CHAPTER – III

EDGE AND VERTEX INTERSECTION OF PATHS IN A GRAPH

In section 3.1 we study vertex intersection graph of a path cover of a graph. In 3.2 we make an analogous study of edge intersection graph of paths of length 2 in a graph. This is in analogy with the work done by H.J. Broersma and C. Hoede [18]. In 3.3 we introduce path partition graph.

3.1. Path Graph

In [1] B.D. Acharya and E. Sampathkumar define a graph G as a graphoidal graph if G is the intersection graph of a graphoidal cover of a graph H. A path cover (path partition as in Harary [29]) of a graph G is a collection \( \mathcal{P} \) of paths in G such that every edge of G lies in exactly one path in \( \mathcal{P} \). We define a graph G as a path graph if G is the intersection graph of a path cover of a graph H. In [1], it was conjectured that every graph is graphoidal. In [6], C. Pakkiam and S. Arumugam disproved it. In this connection we have the following for path graphs.

3.1.1. Theorem: Every graph is a path graph.

Proof: Let \( V(G) = \{v_1, v_2, \ldots, v_p\} \) and \( E(G) = \{e_1, e_2, \ldots, e_q\} \). Let \( S_k = \{v_k\} \cup \{v: e \text{ is incident with } v_k\}, 1 \leq k \leq p \). In \( S_k \), the first element should be \( v_k \) and the edges can be ordered in any way. Every edge of G belongs to exactly two \( S_k \)'s and \( v_i \) belongs to \( S_i \) only. Moreover \( S_i \cap S_j = \{e\} \), where \( e = (v_i, v_j) \). The vertex set \( V(H) = V(G) \cup E(G) \) and the edge set \( E(H) \) is defined in such a way
that each $S_k$ is a path in $H$. Clearly $\Psi = \{S_1, S_2, ..., S_p\}$ is a path cover of $H$ and the map $f: V(G) \to V(\Omega(\psi)) = \Psi$ defined by $f(v_i) = S_i$ is an isomorphism.

Hence $G$ is a path graph.

The following example illustrates the above theorem. Consider the graph $G$ given in figure 1.

3.1.2. Example

Let $S_1 = (v_1, e_1, e_2, e_3)$

$S_2 = (v_2, e_1, e_4, e_6, e_7, e_9)$

$S_3 = (v_3, e_2, e_4, e_5)$

$S_4 = (v_4, e_3, e_5, e_6, e_8, e_{11})$

$S_5 = (v_5, e_7, e_8, e_{10})$

$S_6 = (v_6, e_9, e_{10}, e_{11}, e_{12})$

$S_7 = (v_7, e_{12})$

Fig. 1

The graph $H$ in which each $S_k$ is a path and $V(H) = E(G) \cup V(G)$ is given in figure 2.

Fig. 2
Clearly $\Psi = \{ S_1, S_2, S_3, S_4, S_5, S_6, S_7 \}$ is a path cover of $H$ and $G \equiv \Omega(\Psi)$.

Analogous to the result for graphoidal covers in [6], we have the following result for path covers.

3.1.3. Theorem: Let $\Psi$ be a path cover of $G$. Let $P = (v_1, v_2, \ldots, v_k)$ be a path in $\Psi$ such that $\deg(v_i) > 2$, $\deg(v_k) > 2$, $\deg(v_i) = 2$ for $i = 2, 3, \ldots, k-1$ and $G' = G \setminus \{v_2, v_3, \ldots, v_{k-1}\}$ is disconnected. Then $P$ is a cut vertex of $\Omega(\Psi)$.

Proof: Suppose $\Omega(\Psi) \setminus P$ is connected. Since $\deg(v_i) = 2$ for $i=2,3,\ldots, k-1$, the vertices $v_2, v_3, v_4, \ldots, v_{k-1}$ do not lie in any path other than $P$ in $\Psi$. Let $Q_1, Q_2$ be paths in $\Psi$ different from $P$ such that $v_1$ lies on $Q_1$ and $v_k$ lies on $Q_2$.

Let $(Q_1 = P_1, P_2, \ldots, P_n = Q_2)$ be a path in $\Omega(\Psi) \setminus P$. Let $w_i$ be a vertex common to $P_i$ and $P_{i+1}$, $i = 1, 2, \ldots, n-1$. Then the $v_1 - w_1$ section of $P_1$ followed by the $w_1 - w_2$ section of $P_2$, $w_2 - w_3$ section of $P_3, \ldots$, $w_{n-1} - v_k$ section of $Q_2$ forms a $v_1 - v_k$ walk in $G'$. Hence $v_1$ and $v_k$ can be joined by a path in $G'$ which is a contradiction, since $v_1$ and $v_k$ are in different components of $G'$. Hence $\Omega(\Psi) \setminus P$ is disconnected and $P$ is a cut vertex of $\Omega(\Psi)$.

Similar to the parameters introduced in [1], we introduce the following parameters for a graph $G$:
\[ \mu_0(G) = \inf_{H \in \theta(G)} |V(H)| \]
\[ \mu_1(G) = \inf_{H \in \theta(G)} |E(H)|, \] where \( \theta(G) \) is the collection of all graphs \( H \) such that \( G \) is the intersection graph of some path cover of \( H \).

3.1.4. Theorem

Let \( G \) be a graph and \( \theta(G) \) be the class of all graphs \( H \) such that \( \Omega(\Psi) \equiv G \) for some path cover \( \Psi \) of \( H \). Then \( \theta(G) \) is infinite.

Proof: Since every graph is a path graph, \( \theta(G) \) is not empty. Let \( H \in \theta(G) \). Let \( \Psi \) be a path cover of \( H \) such that \( \Omega(\Psi) \equiv G \). Let \( S(H) \) denote the subdivision graph of \( H \). Then \( S(\Psi) = \{ S(P) : P \in \Psi \} \) is a path cover of \( S(H) \) and \( \Omega(S(\Psi)) \equiv \Omega(\Psi) \equiv G \). Hence \( S(H) \in \theta(G) \). It follows that \( S^n(H) \in \theta(G) \) for all integers \( n \). Hence \( \theta(G) \) is infinite.

We first determine \( \mu_0 \) and \( \mu_1 \) for complete graphs.

3.1.5. Theorem: \( \mu_0(K_n) = n+1 \) and \( \mu_1(K_n) = n \).

Proof: Let \( H = K_{1,n} \). Clearly \( \Psi = E(H) \) is a path cover of \( H \) and \( \Omega(\Psi) \equiv K_n \). Hence \( H \in \theta(K_n) \) so that \( \mu_0(K_n) \leq n+1 \) and \( \mu_1(K_n) \leq n \). Let \( \Psi_1 \) be any path cover of a graph \( H \) such that \( \Omega(\Psi_1) \equiv K_n \). Then \( \Psi_1 \) is a collection of \( n \) paths which are mutually edge disjoint and any two paths in \( \Psi_1 \) have a common vertex. Hence \( |V(H)| \geq n+1 \) and \( |E(H)| \geq n \) so that \( \mu_0(K_n) \geq n+1 \) and \( \mu_1(K_n) \geq n \). Hence \( \mu_0(K_n) = n+1 \) and \( \mu_1(K_n) = n \).
We now turn to the case of bipartite graphs.

3.1.6. **Theorem**: If \( G \) is a bipartite graph then \( \mu_0(G) = q + n \) and \( \mu_1(G) = 2q - p + n \) where \( n \) is the number of vertices of degree 1, \( p \) is the number of vertices of \( G \) and \( q \) is the number of edges of \( G \).

**Proof**: Let \( X = \{ x_1, x_2, \ldots, x_r \} \) and \( Y = \{ y_1, y_2, \ldots, y_s \} \) be the bipartition of \( G \). Let \( k \) and \( l \) denote the number of vertices of degree 1 in \( X \) and \( Y \) respectively. Without loss of generality assume that \( x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l \) are the vertices of degree 1 in \( G \). Let \( \text{deg}(x_i) = d_i \) and \( \text{deg}(y_j) = f_j \). Let \( P_i = \{ x_i \} \cup \{ e : e \text{ is incident with } x_i \}, 1 \leq i \leq k \). Let \( P_i = \{ e : e \text{ is incident with } x_i \}, k+1 \leq i \leq r \). Let \( Q_j = \{ y_j \} \cup \{ e : e \text{ is incident with } y_j \}, 1 \leq j \leq l \) and let \( Q_j = \{ e : e \text{ is incident with } y_j \}, l+1 \leq j \leq s \). In \( P_i \) and \( Q_j \), the first element should be \( x_i \) and \( y_j \) respectively (\( 1 \leq i \leq k \) and \( 1 \leq j \leq l \)). The edges of \( G \) in the sets \( P_i (k+1 \leq i \leq r) \) and \( Q_j (l+1 \leq j \leq s) \) are in any order. Clearly \( P_1, P_2, \ldots, P_r \) are mutually disjoint sets and \( Q_1, Q_2, \ldots, Q_s \) are also mutually disjoint sets. We now construct a graph \( H \) as follows: Let \( V(H) = \{ x_1, x_2, \ldots, x_k, y_1, \ldots, y_l \} \cup E(G) \) and the edge set \( E(H) \) be defined in such a way that \( P_i \) and \( Q_j \) are paths in \( H \) (as in example 3.1.2).

Clearly \( \Psi = \{ P_1, P_2, \ldots, P_r, Q_1, Q_2, \ldots, Q_s \} \) is a path cover of \( H \) and the map \( f : V(G) \to V(\Omega(\Psi)) = \Psi \) defined by \( f(x_i) = P_i (i = 1, 2, \ldots, r) \) and \( f(y_j) = Q_j (j = 1, 2, \ldots, s) \) is an isomorphism and so \( H \in \Theta(G) \).
Now, $|V(H)| = k + l + q = q + n$ and

$$|E(H)| = \sum_{i=k+1}^{r} \text{length}(P_i) + k + \sum_{j=l+1}^{s} \text{length}(Q_j) + l$$

$$= \sum_{i=k+1}^{r} (d_i - 1) + \sum_{i=1}^{k} d_i + \sum_{j=l+1}^{s} (f_j - 1) + \sum_{j=1}^{l} f_j$$

$$= \sum_{i=1}^{r} d_i - (r - k) + \sum_{j=1}^{s} f_j - (s - l)$$

$$= q - r + k + q - s + l$$

$$= 2q - (r + s) + k + l$$

$$= 2q - p + n$$

Therefore $\mu_0(G) \leq q + n$ and $\mu_1(G) \leq 2q - p + n$. Now let $\Psi_1$ be any path cover of a graph $H$ such that $\Omega(\Psi_1) \equiv G$. Then $\Psi_1$ is a collection of $r$ disjoint paths together with $s$ disjoint paths satisfying the following:

If $d_i \geq 2$, then the $i^{th}$ path of the $r$ disjoint paths intersects with $d_i$ paths of the $s$ disjoint paths. If $f_j \geq 2$ then the $j^{th}$ path of the $s$ disjoint paths intersects with $f_j$ paths of the $r$ disjoint paths.

If $d_i = 1$, then the $i^{th}$ path of the $r$ disjoint paths is exactly of length one; if $f_j = 1$, then the $j^{th}$ path of the $s$ disjoint paths is exactly of length one.

Therefore, $\mu_0(G) \geq \sum_{i=1}^{r} d_i + n = q + n$ and

$$\mu_1(G) \geq \sum_{i=1}^{r} (d_i - 1) + \sum_{j=1}^{s} (f_j - 1) + n$$

$$= 2q - p + n$$

Hence $\mu_0(G) = q + n$ and $\mu_1(G) = 2q - p + n$. 

70
3.1.7. Corollary: Let $m,n \geq 2$. Then $\mu_0(K_{m,n}) = mn = q$ and $\mu_1(K_{m,n}) = 2mn - (m+n) = 2q - p$.

3.1.8. Corollary: For a tree $T$, $\mu_0(T) = p + n - 1$ and $\mu_1(T) = p + n - 2$, where $n$ is the number of pendant vertices.

**Proof:** For a tree, $q = p - 1$ and the result follows.

3.2. Edge intersection graph of paths in a graph (EPG graph)

Let $\Psi$ be a collection of non-trivial simple paths in a graph $G$. We define its edge intersection graph $\Gamma(\Psi)$ to be the graph whose vertices are members of $\Psi$ and where two vertices are joined by an edge if and only if the corresponding paths share an edge in $G$. A graph $G$ is called edge intersection graph of paths in a graph $H$ or EPG graph if $G \cong \Gamma(\Psi)$ for some collection $\Psi$ of paths in a graph $H$.

In [25, 26] edge intersection graph of paths in a tree (EPT graph) is studied. In [18], H.J. Broersma and C. Hoede characterize $P_3$ - graphs where they define a $P_3$ - graph as representing some paths $P_3$ by vertices and joining two vertices whenever the corresponding paths form a path $P_4$ or a cycle $C_3$. In this section we study the properties of edge intersection graph of paths in any graph $G$.

In particular we concentrate on the edge intersection graph of the collection of all paths $P_3$. As in 3.1 we introduce parameters $\mu'_0(G)$ and $\mu'_1(G)$ and study the properties of EPG graphs as for path graphs.
We denote a path \( P_n = (u_1, u_2, \ldots, u_n) \) also by \((e_1, e_2, \ldots, e_n)\) where 
\[ e_i = (u_{i-1}, u_i), \quad i = 1, 2, \ldots, n. \]
If \( e = uv \), define \( \text{deg}(e) = \text{deg}(u) + \text{deg}(v) - 2 \).

Analogous to 3.1.4 and 3.1.5 we prove the following results for edge intersection graphs.

3.2.1. **Theorem**: Let \( G \) be a graph. Let \( \theta(G) \) denote the class of all graphs \( H \) such that \( \Gamma(\Psi) \cong G \) for some collection of paths \( \Psi \) of \( H \). Then \( \theta(G) \) is infinite.

**Proof**: In [26], it is shown that, “Every graph is an edge intersection graph of a collection of paths of some graph \( H \)”. So \( \theta(G) \) is not empty. Let \( H \in \theta(G) \).

Let \( \Psi \) be a collection of paths of \( H \) such that \( \Gamma(\Psi) \cong G \). Let \( H^1 \) be the graph obtained from \( H \) by attaching a star \( K_{1,s} \) at a vertex \( u \) of \( H \) if \( u \) is an end vertex of \( s \) paths in \( \Psi \). Now, \( H^1 \) has \( 2|\Psi| \) vertices and \( 2|\Psi| \) edges more than that of \( H \). Let \( \Psi^1 \) be the path cover of \( H^1 \) obtained from \( \Psi \) by adding an edge of \( E(H^1) - E(H) \) at each end of every path of \( \Psi \). Note that \( \Gamma(\Psi^1) \cong \Gamma(\Psi) \cong G \). Hence \( H^1 \in \theta(G) \). It follows that \( H^n \in \theta(G) \) for all \( n \) where \( H^2 = (H^1)^1, \ldots, H^n = (H^{n-1})^1 \). Hence \( \theta(G) \) is infinite. 

As in 3.1 we introduce the following parameters for edge intersection graphs.

\[
\mu'_0(G) = \inf_{H \in \theta(G)} |V(H)|
\]
\[ \mu_1'(G) = \inf_{H \in \Theta(G)} |E(H)|, \]
where \( \Theta(G) \) is the collection of all graphs \( H \) such that \( G \) is the edge intersection graph of some collection of paths of \( H \).

**3.2.2. Theorem:** \( \mu_0'(K_n) = n+1 \) and \( \mu_1'(K_n) = n \).

**Proof:** Let \( H = K_{1,n} \) and \( V(H) = \{a, b_1, b_2, \ldots, b_n\} \), \( E(H) = \{(a,b_i) : 1 \leq i \leq n\} \). Let \( \Psi = \{(a,b_1)\} \cup \{(b_i ab_j) : 2 \leq i < j \leq n\} \) be a collection of paths of \( H \). Then \( \Gamma(\Psi) \cong K_n \).

Hence \( H \in \Theta(K_n) \) so that \( \mu_0'(K_n) \leq n+1 \) and \( \mu_1'(K_n) \leq n \). Now let \( \Psi_1 \) be any collection of paths of a graph \( H \) such that \( \Gamma(\Psi_1) \cong K_n \). Then \( \Psi_1 \) is a collection of \( n \)-paths and any two paths in \( \Psi_1 \) have a common edge. Hence \( |V(H)| \geq n+1 \) and \( |E(H)| \geq n \) so that \( \mu_0'(K_n) = n+1 \) and \( \mu_1'(K_n) = n \).

The following result is useful in studying edge intersection graphs.

**3.2.3. Theorem:** Let \( G \) be a graph having no isolated vertices and let \( \Psi \) be the collection of all paths in \( G \). Then \( G \) is connected if and only if \( \Gamma(\Psi) \) is connected.

**Proof:** Suppose \( G \) is connected. Let \( P, Q \in \Psi \). If \( P \) and \( Q \) have a common edge in \( G \) then \( P \) and \( Q \) are adjacent in \( \Gamma(\Psi) \). If not, choose \( e \in P \) and \( e' \in Q \). Since \( G \) is connected, there exists an \( e - e' \) path, say \( R \), in \( \Psi \). Now, \( PRQ \) is a path in \( \Gamma(\Psi) \) joining \( P \) and \( Q \). Hence \( \Gamma(\Psi) \) is connected. Clearly, if \( G \) is not connected, then \( \Gamma(\Psi) \) is also not connected.
In the following we study edge intersection graph of paths of length two in a graph.

3.2.4. Theorem: Let G be a graph. Let Ψ be the collection of all paths of length two. Then \( \Gamma(\Psi) \) has the following properties.

(i) Number of vertices in \( \Gamma(\Psi) \) is:

\[
|\Psi| = \sum_{\deg(v) \geq 2} \binom{\deg(v)}{2}
\]

(ii) Number of edges in \( \Gamma(\Psi) \) is:

\[
\sum_{e \in E(G), \deg(e) > 1} \binom{\deg(e)}{2}
\]

(iii) If \( P = (e_1, e_2) \in \Psi \), then \( \deg(P) = \deg(e_1) + \deg(e_2) - 2 \)

Proof: (i) Let \( \deg(v) \geq 2 \). Any two edges at \( v \) determine a path in \( \Psi \).

Hence \( |\Psi| = \sum_{\deg(v) > 1} \binom{\deg(v)}{2} \)

(ii) Let \( e \in E(G) \) and \( \deg(e) > 1 \). Number of edges adjacent with \( e \) in \( G \) is \( \deg(e)\deg(u) + \deg(v) - 2 \). It means that there are \( \deg(e) \) paths in \( \Psi \) having \( e \) as one of its two edges. Also any two of these paths determine an edge in \( \Gamma(\Psi) \). Hence (ii) follows.

(iii) The number of paths in \( \Psi \) different from \( P \) and containing \( e_i (i=1,2) \) is \( \deg(e_i) - 1 \) and these paths are adjacent with \( P \) in \( \Gamma(\Psi) \). Hence (iii) follows.

3.2.5. Theorem: Let \( \Psi \) be the collection of all paths of length 2 in a connected graph \( G \). Then \( G \cong \Gamma(\Psi) \) if and only if \( G \) is a cycle.
Proof: If $G$ is a cycle, then clearly $G \cong \Gamma(\Psi)$. Conversely, let $G \cong \Gamma(\Psi)$.

Suppose $\Delta \geq 3$. A $K_3$ in $G$ gives a $K_3$ in $\Gamma(\Psi)$. Since $\Delta \geq 3$, there exists a subgraph $K_{1,3}$ in $G$ which gives a $K_3$ in $\Gamma(\Psi)$ contradicting $G \cong \Gamma(\Psi)$. This gives $\Delta \leq 2$ and so $G$ is either a cycle or a path. If $G = P_n$, then $\Gamma(\Psi) \cong P_{n-1}$. Hence $G$ is a cycle.

3.2.6. Theorem: A graph $G$ is a connected block graph if and only if $G \cong \Gamma(\Psi)$ where $\Psi$ is a collection of paths of length two in a tree $T$ with the following properties:

i) for every internal vertex $u$ of $T$ there is a path in $\Psi$ with $u$ as internal vertex.

ii) paths in $\Psi$ with common internal vertex share a common edge.

Proof: Suppose $G \cong \Gamma(\Psi)$. Let $P, Q \in \Psi$. If $P$ and $Q$ have a common edge in $T$, then $P$ and $Q$ are adjacent in $\Gamma(\Psi)$. If not, choose $e \in P$ and $e' \in Q$. Since $T$ is a tree there exists a unique $e - e'$ path in $G$ say $(e = e_1, e_2, \ldots, e_k = e')$. Let $e_i = (w_i, w_{i-1})$, $i = 1, \ldots, k - 1, k$. Let $P_1$ and $P_2$ be paths in $\Psi$ containing the edges $e_1$ and $e_2$ respectively with a common internal vertex $w_1$. Let $P_2'$ be a path in $\Psi$ containing the edge $e_2$ and $w_2$ be the internal vertex of $P_2'$. Let $P_3$ be a path in $\Psi$ containing the edge $e_3$ and $w_2$ be the internal vertex of $P_3$. Continuing this process we obtain a path $P_k$ in $\Psi$ containing the edge $e_k$ and $w_{k-1}$ is the internal vertex of $P_k$. Consider the collection of paths $P, P_1, P_2, P_2'$.
Let $i_1$ be the least positive integer such that $P_{i_1} \neq P$. Let $i_2 > i_1$ be the least positive integer such that $P_{i_2} \neq P_{i_1}$. Continuing this process we obtain a path $P_{i_n} \neq Q$ such that $P_{i_n}$ and $Q$ have a common edge.

Now $\{P, P_{i_1}, P_{i_2}, \ldots, P_{i_n}, Q\}$ is a path in $\Gamma(\Psi)$ joining $P$ and $Q$. Hence $G$ is connected. For any vertex $v$ in $T$ such that $\deg(v) > 2$, there exist $\deg(v) - 1$ paths in $\Psi$ containing the common edge at $v$, say $e$, and these paths form a complete subgraph of $\Gamma(\Psi)$. Let $e_1$ and $e_2$ be edges in $T$. Let $A_1$ and $A_2$ be the induced subgraphs of $\Gamma(\Psi)$ whose vertices consist of all those two–paths in $\Psi$ that contain the edges $e_1$ and $e_2$ respectively. By definition, $A_1$ and $A_2$ are complete. Since $T$ is a tree, $A_1$ and $A_2$ can share at most one vertex. Thus each block in $\Gamma(\Psi)$ is complete and hence by theorem 1.22, $\Gamma(\Psi)$ is a block graph.

Conversely, suppose $G$ is a connected block graph. We prove the result by induction on the number of blocks of $G$. If $G$ has just one block, then $G$ is the complete graph $K_n$. Hence a collection $\Psi$ of paths of length two have the same edge in common with the star $K_{1,n+1}$. If $V(K_{1,n+1}) = \{v, u_1, u_2, \ldots, u_n, u_{n+1}\}$, then $\Psi = \{(u_1 v, u_{n+1}), (u_2 v, u_{n+1}), \ldots, (u_n v, u_{n+1})\}$ and $G \cong \Gamma(\Psi)$.

We now assume that the result is true for any connected block graph with $k$ blocks where $k \geq 1$. Let $G$ be a connected block graph with $k+1$ blocks. Let $B = K_m$ be a block in $G$ such that $B$ contains exactly one cut vertex $v$ of $G$. 

76
The removal of the vertices of \( B \) other than \( v \) from \( G \) gives a connected block graph \( G_1 \) with \( k \) blocks. By induction hypothesis, there exists a tree \( T_1 \) and a collection \( \Psi_1 \) of paths of length two satisfying (*) in \( T_1 \) such that \( G_1 \cong \Gamma(\Psi) \).

Let \( P_1 \) be a path in \( \Psi_1 \) corresponding to a vertex \( v \in G_1 \). Since \( G_1 \) is connected, at least one end of this path, say \( w \), is a pendant vertex of \( T_1 \). Let \( P_1 = (w \ w' \ w) \). Let \( T \) be the tree obtained by adding the vertices \( u_1, u_2, \ldots, u_{m-1} \) and the edges \( wu_1, wu_2, \ldots, wu_{m-1} \) to \( T_1 \). Let \( P'_i \) denote the new path \( (w \ w' \ u_i) \), \( i = 1, 2, 3, \ldots, m - 1 \). Then \( \Psi = \Psi_1 \cup \{ P'_1, P'_2, \ldots, P'_{m-1} \} \) is a collection of paths of length two in \( T \) and \( G \cong \Gamma(\Psi) \). This completes the induction and the proof.

The following example illustrates the induction process in the above theorem. Given a connected block graph \( G \), we construct a tree \( T \) and a collection of paths \( \Psi \) of length two of \( T \) with the property that for every internal vertex \( u \) of \( T \) there is a path in \( \Psi \) with \( u \) as internal vertex and paths in \( \Psi \) with a common internal vertex share a common edge so that \( G \cong \Gamma(\Psi) \).
The Strong Perfect Graph Conjecture (SPGC) may be stated as follows. \( G \) is perfect if and only if \( G \) contains neither a chordless cycle of length > 5 nor the complement one. The SPGC has been shown to hold for several families of graphs like planar graphs, \( K_4 \)– free graphs, 3–chromatic

Fig. 3
graphs etc. In [26], it is shown that SPGC is satisfied for edge intersection graphs of paths in a tree (EPT graphs). In this direction we have the following result.

3.2.7. Theorem: Let $G \cong \Gamma(\Psi)$ where $\Psi$ is the collection of all paths of length two of a graph $H$. Then SGPC holds for $G$.

Proof: Parthasarathy and Ravindra have shown that SPGC holds for $K_{1,3}$ free graphs. It therefore suffices to show that $G$ is a $K_{1,3}$ – free graph. Assume that $G$ contains an induced copy of $K_{1,3}$. Let $w, x, y, z$ be the vertices of $G$ such that $w$ is the central vertex of $K_{1,3}$. Let $P_w, P_x, P_y$ and $P_z$ be the corresponding paths in $H$. Then $P_x \cap P_w, P_y \cap P_w$ and $P_z \cap P_w$ are not empty. Since $P_w$ is a path of length 2, atleast two of the paths $P_x, P_y$ and $P_z$ (say $P_x$ and $P_y$) have a common edge. It means that $x$ and $y$ are adjacent in $G$. This is a contradiction to our assumption. Hence $G$ is a $K_{1,3}$ – free graph.

3.3. PATH PARTITION GRAPH

In this section we define path partition graphs and study the graphs in some detail.

The following notion is analogous to “Partition graphs” introduced by Duane Detemple et. al. [23].

3.3.1. Definition: A graph $G$ is said to be a path partition graph if there exist a graph $H$ such that $G \cong \Gamma(\Psi)$, where $\Psi$ is a collection of non-trival simple paths of $H$ containing a path cover of $H$. 

79
Let $G$ be an edge intersection graph of a collection of paths of a graph $H$. We call $M \subseteq V(G)$ a partitioning set or a covering set if \{\(P_v \in \Psi : v \in M\}\} partitions $E(H)$. Clearly such a set must be a maximal independent set.

3.3.2. Remark: If no maximal independent set of $G$ is a covering set, then $G$ is not a path partition graph.

We have the following for complete graphs.

3.3.3. Theorem: Every complete graph is a path partition graph.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$. Let $\Psi = \{(e_1, e_2), (e_1, e_2, e_3), \ldots, (e_1, e_2, e_3, \ldots, e_n)\}$ be a collection of paths of $H$ where $H$ is the path $(u_0, u_1, u_2, \ldots, u_n)$ and $e_i = (u_{i-1}, u_i), 1 \leq i \leq n$. $\Psi$ contains a path cover $(e_1, e_2, \ldots, e_n)$ of $H$ and $K_n \equiv \Gamma(\Psi)$ where $v_1$ corresponds to $(e_1, e_2, \ldots, e_i)$.

In the above proof, \{\(v_n\}\} is a covering set and we call $v_n$ a covering vertex of $K_n$.

In connection with Remark 3.3.2 the following result is noteworthy.

3.3.4. Theorem: Let $G$ be a path partition graph and $M$ be a maximal independent set. If $M$ is a covering set then every edge $uv$ in $G - M$ is part of a triangle $uvw$ in $G$ with $w \in M$.

Proof: Suppose $G$ is a path partition graph with $M$ as a covering set. There exists a collection of paths $\Psi$ of a graph $H$ such that $G \equiv \Gamma(\Psi)$ and \{\(P_w : w \in M\}\).
Let \( P \) be a path cover of \( H \). If \( uv \) is an edge of \( G - M \), then \( P_u \cap P_v \neq \emptyset \). Let \( e \in P_u \cap P_v \). If \( e \in P_w \) for some \( w \in M \). Then \( uvw \) is a triangle in \( G \). \( \square \)

3.3.5. **Corollary**: \( C_m \) (\( m > 3 \)) is not a path partition graph when \( m \) is odd.

**Proof**: The result follows from 3.3.2 and 3.3.4. \( \square \)

3.3.6. **Corollary**: \( C_m \times P_n \) (\( m > 3 \) and \( m \) is odd) is not a path partition graph.

**Proof**: The result follows from 3.3.2 and 3.3.4. \( \square \)

Now we turn to the case of bipartite graphs.

3.3.7. **Theorem**: A bipartite graph \( G \) is a path partition graph.

**Proof**: Let \( V(G) = X \cup Y \). Let \( X = \{ u_1, u_2, \ldots, u_m \} \) and \( Y = \{ v_1, v_2, \ldots, v_n \} \).

Let \( S(u_i) = \{ e_{i1}, e_{i1}', e_{i1}'', e_{i2}, e_{i2}', e_{i2}'', \ldots, e_{in}, e_{in}' \} \), \( 1 \leq i \leq m \). Let \( \deg(v_j) = d_j \), \( 1 \leq j \leq n \). Let \( N(v_j) = \{ u_{k_1}, u_{k_2}, \ldots, u_{k_s} \} \), where \( k_1 < k_2 < \ldots < k_s \) and \( s = d_j \).

Let \( S'(v_j) = \{ e_{k_1j}, e_{k_2j}, \ldots, e_{k_sj} \} \). Define \( f_j(k_i) = k_{i+1} \), \( 1 \leq i \leq s-1 \).

Define \( S(v_j) = \{ e_{k_1j}, e_{k_1j}', e_{k_2j}, e_{k_2j}', \ldots, e_{k_sj}, e_{k_sj}' \} \). Construct the graph \( H \) as follows:

Let \( V(H) = \{ w_{ij}, w_{ij}' : 1 \leq i \leq m, 1 \leq j \leq n \} \cup \{ w_{m+1,j} : 1 \leq j \leq n \} \).

For each \( j \) and for \( 1 \leq i \leq m \), define

\[
  w_{ij}'' = \begin{cases} 
    w_{f_j(k_i), j} & \text{if } i = k_r \text{ and } r < d_j \\
    w_{i+1,j} & \text{otherwise}
  \end{cases}
\]

81
E(H) = \{e_{ij}, e_{ij}': 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{e_{ij}'' : 1 \leq i \leq m, 1 \leq j \leq n-1\}

where \(e_{ij} = (w_{ij}, w_{ij}')\), \(e_{ij}' = (w_{ij}, w_{ij}'')\) and \(e_{ij}'' = (w_{ij}, w_{i,j+1})\). Clearly \(\Psi\) is a collection of paths \(\{S(u_1), S(u_2), \ldots, S(u_m), S(v_1), S(v_2), \ldots, S(v_n)\}\) such that \(\Gamma(\Psi) \cong G\) and \(\{S(u_1), S(u_2), \ldots, S(u_m)\}\) is a path cover for \(H\).

Hence the following bipartite graphs are path partition graphs: a tree, a cube, \(P_m \times P_n\), \(C_m \times P_n\) where \(m\) is even and \(m > 3\).

The following example illustrates the above construction.

**3.3.8. Example**

Consider a bipartite graph \(G\):

For \(1 \leq i \leq 4\), we define

\[S(u_i) = (e_{i1}, e_{i1}', e_{i1}'', e_{i2}, e_{i2}', e_{i2}'', e_{i3}, e_{i3}', e_{i3}'', e_{i4}, e_{i4}', e_{i4}'', e_{i5}, e_{i5}')\]

Now \(N(v_1) = \{u_1, u_2, u_4\}\), \(N(v_2) = \{u_2, u_3\}\)
Let $S'(v_1) = \{ e_{11}, e_{21}, e_{41} \}$

$S'(v_2) = \{ e_{22}, e_{32} \}$

$S'(v_3) = \{ e_{13}, e_{33}, e_{43} \}$

$S'(v_4) = \{ e_{14}, e_{24} \}$ and

$S'(v_5) = \{ e_{35}, e_{45} \}$

Now we define

$S(v_1) = \{ e_{11}, e_{11}', e_{21}, e_{21}', e_{41}, e_{41}' \}$

$S(v_2) = \{ e_{22}, e_{32}, e_{32}' \}$

$S(v_3) = \{ e_{13}, e_{13}', e_{33}, e_{33}', e_{43}, e_{43}' \}$

$S(v_4) = \{ e_{14}, e_{24}, e_{24}' \}$

$S(v_5) = \{ e_{35}, e_{35}', e_{45}, e_{45}' \}$

We construct $H$ as follows:

Let $V(H) = \{ w_{i,j}, w_{i,j}' : 1 \leq i \leq 4, 1 \leq j \leq 5 \} \cup \{ w_{5,j} : 1 \leq j \leq 5 \}$. Define $f_1(1)=2, f_1(2)=4, f_2(2)=3, f_3(1)=3, f_3(3)=4, f_4(1)=2$ and $f_5(3)=4$. Then we have $w_{1,1}'' = w_{2,1}'$, $w_{2,1}' = w_{4,1}, w_{2,2}' = w_{3,2}, w_{1,3}'' = w_{3,3}, w_{3,3}'' = w_{4,3}, w_{1,4}'' = w_{2,4}$, and $w_{3,5}'' = w_{4,5}$. For other values of $i$ and $j$, $w_{ij}'' = w_{i+1,j}'$ for $1 \leq i \leq 4, 1 \leq j \leq 5$. Let $E(H) = \{ e_{ij}, e_{ij}' : 1 \leq i \leq 4, 1 \leq j \leq 5 \} \cup \{ e_{ij}'' : 1 \leq i \leq 4, 1 \leq j \leq 4 \}$, where $e_{ij} = (w_{ij}, w_{ij}')$, $e_{ij}' = (w_{ij}', w_{ij}'')$ and $e_{ij}'' = (w_{ij}'', w_{i,j+1})$. 

83
Clearly $\Psi = \{S(u_1), S(u_2), S(u_3), S(u_4), S(v_1), S(v_2), S(v_3), S(v_4)\}$ is a collection of paths in $H$ such that $\Gamma(\Psi) \cong G$ and $\{S(u_1), S(u_2), S(u_3), S(u_4)\} \subset \Psi$ is a path cover for $H$. The isomorphism is given by the map $f: V(G) \to V(\Gamma(\Psi)) = \Psi$ where $f(x) = S(x)$. 