \alpha(x)\ \sin nx\ dx\ ,$$

and the series $\sum_{n=1}^{\infty} b_n\, q_n$ converges absolutely.

The object of this chapter is to obtain certain generalisation of these theorems. In what follows we assume that $\alpha(x)$, $\beta(x)$ and $\psi(x)$ are positive functions defined on $(0,\pi)$ such that $\alpha(x)\,\beta(x) \in L(0,\pi)$ and $\alpha(x)\,\psi(x)$ is either even or odd.

7.2 We prove the following theorems.

Theorem 1. Let $f(x)$ be an even function and continuous on $(0,\pi)$ and let its Fourier series be (7.1.1). Suppose that $\alpha(x)\dfrac{\psi(x)}{\beta(x)}$ is non-increasing on $(0,\pi)$ and that there is a positive number $\eta \leq \pi$ such that

$$(7.2.1) \qquad \int_0^t \beta(x)\,\alpha(x)\,dx \leq C\;\Phi(t)\,\alpha(t)$$

for all t, $0 < t \leq \eta$, where $\Phi(t) = \int_0^t \beta(x)\,dx$. If the series

$$\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx$$

converges absolutely, then the Fourier series (7.1.1) converges uniformly to $f(x)$ and

$$(\,f(x) - f(s)\,)\,\alpha\,(x-s)\,\psi(x-s) \in L(0,\pi)$$

for each s, $0 < s < \pi$, where $x\,\dfrac{\psi(x)}{\beta(x)} < M^{*}$, $0 < x \leq \pi$

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* where M is a positive constant.

and

$$(7.2.2) \qquad \frac{\phi(x)\ \Phi(x)}{\psi(x)\,(\Phi(x)-\Phi(x/2))} = O(x), \quad x \to 0.$$

Theorem 2. Let $g(x)$ *be an odd function, continuous on* $(0,\pi)$ *and let its Fourier series be* (7.1.2). *Suppose that* $\phi(x)\ \alpha(x)$ *is non-increasing on* $(0,\pi)$ *and that there is a positive number* $\eta \leq \pi$ *such that*

$$(7.2.3) \qquad \int_t^{\eta} \alpha(x)\psi(x)\ dx \leq C\ \phi(t)\ \alpha(t)$$

for all t, $0 < t \leq \eta$, *where*

$$x\,\frac{\psi(x)}{\phi(x)} < M, \qquad 0 < x \leq \pi .$$

If the series

$$\sum_{n=1}^{\infty} n\, b_n \int_0^{1/n} \phi(x)\ \alpha(x)\ dx$$

converges absolutely, then the Fourier series (7.1.2) *converges uniformly to* $g(x)$ *and* $(g(x) - g(s))\ \alpha(x-s)\ \psi(x-s) \in L(0,\pi)$ *for each* s, $0 \leq s \leq \pi$. *In particular for* $s=0$, *Parseval's formulae*

$$\frac{2}{\pi}\int_0^{\pi} g(x)\ \alpha(x)\ \psi(x)\ dx = \sum_{n=1}^{\infty} b_n\, q_n$$

<u>holds and the series</u> $\sum_{n=1}^{\infty} b_n q_n$ <u>converges absolutely, where</u>

(7.2.4) $$q_n = \frac{2}{\pi}\int_0^{\pi} \alpha(x)\ \psi(x)\ \sin nx\ dx.$$

<u>Remarks</u>. By putting $\psi(x) = 1$ and $\phi(x) = x$ in Theorems 1 and 2, we get Theorems A and B respectively. On the other hand, if we take $\phi(x) = x$ and write $\psi(x)\ \alpha(x) = \beta(x)$ in Theorem 1, we get Theorem A in which one of our conditions namely " $\frac{\beta(x)}{x}\downarrow$ " is lighter than the corresponding condition "$\beta(x)\downarrow$ " of Theorem A. By taking $\psi(x) = 1+x$, and $\phi(x) = x$ a similar remark is applicable to Theorem B. Also we extend the scope of our theorems by assuming that $\beta(x)$ is either even or odd instead of restricting it to be even in Theorem A and odd in Theorem B. Other interesting cases for which our theorems hold are

(i) $\psi(x) = \sin\frac{x}{2}$, $\phi(x) = x^2$.

(ii) $\psi(x) = \cos\frac{x}{2}$, $\phi(x) = x$.

7.3 <u>Proof of Theorem 1</u>. Since $\sum_{n=1}^{\infty} a_n \int_{1/n}^{\pi} \alpha(x)\ \psi(x)\ dx$

converges absolutely, we have

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left\{ |a_n| \frac{\int_{1/n}^{\pi} \alpha(x)\ \psi(x)\ dx}{\int_{1/n}^{\pi} \alpha(x)\ \psi(x) dx} \right\}$$

$$\leq \left(\int_1^{\pi} \alpha(x)\ \psi(x)\,dx \right)^{-1} \sum_{n=1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\,\psi(x)\,dx$$

$$< \infty .$$

By hypothesis $f(x)$ is continuous and hence in view of well known result* the Fourier series (7.1.1) converges uniformly to $f(x)$.

Since $\dfrac{\alpha(x)\ \psi(x)}{\phi(x)}$ is non-increasing, we have

$$\int_{1/n}^{2/n} \alpha(x)\ \psi(x)\ dx$$

$$\geq \frac{\alpha(2/n)\ \psi(2/n)}{\phi(2/n)} \int_{1/n}^{2/n} \phi(x)\ dx$$

$$= \frac{\alpha(2/n)\ \psi(2/n)}{\phi(2/n)} \left(\Phi(2/n) - \Phi(1/n) \right),$$

or,

$$\alpha(2/n) \leq \frac{\phi\ (2/n)}{\psi(\frac{2}{n})\ \left(\Phi(\frac{2}{n}) - \Phi(\frac{1}{n}) \right)} \int_{1/n}^{2/n} \alpha(x)\ \psi(x)\,dx$$

$$\leq \frac{\phi(2/n)}{\psi(\frac{2}{n})\ \left(\Phi(\frac{2}{n}) - \Phi(\frac{1}{n}) \right)} \int_{1/n}^{\pi} \alpha(x)\ \psi(x)\ dx.$$

Hence from (7.2.1) and (7.2.2) we have

*see for example Zygmund [1] .

$$(7.3.1)\qquad \int_0^{1/n} \phi(x)\alpha(x)\,dx \le \int_0^{2/n} \phi(x)\alpha(x)\,dx$$

$$\le C\,\alpha\,(2/n)\ \Phi\,(2/n)$$

$$\le C\,\frac{\phi(\frac{2}{n})\ \ \Phi\,(\frac{2}{n})}{\Psi(\frac{2}{n})\left(\Phi(\frac{2}{n}) - \ \Phi(\frac{1}{n})\right)}\cdot \int_{1/n}^{\pi} \alpha(x)\ \ \Psi(x)\,dx$$

$$\le \frac{C_1}{n} \int_{1/n}^{\pi} \alpha(x)\ \ \Psi(x)\,dx$$

for all $n \ge N$, where N denotes the smallest integer $\ge \frac{1}{\eta}$. Since $\phi(x)\ \alpha(x) \in L(0,\pi)$ and $\Psi(x)\ \alpha(x)$ is either even or odd, we have for $0 < s < \pi$

$$\int_0^{\pi} \left|(f(x) - f(s)\ )\ \alpha(x-s)\ \ \Psi\,(x-s)\right|\ dx$$

$$= \int_0^{\pi} \left|(\ \sum_{n=1}^{\infty} a_n(\cos nx - \cos ns))\ \alpha(x-s)\ \ \Psi\,(x-s)\right|\ dx$$

$$\le \sum_{n=1}^{\infty} |a_n| \int_0^{\pi} \left|(\cos nx - \cos ns)\ \alpha(x-s)\ \ \Psi\,(x-s)\right|\ dx$$

$$\le 2 \sum_{n=1}^{\infty} |\,a_n| \int_0^{\pi} \left|\,\sin \frac{n}{2}\ (x-s)\ \ \alpha(x-s)\ \ \Psi\,(x-s)\right|\ dx$$

$$= 2 \sum_{n=1}^{\infty} |\,a_n| \int_{-s}^{\pi-s} \left|\,\sin \frac{nx}{2}\,\alpha(x)\ \ \Psi\,(x)\right|\ dx$$

$$= 2 \sum_{n=1}^{\infty} |a_n| \left(\int_{-s}^{0} + \int_{0}^{\pi-s} \right) \left| \sin\frac{nx}{2}\, \alpha(x)\, \psi(x) \right| dx$$

$$\leq 2 \sum_{n=1}^{\infty} |a_n| \left(\int_{0}^{s} + \int_{s}^{\pi-s} \right) \left| \sin\frac{nx}{2} \right| \alpha(x)\, \psi(x)\, dx$$

$$\leq 4 \sum_{n=1}^{\infty} |a_n| \int_{0}^{\pi} \left| \sin\frac{nx}{2} \right| \alpha(x)\, \psi(x)\, dx$$

$$= 4 \sum_{n=1}^{N} |a_n| \int_{0}^{\pi} \left| \sin\frac{nx}{2} \right| \alpha(x)\, \psi(x)\, dx$$

$$+ 4 \sum_{n=N+1}^{\infty} |a_n| \left(\int_{0}^{1/n} + \int_{1/n}^{\pi} \right) \left| \sin\frac{nx}{2} \right| \alpha(x)\, \psi(x)\, dx$$

$$= I_1 + I_2 , \text{ say.}$$

Now

$$I_2 \leq 2 \sum_{n=N+1}^{\infty} n\, |a_n| \int_{0}^{1/n} x\, \alpha(x)\, \psi(x)\, dx +$$

$$+ 4 \sum_{n=N+1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\, \psi(x)\, dx$$

$$\leq C \sum_{n=N+1}^{\infty} n\, |a_n| \int_{0}^{1/n} \alpha(x)\phi(x)\, dx + 4 \sum_{n=N+1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\, \psi(x)\, dx$$

$$\leq C_1 \sum_{n=N+1}^{\infty} |a_n| \int_{1/n}^{\pi} \alpha(x)\, \psi(x)\, dx$$

$$< \infty ,$$

by (7.3.1) and the hypothesis.

Also,

$$I_1 \leq 4 \left(\int_0^{\pi} x \, \psi(x) \, \alpha(x) \, dx \right) \sum_{n=1}^{N} n \, |a_n|$$

$$\leq C \left(\int_0^{\pi} \alpha(x) \, \phi(x) \, dx \right) \sum_{n=1}^{N} n \, |a_n|$$

$$< \infty .$$

Hence $|f(x) - f(s)| \, \alpha(x-s) \, \psi(x-s) \in L \, (0,\pi)$.

Thus Theorem 1 is established.

7.4 <u>Proof of Theorem 2</u>. Since $\phi(x) \, \alpha(x)$ is positive, non-increasing and Lebesgue integrable on $(0,\pi)$, we have

$$\int_0^{1/n} \phi(x) \, \alpha(x) \, dx \geq \frac{1}{n} \phi\left(\frac{1}{n}\right) \alpha\left(\frac{1}{n}\right) \geq \frac{1}{n} \phi(1) \, \alpha(1).$$

According to the hypothesis, $\sum_{n=1}^{\infty} n \, b_n \int_0^{1/n} \phi(x) \, \alpha(x) \, dx$ converges absolutely and hence we have

$$\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} |b_n| \frac{n \int_0^{1/n} \phi(x) \, \alpha(x) \, dx}{n \int_0^{1/n} \phi(x) \, \alpha(x) \, dx}$$

$$\leq \frac{1}{\phi(1) \alpha(1)} \sum_{n=1}^{\infty} n \, |b_n| \int_0^{1/n} \alpha(x) \, \phi(x) \, dx$$

$$< \infty .$$

consequently, as in the proof of Theorem 1, the Fourier series (7.1.2) converges uniformly to $g(x)$.

Now using condition (7.2.3)

$$(7.4.1) \qquad \int_{1/n}^{\eta} \alpha(x)\,\psi(x)\,dx \le C\,\phi\!\left(\tfrac{1}{n}\right)\alpha\!\left(\tfrac{1}{n}\right) \le C\,n\int_0^{1/n}\alpha(x)\,\phi(x)\,dx$$

for all $n \ge N$, where N denotes the smallest integer $\ge \frac{1}{\eta}$. We first assume that $0 < s < \pi$. Then

$$\int_0^{\pi} \left|(g(x) - g(s))\,\alpha(x-s)\,\psi(x-s)\right| dx$$

$$= \int_0^{\pi} \left|\left(\sum_{n=1}^{\infty} b_n(\sin nx - \sin ns)\right)\alpha(x-s)\,\psi(x-s)\right| dx$$

$$\le 2\sum_{n=1}^{\infty}|b_n|\int_0^{\pi}\left|\sin\frac{n(x-s)}{2}\,\alpha(x-s)\,\psi(x-s)\right| dx$$

$$= 2\sum_{n=1}^{\infty}|b_n|\int_{-s}^{\pi-s}\left|\sin\frac{nx}{2}\,\alpha(x)\,\psi(x)\right| dx$$

$$= 2\sum_{n=1}^{\infty}|b_n|\left(\int_{-s}^{0}+\int_0^{\pi-s}\right)\left|\sin\frac{nx}{2}\,\alpha(x)\,\psi(x)\right| dx$$

$$\le 2\sum_{n=1}^{\infty}|b_n|\left(\int_0^{s}+\int_0^{\pi-s}\right)\left|\sin\frac{nx}{2}\right|\alpha(x)\,\psi(x)\,dx$$

$$(7.4.2) \qquad \le 4\sum_{n=1}^{\infty}|b_n|\int_0^{\pi}\left|\sin\frac{nx}{2}\right|\alpha(x)\,\psi(x)\,dx$$

$$= 4\left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) |b_n| \int_0^{\pi} \left|\sin \frac{nx}{2}\right| \alpha(x)\ \psi(x)\ dx$$

$$= J_1 + J_2, \quad \text{say.}$$

Now,

$$J_2 = 4 \sum_{n=N+1}^{\infty} |b_n| \left(\int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{\pi} \right) \left|\sin \frac{nx}{2}\right| \alpha(x)\ \psi(x)\ dx$$

$$\leq 2 \sum_{n=N+1}^{\infty} n\, |b_n| \int_0^{1/n} x\, \alpha(x)\ \psi(x)\ dx\ +$$

$$+\ 4 \sum_{n=N+1}^{\infty} |b_n| \int_{1/n}^{\eta} \alpha(x)\ \psi(x)\ dx\ +$$

$$+\ 4 \sum_{n=N+1}^{\infty} |b_n| \int_{\eta}^{\pi} \alpha(x)\ \psi(x)\ dx$$

$$\leq C \sum_{n=N+1}^{\infty} n\, |b_n| \int_0^{1/n} \alpha(x)\ \phi(x)\ dx$$

$$+\ C_1 \sum_{n=N+1}^{\infty} n\, |b_n| \int_0^{1/n} \alpha(x)\ \phi(x)\, dx + C_2$$

$$< \infty,$$

by virtue of (7.4.1) and the hypothesis.

Also,

$$J_1 \leq 2 \sum_{n=1}^{N} n\, |b_n| \int_0^{\pi} x\, \alpha(x)\ \psi(x)\ dx$$

$$\leq C \left(\sum_{n=1}^{N} n \, |b_n| \right) \int_0^{\pi} \alpha(x)\, \phi(x)\, dx$$

$$< \infty .$$

Hence $(g(x) - g(s))\, \alpha(x-s)\, \psi(x-s) \in L\,(0,\pi)$ for $0 < s < \pi$.

The cases $s=0$ and $s=\pi$ can be easily disposed off in a similar manner.

To prove the last part of our assertion we notice that

(7.4.3) $$\sum_{n=1}^{\infty} |b_n\, q_n|$$

$$\leq \sum_{n=1}^{\infty} |b_n| \int_0^{\pi} \alpha(x)\, \psi(x)\, |\sin nx|\, dx$$

$$< \infty ,$$

as shown in (7.4.2).

Thus $\sum_{n=1}^{\infty} b_n\, q_n$ converges absolutely and hence

$$\sum_{n=1}^{\infty} b_n\, q_n = \sum_{n=1}^{\infty} b_n . \frac{2}{\pi} \int_0^{\pi} \alpha(x)\, \psi(x)\, \sin nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \alpha(x)\, \psi(x) \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \alpha(x)\, \psi(x)\, g(x).$$

The term by term integration is justified in view of (7.4.3).

This completes the proof of Theorem 2.