CHAPTER II

DECOMPOSITION OF GRAPHS INTO INTERNALLY DISJOINT TREES

2.1. INTRODUCTION

In [21] F.R.K. Chung defines the parameter \( \gamma(G) \) of a graph \( G \) as the minimum number of trees covering all the edges of \( G \). In [24] Foregger, M.F. and Foregger, T.H. define a parameter \( \gamma'(G) \) as the minimum number of subsets into which the vertex set of \( G \) can be partitioned so that each subset induces a tree. Here we define a graphoidal tree cover of \( G \) as a partition of edges into internally vertex disjoint trees and \( \gamma'_T(G) \) as the minimum number of internally vertex disjoint trees covering all the edges of \( G \).

In this chapter we find \( \gamma'_T(G) \) for some standard graphs in section 2.2. In section 2.3 we study some relationship between \( \gamma'_T(G) \) and \( \gamma'(G) \). In [1] Acharya and Sampathkumar defined a graphoidal cover as a partition of edges into internally disjoint (not necessarily open) paths. We consider only open paths and call a partition of edges into internally disjoint paths as a graphoidal path cover (acyclic graphoidal cover in[8]). We define a graphoidal tree \( d \)-cover as a partition of edges into internally disjoint trees in which each tree has a maximum degree bounded by \( d \). The minimum cardinality of such \( d \)-covers is denoted by \( \gamma'_T^{(d)}(G) \). Clearly a graphoidal tree 2-cover is a graphoidal path cover. We find \( \gamma'_T^{(d)}(G) \) for some standard graphs.
2.2. GRAPHOIDAL TREE COVER

2.2.1. Definition – A graphoidal tree cover \( \mathcal{G} \) of \( G \) is a collection of non-trivial trees in \( G \) such that

(i) Every vertex is an internal vertex of at most one tree.

(ii) Every edge is in exactly one tree.

Let \( G \) denote the set of all graphoidal tree covers of \( G \). Since \( E(G) \) is a graphoidal tree cover, we have \( G \neq \emptyset \).

Let \( \gamma_r(G) = \min_{\mathcal{G}} |\mathcal{G}| \).

Then \( \gamma_r(G) \) is called the graphoidal tree covering number of \( G \). Any graphoidal tree cover \( \mathcal{G} \) of \( G \) for which \( |\mathcal{G}| = \gamma_r(G) \) is called a minimum graphoidal tree cover.

2.2.2. Example : Consider a graph \( G \) given in the figure.

![Graph Diagram](image)

Let \( T_1, T_2 \) and \( T_3 \) be the trees given below:
Clearly $\mathcal{S} = \{T_1, T_2, T_3\}$ is a graphoidal tree cover of $G$. Moreover it is a minimum graphoidal tree cover. Hence $\gamma_T(G)=3$.

2.2.3. Remark: If $\delta(G) > \gamma_T$, then every vertex is an internal vertex of some tree in a minimum graphoidal tree cover.

2.2.4. Observation: If $G$ is a $(p,q)$ graph then $\gamma_T(G) \geq \left\lceil \frac{q}{p-1} \right\rceil$.

2.2.5. From the above observation we have the following:

If $\delta(G) >0$, then $\gamma_T(G) > \frac{\delta}{2}$.

We first determine the graphoidal tree covering number of a complete graph.

2.2.6. Theorem [43]: $\gamma_T(K_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof: From 2.2.5, it follows that $\gamma_T(K_n) \geq \left\lceil \frac{n}{2} \right\rceil$. We give a construction for the reverse inclusion. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ First, let $n$ be even, say $n=2k$. 

For \( i=1,2,\ldots, k \), let \( T_i \) be the tree shown in fig. 3. In \( T_i \), \( v_i \) is adjacent to \( v_{i+1} \), \( \ldots, v_{i+k-1}, v_{i+k} \) and \( v_{i+k} \) is adjacent to \( v_{i+k+1}, \ldots, v_{i+2k-1} \) (subscripts modulo \( n \)).

Clearly, \( \mathcal{Z} = \{T_1, T_2, \ldots, T_k\} \) is a graphoidal tree cover for \( K_n \). Therefore, \( \gamma_T(K_n) = \left\lceil \frac{n}{2} \right\rceil \) if \( n \) is even. (See fig. 4 for the case \( n=8 \)). Here we note that each \( T_i \) in the above graphoidal tree cover \( \mathcal{Z} \) is a spanning tree.

![Fig 3](image)

For \( n \) odd, say \( 2k+1 \), we have \( \left\lceil \frac{2k + 2}{2} \right\rceil = k+1 \) spanning trees which form a graphoidal tree cover for \( K_{n+1} \). Fix a vertex and delete it from each tree in

![Fig 4](image)
the above tree decomposition for $K_{n+1}$. Leaving out the isolated vertices, we get a graphoidal tree cover for $K_n$ with $k+1=\left\lfloor \frac{2k + 1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$ trees. Hence $\gamma_\tau(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$ if $n$ is odd.

Next we consider two relatively easy cases for which $\gamma_\tau(G) = 2$.

2.2.7. Theorem: For a unicyclic graph $G$, $\gamma_\tau(G) = 2$.

Proof: Since $G$ contains a cycle, $\gamma_\tau(G) \geq 2$. Also $\Psi = \{G-e, e\}$ is a graphoidal tree cover for $G$, where $e$ is an edge on the cycle of $G$. Hence $\gamma_\tau(G) = 2$.

2.2.8. Theorem: For the wheel $W_n (n \geq 4)$, $\gamma_\tau(W_n) = 2$.

Proof: The graph $W_n (n \geq 4)$ is shown in fig. 5.

Now $\{T_1, T_2\}$ is a graphoidal tree cover for $W_n$, where $T_1 = \{(x_0, x_i) : 1 \leq i \leq n-2\} \cup \{(x_{n-2}, x_{n-1})\}$ and $T_2 = (x_0, x_{n-1}, x_1, x_2, \ldots, x_{n-2})$. Since $\gamma_\tau(W_n) \geq 2$, it follows that $\gamma_\tau(W_n) = 2$. •
We now turn to the case of complete bipartite graphs, beginning with a general result on the diameter of trees in a minimum graphoidal tree cover. The following standard notation is used for the partite sets of \( K_{m,n} \) with \( m \leq n \): \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \).

2.2.9. Lemma: If a minimum graphoidal tree cover \( \mathcal{T} \) of \( K_{m,n} \) contains a tree with a path of length \( \geq 5 \), then it also contains a tree with exactly one edge.

Proof: Let \( T \in \mathcal{T} \) contain a path \( P = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots) \) where \( x_1 \in X \) and \( y_j \in Y \). Since \( y_1 \) and \( x_3 \) are internal in \( T \), these cannot be internal in any other member of \( \mathcal{T} \). Therefore \( T_1 = \{(y_1, x_3)\} \in \mathcal{T} \). Hence the lemma.

2.2.10. Lemma: If \( m \leq n \leq 2m-3 \), \( \gamma_T(K_{m,n}) \geq \left\lceil \frac{m+n}{3} \right\rceil \).

Proof: Suppose \( \gamma_T(K_{m,n}) = r \) with \( r < \left\lceil \frac{m+n}{3} \right\rceil \). Let \( \mathcal{T} \) be a minimum graphoidal tree cover of \( K_{m,n} \). Since \( m > \frac{m+n}{3} > r \) (as \( n \leq 2m-3 \)), by 2.2.3, we have every vertex is an internal vertex of a tree in \( \mathcal{T} \).

Claim 1: No tree in \( \mathcal{T} \) can have more than two internal vertices from \( X \) with a common neighbour from \( Y \).

Suppose \( x_1, x_2, \ldots, x_k \) (\( k \geq 3 \)) are all adjacent to \( y_1 \) in \( T_1 \) of \( \mathcal{T} \). Then the sum of degrees of \( x_1, x_2, \ldots, x_k \) in \( T_1 \) is at most \( n+k-1 \). But each \( x_i \) (\( i=1,2,\ldots, k \))
is an end vertex in at most $r-1$ other members of $I$. So they have at most $n+k-1+k(r-1)$ total adjacencies in $I$. Since $r < \frac{m+n}{3}$, $n+k-1 + k(r-1) < \frac{3n+3k-3+k(m+n)-3k}{3} < \frac{n(2k+3)}{3} - 1$ ($m \leq n$)

\[= nk - \left(\frac{n(k-3)}{3} + 1\right) < nk (k \geq 3),\]

a contradiction. Hence we have claim 1.

**Claim 2**: There exists a minimum graphoidal tree cover $I'$ such that no tree in $I'$ has a path of length $\geq 5$.

Suppose $T_1 \in I$ has a path $(x_1, y_1, x_2, y_2, x_3, y_3, ...)$). Then by the previous lemma, a tree $T_2$ in $I$ has just the single edge $y_1x_3$. Let $T_1'$ be the tree containing $x_2$ obtained by removing the edge $y_1x_2$ from $T_1$. Let $T_2'$ be the tree $(T_1-T_1') \cup T_2$. Let $I'$ be the graphoidal tree cover obtained from $I$ after replacing $T_1,T_2$ by $T_1', T_2'$ respectively. If a tree in $I'$ again contains a path of length $\geq 5$ we repeat the above process and so on. Finally we get the required minimum graphoidal tree cover $I'$. Hence claim 2.

Now we can assume that no tree in $I$ has a path of length $\geq 5$.

**Claim 3**: No tree in $I$ can have more than two internal vertices from $Y$ with a common neighbour from $X$. 

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Suppose there is a tree \( T_1 \) in \( \mathcal{I} \) containing \( k \) internal vertices \( y_1, y_2, \ldots, y_k \) \((k \geq 3)\) with a common neighbour \( x_1 \). Since \( m > r_1 \) and every vertex is an internal vertex of a tree in \( \mathcal{I} \), there is a tree in \( \mathcal{I} \), say, \( T_2 \) containing at least two vertices from \( X \) as internal vertices. By claim 2, the internal vertices of a tree in \( \mathcal{I} \) form a star and so the internal vertices from \( X \) in \( T_2 \) have a common neighbour from \( Y \). By claim 1, \( T_2 \) has exactly two internal vertices \( x_2 \) and \( x_3 \) from \( X \) with a common neighbour \( y \) from \( Y \). Between \( x_2, x_3 \) and \( y_1, y_2, \ldots, y_k \) there are \( 2k \) edges in \( K_{m,n} \). Clearly \( x_2 \) and \( x_3 \) can be made adjacent with two \( y \)'s in \( T_1 \). Let it be \( y_1 \) and \( y_2 \). Now \( y_1, y_2, \ldots, y_k \) can be made adjacent with \( x \)'s in \( T_2 \). But it will cover exactly \( k+2 \) edges (out of \( 2k \) edges) and so by the definition of graphoidal tree cover, each uncovered edge is a tree in \( \mathcal{I} \). Without loss of generality let \( T_3, T_4, \ldots, T_k \) be the trees with edges \(( y_3, x_1 \), \(( y_4, x_1 \), \ldots, \(( y_k, x_3 \) respectively where \( l_i \in \{2,3\}, 3 \leq i \leq k \). By claim 2 the internal vertices of \( T_1 \) form a star. Let \( T'_1 \) be the tree obtained from \( T_1 \) by removing all the edges incident with \( y_i \) \((3 \leq i \leq k)\). Let \( T'_1 \) be the tree formed by the remaining edges of \( T_1 \) after the removal. Now each \( T_i \) in \( \mathcal{I} \) is replaced by \( T'_i \cup T_i \) for \( 3 \leq i \leq k \). Also replace \( T_1 \) in \( \mathcal{I} \) by \( T'_1 \). If \( \mathcal{I} \) again contains a tree having more than two internal vertices from \( Y \) with a common neighbour from \( X \). We repeat the process and so on. Hence we have claim 3.
From Claim 1, Claim 2 and Claim 3, it follows that no tree in \( \mathfrak{T} \) has more than three internal vertices. Since every vertex of \( K_{m,n} \) must be an internal vertex of a tree in \( \mathfrak{T} \) and \( \gamma_T(K_{m,n}) = r \), we have only \( 3r \) \((<m+n)\) internal vertices in \( \mathfrak{T} \). This is a contradiction. Hence \( \gamma_T(K_{m,n}) \geq \left\lceil \frac{m+n}{3} \right\rceil \).

2.2.11. Theorem: If \( m \leq n \leq 2m-3 \), then \( \gamma_T(K_{m,n}) = \left\lceil \frac{m+n}{3} \right\rceil \). Furthermore, if \( n > 2m-3 \), then \( \gamma_T(K_{m,n}) = m \).

Proof: By 2.2.10, \( \gamma_T(K_{m,n}) \geq \left\lceil \frac{m+n}{3} \right\rceil \). Next we proceed to prove \( \gamma_T(K_{m,n}) \leq \left\lceil \frac{m+n}{3} \right\rceil \) where \( 3 \leq m \leq n \leq 2m-3 \).

Let the partite set of \( K_{m,n} \) be \( V_1 \) and \( V_2 \) where \( V_1 = \{x_1, x_2, ..., x_m\} \) and \( V_2 = \{y_1, y_2, ..., y_n\} \).

Let \( r = \left\lfloor \frac{2m-n}{3} \right\rfloor = \frac{2m-n-k}{3} \) where \( k \) is 0, 1 or 2

Define for \( 1 \leq i \leq r \)

\[
P_i = \{(x_i, y_i, x_{r+i})\} \cup \{(y_i, x_j) : j \neq i, r+i; 1 \leq j \leq m - k\}
\]

\[
\cup \{(x_i, y_j) : r < j \leq m - r - k\} \cup \{(x_{r+i}, y_j) : m - r - k < j \leq n - k\}
\]

For \( 1 \leq i \leq m - 2r - k \) we define

\[
P_{i+r} = \{(y_{r+i}, x_{2r+i}, y_{m-r-k+i})\} \cup \{(x_{2r+i}, y_j) : j \neq r+i, m - r - k+i, r < j \leq n - k\}
\]

\[
\cup \{(y_{r+i}, x_j) : r+1 \leq j \leq 2r\} \cup \{(y_{m-r-k+i}, x_j) : 1 \leq j \leq r\}
\]
When \( k = 1 \)

\[
P_{m-r} = \{(x_m, y_n)\} \cup \{(x_m, y_j) : 1 \leq j \leq n-1\} \cup \{(y_n, x_j) : 1 \leq j \leq m-1\}
\]

When \( k = 2 \)

\[
P_{m-r-1} = \{(x_{m-1}, y_{n-1})\} \cup \{(x_{m-1}, y_j) : 1 \leq j \leq n-2\} \cup \{(y_{n-1}, x_j) : 1 \leq j \leq m-2\}
\]

\[
P_{m-r} = \{(x_m, y_n)\} \cup \{(x_m, y_j) : 1 \leq j \leq n-1\} \cup \{(y_n, x_j) : 1 \leq j \leq m-1\}
\]

Clearly \( \mathcal{I} = \{P_1, P_2, \ldots, P_{m-r}\} \) is a graphoidal tree cover for \( K_{m,n} \).

Therefore \( \gamma_T(K_{m,n}) \leq m - r = \frac{m + n + k}{3} = \left\lfloor \frac{m + n}{3} \right\rfloor \)

Let \( n \geq 2m - 2 \). Let \( V(K_{m,n}) = X \cup Y \) where \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \). Let \( n = 2m - 2 + k, k \geq 0 \). Suppose that \( \gamma_T(K_{m,n}) \neq m \). Then there exists a graphoidal tree cover \( \mathcal{I} \) with at most \( m-1 \) trees. Since \( \delta > m-1 \), it follows that every vertex is an internal vertex of a tree in \( \mathcal{I} \). If \( x_i \) is an internal vertex of a tree \( T \) in \( \mathcal{I} \) then \( \deg_T(x_i) \geq 2m - 2 + k - (m-2) = m+k \). This implies that in any minimum graphoidal tree cover exactly one vertex of \( X \) should be internal in a tree. But there are \( m \) vertices and \( |\mathcal{I}| \leq m - 1 \). This leads to a contradiction. Hence \( \gamma_T(K_{m,n}) \geq m \). Clearly, \( \gamma_T(K_{m,n}) \leq m \) and so \( \gamma_T(K_{m,n}) = m \).

The following examples illustrate \( \gamma_T(K_{m,n}) = \left\lfloor \frac{m + n}{3} \right\rfloor \).
2.2.12. Examples

(i) Consider \( K_{4,5} \). Clearly \( k=0 \) and \( r=1 \).

(ii) Consider \( K_{8,9} \). Clearly \( r = 2 \) and \( k = 1 \).
(iii) Consider $K_{12,13}$. Clearly $r = 3$ and $k = 2$. 

Fig. 8
Our first result on the graphoidal tree covering number of graph products is for grid graphs.

2.2.13. Theorem: For $m,n \geq 2$, $\gamma_T(\mathbf{P}_m \times \mathbf{P}_n) = 2$

Proof: Let $V(\mathbf{P}_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(\mathbf{P}_n) = \{v_1, v_2, \ldots, v_n\}$

Define $w_{ij} = \{(u_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$

Case (i) when $n$ is odd.

Let $n>3$. (For $n=3$ see Fig.9)

$T_1 = \{(w_{ij}, w_{ij+1}) : 1 \leq j \leq n-1\} \cup \{(w_{ij}, w_{i+1,j}) : 1 \leq i \leq m-1, j \equiv 1 \pmod{2}, 1 \leq j \leq n\}$

$\cup \{(w_{ij}, w_{ij+1}) : 1 \leq j \leq n - 2, j \equiv 1 \pmod{2}, 2 \leq i \leq m-1\} \cup \{(w_{m1}, w_{m2})\}$

$\cup \{(w_{mn-1}, w_{mn})\}$

$T_2 = \{(w_{mj}, w_{mj+1}) : 2 \leq j \leq n - 2\} \cup \{(w_{ij}, w_{i+1,j}) : 1 \leq i \leq m - 1, j \equiv 0 \pmod{2}, 2 \leq j \leq n - 1\} \cup \{(w_{mj}, w_{mj+1}) : 2 \leq j \leq n - 1, j \equiv 0 \pmod{2}, 2 \leq i \leq m - 1\}$

Case (ii) when $n$ is even

$T_1 = \{(w_{ij}, w_{i+1,j}) : 1 \leq j \leq n - 2\} \cup \{(w_{ij}, w_{i+1,j}) : 1 \leq i \leq m-1, j \equiv 1 \pmod{2}, 1 \leq j \leq n-1, j \equiv 1 \pmod{2}, 2 \leq i \leq m-1\} \cup \{(w_{m1}, w_{m2})\}$

$T_2 = \{(w_{mj}, w_{mj+1}) : 2 \leq j \leq n - 1\} \cup \{(w_{ij}, w_{i+1,j}) : 1 \leq i \leq m - 1, j \equiv 0 \pmod{2}, 2 \leq j \leq n - 2, j \equiv 0 \pmod{2}, 2 \leq i \leq m - 1\} \cup \{(w_{1,n-1}, w_{1,n})\}$

Now $\mathcal{Z} = \{T_1, T_2\}$ is a graphoidal tree cover for $\mathbf{P}_m \times \mathbf{P}_n$ and so $\gamma_T(\mathbf{P}_m \times \mathbf{P}_n) = 2$. 

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Fig. 9

$P_2 \times P_3$

$P_m \times P_3, m > 2$

Fig. 10

$P_2 \times P_{10}$

Fig. 11

$P_4 \times P_4$
In Fig. 10 and Fig. 11 thick lines form the tree $T_1$ and dotted lines form the tree $T_2$ for the two cases above.

2.2.14. Remark

In the above theorem the maximum degree for each tree in the graphoidal tree cover constructed is 3 and also every vertex of $P_m \times P_n$ (except when $m=2$ and $n=3$) is either internal in $T_1$ or internal in $T_2$.

2.2.15. Theorem $\gamma_t(P_n \times C_m) = 2$ for $m \geq 3$, $n \geq 2$

Proof: Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(C_m) = \{v_1, v_2, \ldots, v_m\}$

Define $w_{ij} = \{(u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$

Case (i) when $m$ is even

Let $T_1 = \{(w_{ij}, w_{i+1,j}) : 1 \leq i \leq n-1, j \equiv 1 \pmod{2}, 1 \leq j \leq m-1\}$

$\cup \{(w_{ij}, w_{i+1,j}) : j \equiv 1 \pmod{2}, 1 \leq j \leq m-1, 1 \leq i \leq n-1\}$

$\cup \{(w_{1,j}, w_{1,j+1}) : j \equiv 0 \pmod{2}, 1 \leq j \leq m-2\} \cup \{(w_{ij}, w_{i+1,j}) : j \equiv 0 \pmod{2}, 1 \leq j \leq m-2\}$

Let $T_2 = \{(w_{i,j}, w_{i,j+1}) : j \equiv 0 \pmod{2}, 1 \leq j \leq m-2, 2 \leq i \leq n\}$

$\cup \{(w_{i,j}, w_{i+1,j}) : 1 \leq i \leq n-1, j \equiv 0 \pmod{2}, 1 \leq j \leq m\}$

$\cup \{(w_{n,j}, w_{n,j+1}) : j \equiv 1 \pmod{2}, 3 \leq j \leq m-1\}$

$\cup \{(w_{i,m}, w_{i,1}) : 1 \leq i \leq n\}$

Case (ii) when $m$ is odd

Let $T_1 = \{(w_{i,j}, w_{i,j+1}) : 1 \leq j \leq m-1, i \equiv 1 \pmod{2}, 1 \leq i \leq n\}$

$\cup \{(w_{i,j}, w_{i+1,j}) : i \equiv 1 \pmod{2}, 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$

$\cup \{(w_{i,1}, w_{i+1,1}) : i \equiv 0 \pmod{2}, 1 \leq i \leq n-1\}$
Let $T_2 = \{(w_{i,j}, w_{i,j+1}) : 1 \leq j \leq m - 1, i \equiv 0 \pmod{2}, 1 \leq i \leq n\}$

$\cup \{(w_{i,j}, w_{i+1,j}) : i \equiv 0 \pmod{2}, 2 \leq i \leq n - 1, 2 \leq j \leq m\}$

$\cup \{(w_{i,m}, w_{i+1,m}) : 1 \leq i \leq n - 1, i \equiv 1 \pmod{2}\}$

$\cup \{(w_{i,m}, w_{i+1}) : 1 \leq i \leq n, i \equiv 1 \pmod{2}\}$

In cases (i) and (ii), $\mathcal{G} = \{T_1, T_2\}$ is a graphoidal tree cover for $P_n \times C_m$. Hence $\gamma_T(P_n \times C_m) = 2$.

Here again, in Fig. 12 and Fig. 13 thick lines form the tree $T_1$ and dotted lines form the tree $T_2$ for the two cases above.

2.2.16. Remark: In the above theorem, the maximum degree of each tree in the graphoidal tree cover constructed is 3 and also every vertex of $P_n \times C_m$ ($m \geq 3$) is internal in $T_1$ or $T_2$. 

![Diagram of $P_n \times C_m$ with thick and dotted lines forming the trees $T_1$ and $T_2$.]
2.2.17. Theorem: $\gamma_T(C_m \times C_n) = 3$, $m,n \geq 3$.

Proof: Since $C_m \times C_n$ is 4-regular, by 2.2.5 it follows that $\gamma_T(C_m \times C_n) > 2$.

Let $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$.

Define $w_{i,j} = \{(u_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. Now consider the minimum graphoidal tree cover $\mathcal{T} = \{T_1, T_2\}$ of $P_{m-1} \times C_n$ as in 2.2.15.

Case (i) When $n$ is even

To the tree $T_2$, add vertices $w_{m,2}$ and $w_{m,4}$. Then add a vertex $w_{m,3}$ adjacent to $w_{1,3}$, $w_{m-1,3}$, $w_{m,2}$ and $w_{m,4}$. The only additional internal vertex this
creates is $w_{m,3}$. Now take a third tree as $T_3 = \{(w_{m,i-1}, w_{m,i}) : 2 \leq i \leq n ; i \neq 3,4\}$
$\cup \{(w_{m,1}, w_{m,n})\} \cup \{ (w_{m,i}, w_{1,i} , (w_{m,i}, w_{m-1,i}) : 1 \leq i \leq n ; i \neq 3\}.

Case (i) When $n$ is odd

To the tree $T_2$, add vertices $w_{m,1}$ and $w_{m,3}$. Then add a vertex $w_{m,2}$ adjacent to $w_{1,2}, w_{m-1,2}, w_{m,1}, w_{m,3}$. The only additional internal vertex this creates is $w_{m,2}$. Now take a third tree as $T_3 = \{(w_{m,i-1}, w_{m,i}) : 4 \leq i \leq n\} \cup \{(w_{m,1}, w_{m,n})\} \cup \{ (w_{m,i}, w_{1,i} , (w_{m,i}, w_{m-1,i}) : 1 \leq i \leq n ; i \neq 2\}.$

2.2.18. Theorem: For any non-trivial tree $T$, $\gamma_T(T \times K_2) = 2$.

Proof: Let $T_1$ be one copy of $T$ and let $T_2$ be the tree got from $T \times K_2$ by removing the edges of $T_1$. $\mathcal{E} = \{T_1, T_2\}$ is clearly a graphoidal tree cover for $T \times K_2$ and hence $\gamma_T(T \times K_2) = 2$.

We recall that $\pi(G)$ is the minimum number of subsets into which $E(G)$ can be partitioned so that the graph formed by each of the subsets is a tree. $\gamma(G)$ is the minimum number of subsets into which $E(G)$ can be partitioned so that the graph formed by each of the subsets is a forest. Clearly $\gamma(G) \leq \pi(G) \leq \gamma_T(G)$.

2.2.19. Theorem: Let $G$ be a graph having $n$ vertices, $e$ edges and $k$ components, $k \leq n$. Then $k - 1 + \left\lceil \frac{e}{n - k} \right\rceil \leq \gamma_T(G)$.

Proof follows from 1.26.

From 1.33, 1.34 and 1.35 we get the following.
2.2.20. Theorem

1. \( \gamma_T(G) \geq \left\lceil \frac{n+1}{2} \right\rceil \) for an \( n \)-regular graph \( G \)

2. \( \gamma_T(L(K_n)) \geq n - 1 \)

3. \( \gamma_T(L(K_{m,n})) \geq \left\lceil \frac{m+n-1}{2} \right\rceil \)

4. \( \gamma(K_n) = \tau(K_n) = \gamma_T(K_n) = \left\lceil \frac{n}{2} \right\rceil \)

From the definition of graphoidal tree cover, it is also observed that \( \tau(G) = \gamma_T(G) \) for all graphs with \( \Delta \leq 3 \). so we have the following results from 1.27, 1.28 and 1.30.

2.2.21. Theorem: If \( G \) is a 2-connected cubic graph, \( p \geq 8 \), then \( \gamma_T(G) \leq \left\lfloor \frac{p}{4} \right\rfloor \).

2.2.22. Theorem: If \( G \) is a 3-connected cubic graph, \( p \geq 12 \), then \( \gamma_T(G) \leq \left\lfloor \frac{p}{6} \right\rfloor \).

2.2.23. Theorem: If \( G \) is a cyclically 4-edge connected cubic graph with \( p \) vertices, \( 8 \leq p \leq 16 \) then \( \gamma_T(G) = 2 \).

2.2.24. Theorem: If a graph \( G \) has a perfect matching, then \( \gamma_T(G) \leq \left\lfloor \frac{p}{2} \right\rfloor \).

Proof: Let \( M = \{e_1, e_2, \ldots, e_k\} \) be a perfect matching of \( G \). Clearly \( p = 2k \).

Without loss of generality assume that \( e_i = (v_{i}, v_{k+i}) \), \( 1 \leq i \leq k \). Let \( K_p \) be the complete graph with the vertex set \( \{v_1, v_2, \ldots, v_p\} \). \( G \) is a subgraph of \( K_p \). Let \( T_i, 1 \leq i \leq k \), be the trees as in Fig. 3. In \( T_i \), \( v_i \) is adjacent to \( v_{i+1}, v_{i+2}, \ldots, v_{i+k-1} \), \( v_{i+k} \) and \( v_{i+k} \) is adjacent to \( v_{i+k+1}, v_{i+k+2}, \ldots, v_{i+2k-1} \) (subscripts modulo \( n \)).

Clearly \( \mathcal{F} = \{T_1, T_2, \ldots, T_k\} \) is a graphoidal tree cover for \( K_p \). Remove the edges
of $E(K_p) - E(G)$ from each $T_i$ ($1 \leq i \leq k$) of $\mathcal{I}$ and the resulting isolated vertices to get a graphoidal tree cover for $G$. This cover still contains $k$ trees since $e_i \in E(G), 1 \leq i \leq k$. Hence $\gamma_T(G) \leq k = \frac{p}{2}$. 

2.2.25. Corollary: If $G$ is a hamiltonian graph, then $\gamma_T(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Proof: Case (i): Let $|V(G)| = 2k$.

Clearly $G$ has a perfect matching. By the above theorem, $\gamma_T(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Case (ii): Let $|V(G)| = 2k+1$.

Let $v \in V(G)$. $|V(G-v)| = 2k$. Let $C$ be a hamiltonian cycle of $G$. $C - v$ is a hamiltonian path on $2k$ vertices. So we can find a perfect matching for $G - v$.

By the above theorem we can find a graphoidal tree cover $\mathcal{I} = \{T_1, T_2, ..., T_k\}$ for $G - v$. Clearly, $T_{k+1} = G - \bigcup_{i=1}^{k} T_i$ is a star at $v$. Then $\{T_1, T_2, ..., T_k, T_{k+1}\}$ is a graphoidal tree cover for $G$. Hence $\gamma_T(G) \leq k+1 = \left\lfloor \frac{p}{2} \right\rfloor$.

We note down the following corollary to be used later.

2.2.26. Corollary: If for all points $v$ of $G$, $\deg(v) \geq \frac{p}{2}$ where $p \geq 3$, then

$$\gamma_T(G) \leq \left\lfloor \frac{p}{2} \right\rfloor.$$ 

Proof: By theorem 1.36, $G$ is hamiltonian and hence the result follows.
We believe that the following is true

**Conjecture:** \( \gamma_T(G) \leq \left\lfloor \frac{p}{2} \right\rfloor \).

### 2.3. RELATIONSHIP BETWEEN \( \tau'(G) \) AND \( \gamma_T(G) \)

In [24] Foregger, M F and Foregger, T.H defined \( \tau'(G) \) as the minimum number of subsets into which \( V(G) \) can be partitioned so that each subset induces a tree. In this section we try to find some relationship between \( \tau'(G) \) and \( \gamma_T(G) \).

#### 2.3.1. Theorem:
Let \( p \geq 4 \) and let \( \mathcal{S} = \{T_1, T_2, T_3, \ldots, T_n\} \) be a minimum graphoidal tree cover of a graph \( G \) with \( |E(T_j)| = 1 \) for some \( j \) and \( |E(T_i)| > 1 \) for all \( i \neq j \). Then we can always find a minimum graphoidal tree cover \( \mathcal{S}' = \{T_1', T_2', \ldots, T_n'\} \) with \( |E(T_i')| > 1 \) for all \( i \).

**Proof:** Let \( T_j = \{xy\} \)

**Case (i):** Suppose at least one of the vertices \( x \) and \( y \), say \( x \), is internal in a tree of \( \mathcal{S} \).

First assume that \( x \) is internal in a tree \( T_i \) of \( \mathcal{S} \). If \( y \not\in V(T_i) \) then replacing \( T_i \) by \( T_i \cup T_j \) and removing \( T_j \) from \( \mathcal{S} \) we get a graphoidal tree cover \( \mathcal{S}' \) with \( |\mathcal{S}'| < |\mathcal{S}| \). Hence \( y \in V(T_i) \). Let \((w, x, z, \ldots, y)\) be a path in \( T_i \). Let \( C_1 \) and \( C_2 \) be the two components of \( T_i - (xz) \) containing \( x \) and \( y \) respectively. Replace \( T_i \) and \( T_j \) by \( C_2 \cup (xz) \) and \( C_1 \cup T_j \) respectively so that
both of them have at least two edges. Now $\mathcal{I}$ is still a minimum graphoidal tree cover and $|E(T)| > 1$ for every $T \in \mathcal{I}$.

Case (ii)

Suppose both $x$ and $y$ are external vertices in $\mathcal{I}$. If $x \in V(T_i)$ and $y \not\in V(T_i)$ then as in case (i) we get a graphoidal tree cover $\mathcal{I}'$ with $|\mathcal{I}'| < |\mathcal{I}|$.

Hence either $x, y \in V(T)$ or $x, y \not\in V(T)$ for every $T$ in $\mathcal{I}$. Let $x, y \in V(T_r), T_r \in \mathcal{I}$.

Suppose $|E(T_r)| > 2$. Let $e = (xz)$ be an edge in $T_r$. Replace $T_r$ and $T_j$ by $T_r - e$ and $T_j \cup \{e\}$ respectively and the result is true in this case. So let us assume that $|E(T_i)| = 2$ for some $T_i \in \mathcal{I}$ and $x, y \in V(T_i)$. Suppose $T_i = (xyz) \in \mathcal{I}$.

Then $\deg(z) \geq 3$ in $G$. For, suppose $\deg(z) = 2$ in $G$. Since $G$ is connected and $p \geq 4$, we must have at least one of the vertices $x, y$ is of degree $\geq 3$. Since $x$ or $y$ alone can not be a member of a tree in $\mathcal{I}$ and $x, y \in V(T_i), V(T_i)$ we have $\deg(x) \geq 3$ and $\deg(y) \geq 3$.

Let $x$ and $y$ be external vertices in a tree $T_r$ of $\mathcal{I}$ ($r \neq i, j$). Replace $T_r$ and $T_j$ by $T_r \cup (xz)$ and $T_j \cup (zy)$ respectively. Now $\{T_1, T_2, ... , T_{i-1}, T_{i+1}, ... , T_n\}$ is clearly a graphoidal tree cover for $G$. This is a contradiction to the minimality of $\mathcal{I}$. Hence $\deg(z) \geq 3$ in $G$. Now, $z$ must be external in some tree $T_r$ of $\mathcal{I}$. Clearly $x, y \in V(T_r)$. Suppose $x, y \not\in V(T_r)$.

Replace $T_r$ and $T_j$ by $T_r \cup (xz)$ and $T_j \cup (zy)$ respectively in $\mathcal{I}$. Now $\{T_1, T_2, ... , T_{i-1}, T_{i+1}, ... , T_n\}$ is clearly a graphoidal tree cover for $G$. This is a contradiction to the minimality of $\mathcal{I}$. It shows that $x, y \in V(T_r)$. Since $x, y$ and $z$ are external vertices in $T_r$ we have $|E(T_r)| \geq 3$. Let $e$ be an edge in $T_r$.
containing z. Replace \( T_i \) and \( T_j \) by \( \{ T_i - \{xz\} \} \cup \{ e \} \) and \( T_j \cup \{\{xz\}\} \) respectively. Now \( \mathcal{I} \) is a minimum graphodial tree cover and \( |E(T)| > 1 \) for every \( T \in \mathcal{I} \).

2.3.2. Proposition

If \( p \geq 4 \), then there exists a minimum graphoidal tree cover of a connected graph \( G \), in which every tree has more than one edge.

**Proof:** Let \( \mathcal{I} \) be a minimum graphoidal tree cover of \( G \) and let \( \mathcal{I} = \{ T_1, T_2, \ldots, T_n \} \). Let us assume that \( T_i = \{ e_i \} \), \( 1 \leq i \leq k \) and \( |E(T_j)| > 1 \) for \( k+1 \leq j \leq n \). Let \( G' = G - \{ e_1, e_2, \ldots, e_k \} \). Clearly \( \mathcal{I}' = \mathcal{I} - \{ T_1, T_2, \ldots, T_k \} \) is a graphoidal tree cover for \( G' \). Suppose \( G' \) is a disconnected graph. Then the number of components \( \omega(G') \) is greater than one. If \( \omega(G' \cup e_i) = \omega(G') \) for every \( i \in \{ 1, 2, \ldots, k \} \) then \( G \) is disconnected. Hence we can choose \( e_i = (x_i, y_i) \) for some \( i \in \{ 1, 2, \ldots, k \} \) such that \( \omega(G' \cup e_i) < \omega(G') \). Let \( G_1', G_1'' \) be the components of \( G' \) such that \( G_1' \cup G_1'' \cup e_i \) is connected. Without loss of generality assume that \( x_i \in G_1' \), \( y_i \in G_1'' \). If at all \( x_i \) is internal in a tree of \( \mathcal{I} \), let it be in a tree \( T \) (of \( \mathcal{I} \)) in \( G_1' \). Clearly \( \mathcal{I}_1 = \{ \mathcal{I} - \{ T, T_j \} \} \cup \{ T \cup T_i \} \) is a graphoidal tree cover of \( G \) and \( |\mathcal{I}_1| < |\mathcal{I}| \). This is a contradiction. Hence \( G' \) is connected. Take \( G_1 = G' \cup \{ e_i \} \). Clearly \( \mathcal{I}_1 = \mathcal{I}' \cup \{ T_i \} \) is a minimum graphoidal tree cover for \( G_1 \) and \( |\mathcal{I}_1| = n - k + 1 \). For, suppose \( \gamma_r(G_1) < n - k + 1 \) and let \( \mathcal{I}'' \) be a minimum graphoidal tree cover for \( G_1 \). Then \( |\mathcal{I}''| < n - k + 1 \).
Since $G = G_1 \cup \{e_2, \ldots, e_k\}$, $\mathcal{J}''' = \mathcal{J}'' \cup \{T_2, \ldots, T_k\}$ is a graphoidal tree cover for $G$ and $|\mathcal{J}'''| = |\mathcal{J}''| + k - 1 < n - k + 1 + k - 1 = n$. This is a contradiction to the minimality of $\mathcal{J}$. Hence $\gamma_T(G_1) = n - k + 1$. By 2.3.1, there exists a minimum graphoidal tree cover $\mathcal{J}_1'$ of $G_1$ in which every tree has more than one edge and $|\mathcal{J}_1'| = |\mathcal{J}_1| = n - k + 1$. Let $G_2 = G_1 \cup \{e_2\}$. Proceeding as above, we find a minimum graphoidal tree cover $\mathcal{J}_2$ of $G_2$ in which every tree has more than one edge. Finally, we get $G = G_n = G_{n-1} \cup \{e_n\}$ and by a similar argument as above, we find a minimum graphoidal tree cover $\mathcal{J}_n$ of $G = G_n$ in which $|E(T)| > 1$ for every $T \in \mathcal{J}_n$. 

2.3.3. Lemma: Let $p(G) \geq 4$. Let $\mathcal{J}$ be a graphoidal tree cover of $G$ such that $|E(T)| > 1$ for every tree $T \in \mathcal{J}$. Let $i(T)$ be the set of internal vertices of $T$. Then $<i(T)>$ – the subgraph induced by $i(T)$ – is a subgraph of $T$ and it is a tree for every $T \in \mathcal{J}$.

\textbf{Proof}: If $|i(T)| = 1$ then clearly the result is true. Let $|i(T)| > 1$. Let $x, y \in i(T)$ and $xy \in E(G)$. Suppose $xy \notin E(T)$. Then there exists $T'$ of $\mathcal{J}$ such that $T' = \{(xy)\}$, by the definition of graphoidal tree cover. By our assumption this is not possible. Hence $<i(T)>$ is a subgraph of $T$ and it is a tree. Moreover, it is got by removing all the pendant vertices of $T$. 

2.3.4. Theorem: Let $G$ be a $(p, q)$ graph with $p \geq 4$. Then $\tau'(G) \leq \gamma_T(G)$. 

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Proof: By Corollary 2.3.2, there exists a minimum graphoidal tree cover $\mathcal{F}$ such that $|E(T)| > 1$ for all $T \in \mathcal{F}$ and $|\mathcal{F}| = n$ ....... (1)

Let $\mathcal{F} = \{T_1, T_2, T_3, \ldots, T_n\}$.

Case (i): If every vertex is an internal vertex of a tree of $\mathcal{F}$, then $V(G) = i(T_1) \cup \ldots \cup i(T_n)$ is clearly a vertex partition of $G$. By 2.3.3 $<i(T_j)>$ is a sub graph of $T_j$ and is a tree for $1 \leq j \leq n$. Hence $\tau'(G) \leq n \leq \gamma_T(G)$.

Case (ii): Let $x$ be one of the vertices which is not internal in any tree of $\mathcal{F}$. Let $x \in V(T_k)$ and $v \in i(T_k)$ such that $xv \in E(T_k)$. Since $x$ is not internal in any tree of $\mathcal{F}$ and $v$ is not internal in any tree except $T_k$, we have $<i(T_k) \cup \{x\}>$ is a tree. For, if $xu \in E(G)$ and $xu \notin E(T_k)$ where $u \neq v$ in $i(T_k)$, then by the definition of graphoidal tree cover there exists $T'$ of $\mathcal{F}$ such that $T' = \{xu\}$. This is a contradiction to our assumption in (1).

Let $x, y$ be non–internal vertices in any tree of $\mathcal{F}$. Let $x, y \in V(T_k)$. If $xy \in E(G)$ then there exists $T'$ of $\mathcal{F}$ such that $T' = \{xy\}$. This is a contradiction to our assumption in (1). Clearly, in this case $<i(T_k) \cup \{x, y\}>$ is a tree. In this way we adjoin every such vertex to an $i(T_k)$. We make sure that each such vertex is adjoined to only one $i(T_k)$. These induced subgraphs give rise to a partition of $V(G)$ and these induced subgraphs form $n = \gamma_T(G)$ trees. Hence $\tau'(G) \leq n = \gamma_T(G)$. 

\[41\]
From 2.2.6, 2.2.13, 2.2.15 and 2.2.18 it follows that $\gamma_T(G) = \tau'(G)$ for the following graphs $K_n$, $P_m \times P_n$, $P_n \times C_m$ and $T \times K_2$.

### 2.4. GRAPHOIDAL TREE d–COVER

#### 2.4.1. Definition

A **graphoidal tree d–cover** ($d \geq 2$) $\mathcal{Z}$ of $G$ is a collection of non–trivial trees in $G$ such that

(i) Every vertex is an internal vertex of at most one tree

(ii) Every edge is in exactly one tree.

(iii) For every tree $T \in \mathcal{Z}$, $\Delta(T) \leq d$.

Let $\mathcal{G}$ denote the set of all graphoidal tree d–covers of $G$. Since $E(G)$ is a graphoidal tree d–cover, we have $\mathcal{G} \neq \emptyset$.

Let $\gamma_T^{(d)}(G) = \min_{\mathcal{Z} \in \mathcal{G}} |\mathcal{Z}|$.

Then $\gamma_T^{(d)}(G)$ is called the **graphoidal tree d–covering number** of $G$.

Any graphoidal tree d–cover $\mathcal{Z}$ of $G$ for which $|\mathcal{Z}| = \gamma_T^{(d)}(G)$ is called a minimum graphoidal tree d–cover.

Clearly a graphoidal tree 2–cover is a graphoidal path cover and a graphoidal tree d–cover ($d \geq \Delta$) is a graphoidal tree cover. Note that $\gamma_T(G) \leq \gamma_T^{(d)}(G)$ for all $d \geq 2$.

We first determine a lower bound for $\gamma_T^{(d)}(G)$. It is easy to observe that $\gamma_T^{(d)}(G) \geq \Delta - d + 1$. Define $n_d = \min_{\mathcal{Z} \in \mathcal{G}_d} n_3$ where $\mathcal{G}_d$ is a collection of all
graphoidal tree $d$–covers and $n_\gamma$ is the number of vertices which are not internal vertices of any tree in $\mathcal{A}$.

2.4.2. Theorem: For $d \geq 2$, $\gamma^{(d)}_T(G) \geq q - (p - n_d)(d - 1)$.

Proof: Let $Ψ$ be a minimum graphoidal tree $d$–cover of $G$ such that $n_Ψ$ vertices of $G$ are not internal in any tree of $Ψ$. Let $k$ be the number of trees in $Ψ$ having more than one edge. For a tree in $Ψ$ having more than one edge, fix a root vertex which is not a pendant vertex. Assign direction to the edges of the $k$ trees in such a way that the root vertex has in degree zero and every other vertex has in degree 1. In $Ψ$, let $l_1$ be the number of vertices of out degree $d$ and $l_2$ be the number of vertices of out degree less than or equal to $d-1$ (and $> 0$) in these $k$ trees. Clearly $l_1 + l_2$ is the number of internal vertices of trees in $Ψ$ and so $l_1 + l_2 = p - n_Ψ$. In each tree of $Ψ$ there is at most one vertex of out degree $d$ and so $l_1 \leq k$. Hence we have

$$\gamma^{(d)}_T(G) \geq k + q - (l_1d + l_2(d - 1))$$

$$= k + q - (l_1 + l_2)(d - 1) - l_1$$

$$= k + q - (p - n_Ψ)(d - 1) - l_1$$

$$\geq q - (p - n_d)(d - 1).$$

2.4.3. Corollary: $\gamma^{(d)}_T(G) \geq q - p(d - 1)$

Now we determine graphoidal tree $d$–covering number of a complete graph.
2.4.4. Theorem

For \( p \geq 4 \), \( \gamma^{(d)}(K_p) = \begin{cases} 
\frac{p}{2} (p - 2d + 1) & \text{if } d < \frac{p}{2} \\
\left\lfloor \frac{p}{2} \right\rfloor & \text{if } d \geq \frac{p}{2} 
\end{cases} \)

Proof: Let \( d \geq \frac{p}{2} \). We know that \( \gamma^{(d)}(K_p) \geq \gamma_p(K_p) = \left\lfloor \frac{p}{2} \right\rfloor \) by 2.2.6.

Case (i) Let \( p \) be even, say \( p = 2k \). We write \( V(K_p) = \{0, 1, 2, ..., 2k-1\} \).

Consider the graphoidal tree cover \( \mathcal{T} = \{T_1, T_2, ..., T_k\} \) where each \( T_i \) (\( i = 1, 2, ..., k \)) is a spanning tree with edge set defined by

\[ E(T_i) = \{(i - 1, j) : j = i, i+1, ..., i+k-1\} \cup \{(k+i-1, s) : s = j \pmod{2k}, j = i+k, i+k+1, ..., i+2k-2\} \]

Now \( |\mathcal{T}| = k = \frac{p}{2} \). Note that \( \Delta(T_i) = k \leq d \), \( i = 1, 2, ..., k \) and hence

\[ \gamma^{(d)}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor. \]

Case (ii) Let \( p \) be odd, say \( p = 2k + 1 \). We write \( V(K_p) = \{0, 1, 2, ..., 2k\} \).

Consider the graphoidal tree cover \( \mathcal{T} = \{T_1, T_2, ..., T_{k+1}\} \) where each \( T_i \) (\( i = 1, 2, ..., k \)) is a tree with edge set defined by

\[ E(T_i) = \{(i - 1, j) : j = i, i+1, ..., i+k-1\} \cup \{(k+i-1, s) : s = j \pmod{2k+1}, j = i+k, i+k+1, ..., i+2k-1\} \]

\[ E(T_{k+1}) = \{(2k, j) : j = 0, 1, 2, ..., k-1\} \]
Now $|\mathcal{J}_2|=k+1 = \left\lfloor \frac{p}{2} \right\rfloor$. Note that the degree of every internal vertex of $T_i$ is either $k$ or $k+1$ and so $\Delta(T_i) \leq d$, $i=1$, 2, ..., $k+1$. Hence $\gamma_{T_i}^{(d)}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor$ if $d \geq \frac{p}{2}$. Let $d < \frac{p}{2}$. By 2.4.3, $\gamma_{T_i}^{(d)}(K_p) \geq q + p - pd = \frac{p(p-1)}{2} + p - pd = \frac{p}{2} (p - 2d + 1)$.

Remove the edges from each $T_i$ in $\mathcal{J}_1(\mathcal{J}_2)$ when $p$ is even (odd) so that every internal vertex is of degree $d$ in the new tree $T_i'$ formed by this removal. The new trees so formed together with the removed edges form $\mathcal{J}_3$.

If $p$ is even, then $\mathcal{J}_3$ is constructed from $\mathcal{J}_1$ and $|\mathcal{J}_3| = k+q-k(2d-1)$

$$= k+ \frac{2k(2k-1)}{2} - k(2d-1) = k(2k-2d+1) = \frac{p}{2} (p - 2d + 1)$$

If $p$ is odd, then $\mathcal{J}_3$ is constructed from $\mathcal{J}_2$ and $|\mathcal{J}_3| = k+1+q-k(2d-1)-d$

$$= k + 1 + \frac{(2k+1)2k}{2} - 2kd + k - d = (2k + 1) (1 + k-d)$$

$$= \frac{p}{2} (p + 1 - 2d).$$

Hence $\gamma_{T_i}^{(d)}(K_p) = \frac{p}{2} (p + 1 - 2d)$. 

The following examples illustrate the above theorem.

2.4.5. Examples

Consider $K_6$. Take $d=3=\frac{p}{2}$ and $V(K_6) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$
\[ \gamma_T^{(3)}(K_6) = 3 \]

Take \( d = 2 < \frac{p}{2} \)

\[ \gamma_T^{(2)}(K_6) = \frac{6}{2} \cdot (6+1-2\times2) = 9 \]

Consider \( K_7 \). Take \( d = 4 = \left\lceil \frac{p}{2} \right\rceil \) and \( V(K_7) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\} \)

\[ \gamma_T^{(4)}(K_7) = 4 = \left\lceil \frac{p}{2} \right\rceil \].
Take \( d=3 < \frac{p}{2} \).

\[ \gamma_r^{(3)}(K_r) = \frac{7}{2} (7+1-2\times3)=7 \]

We now turn to some cases of complete bipartite graph.

2.4.6. Theorem: If \( n,m \geq 2d \) then \( \gamma_r^{(d)}(K_{m,n}) = p + q - pd= mn - (m+n)(d-1). \)

Proof: By 2.4.3, \( \gamma_r^{(d)}(K_{m,n}) \geq p + q - pd = mn - (m+n)d + m + n. \)

Consider \( G = K_{2d,2d} \). Let \( V(G)=X_1 \cup Y_1 \) where \( X_1={x_1, x_2, \ldots, x_{2d}} \) and \( Y_1={y_1, y_2, \ldots, y_{2d}} \). Clearly \( \deg(x_i) = \deg(y_j) = 2d, 1 \leq i, j \leq 2d. \)

For \( 1 \leq i \leq d \), we define

\[ T_i = \{(x_i, y_j): 1 \leq j \leq d \}, \quad T_{d+i} = \{(x_{i+d}, y_j): d+1 \leq j \leq 2d \} \]
\[ T_{2d+i} = \{(y_i, x_j): d+1 \leq j \leq 2d \} \text{ and } T_{3d+i} = \{(y_{i+d}, x_j): 1 \leq j \leq d \} \]

Clearly \( \mathcal{S} = \{T_1, T_2, \ldots, T_{4d}\} \) is a graphoidal tree \( d \)-cover for \( G. \)

Now consider \( K_{m,n} (m, n \geq 2d) \). Let \( V(K_{m,n}) = X \cup Y \) where \( X = \{x_1, x_2, \ldots, x_m\} \)
and \( Y = \{y_1, y_2, \ldots, y_n\}. \) Now for \( 4d+1 \leq i \leq 4d + m - 2d = m + 2d, \) we define
$T_i = \{(x_{i-2d}, y_j) : 1 \leq j \leq d\}$.

For $m+2d+1 \leq i \leq m+n$, we define $T_i = \{(y_{i-m}, x_j) : 1 \leq j \leq d\}$.

Then $\mathcal{E} = \{T_1, T_2, \ldots, T_{4d}, T_{4d+1}, \ldots, T_{m+2d}, T_{m+2d+1}, \ldots, T_{m+n}\} \cup \{E(G) - E(T_i) : 1 \leq i \leq m+n\}$ is a graphoidal tree $d$–cover for $K_{m,n}$. Hence $|\mathcal{E}| = p+q - pd$ and so $\gamma_{T}(d)(K_{m,n}) \leq p+q - pd = mn - (m+n)(d-1)$ for $m,n \geq 2d$.

The following example illustrates the above theorem.

2.4.7. Example: Consider $K_{8,10}$ and take $d=4$.

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The diagrams illustrate the graphoidal tree $d$–cover for $K_{m,n}$ with $d=4$. The figure shows the graphoidal tree for $K_{8,10}$ with the specified $d$ value.
2.4.8. Theorem: \( \gamma_T^{(d)}(K_{2d-1,2d-1}) = p + q - pd = 2d - 1 \)

Proof: By 2.4.3, \( \gamma_T^{(d)}(K_{2d-1,2d-1}) \geq p + q - pd = 2d - 1 \).

For \( 1 \leq i \leq d - 1 \), we define

\[ T_i = \{ (x_i, y_j) : 1 \leq j \leq d \} \cup \{ (y_i, x_{d+j}) : 1 \leq j \leq d - 1 \} \cup \{ (x_{d+1}, y_{d+j}) : 1 \leq j \leq d - 1 \} \]

Let \( T_d = \{ (x_d, y_j) : 1 \leq j \leq d \} \cup \{ (y_d, x_{d+j}) : 1 \leq j \leq d - 1 \} \)

For \( d + 1 \leq i \leq 2d - 1 \), we define \( T_i = \{ (y_i, x_j) : 1 \leq j \leq d \} \).

Clearly \( T = \{ T_1, T_2, ..., T_{2d-1} \} \) is a graphoidal tree \( d \)-cover of \( G \) and so \( \gamma_T^{(d)}(G) \leq 2d - 1 = (2d - 1)(2d - 1 - 2(d - 1)) = q + p - pd \).

The following example illustrates the above theorem.

2.4.9. Example: Consider \( K_{9,9} \) and \( d = 5 \).
Lemma: $\gamma_t^{(d)}(K_{3r, 3r}) \leq 2r$ where $d \geq 2r$ and $r > 1$.

Proof: Let $V(K_{3r, 3r}) = X \cup Y$ where $X = \{x_1, x_2, \ldots, x_{3r}\}$ and $Y = \{y_1, y_2, \ldots, y_{3r}\}$.

Case (i) When $r$ is even. For $1 \leq s \leq r$, we define

$$T_s = \{(x_i, y_i) : 0 \leq i \leq r-1\} \cup \{(x_i, y_{2r+s}) : 1 \leq i \leq r, i \neq s\} \cup \{(x_{r+s}, y_{2r+s}) : r+s \leq i \leq 3r, i \neq 2r+s\}$$

Then $\mathcal{J} = \{T_1, T_2, \ldots, T_{2r}\}$ is a graphoidal tree $d$–cover for $K_{3r, 3r}$, $\Delta(T_i) \leq 2r$ and $d \geq 2r$. So we have, $\gamma_t^{(d)}(K_{3r, 3r}) \leq 2r$.

Case (ii) When $r$ is odd.

For $1 \leq s \leq r$, we define

$$T_s = \{(x_i, y_i) : 0 \leq i \leq 2r-1\} \cup \{(y_{r+s}, x_i) : r+1 \leq i \leq 3r, i \neq 2r+s\}$$

$$\cup \{(x_{2r+s}, y_i) : 2r+s \leq i \leq 3r\} \cup \{(x_{2r+s}, y_i) : 1 \leq i \leq s - 1, s \neq 1\}$$

Then $\mathcal{J} = \{T_1, T_2, \ldots, T_{2r}\}$ is a graphoidal tree $d$–cover for $K_{3r, 3r}$ and so $\gamma_t^{(d)}(K_{3r, 3r}) \leq 2r$, when $r$ is odd.

The following example illustrates the above lemma for $r=2,3$. 

Theorem 2.4.10: $\gamma_t^{(d)}(K_{3r, 3r}) = 81 + 18 - 90 = 9$
Consider $K_{6,6}$.

```
\begin{align*}
\text{Fig 20}
\end{align*}
```

Consider $K_{9,9}$.

```
\begin{align*}
\text{Fig. 21}
\end{align*}
```
2.4.11. Theorem

\[ \gamma_r^{(d)}(K_{n,n}) = \left\lceil \frac{2n}{3} \right\rceil \] for \( d \geq \left\lceil \frac{2n}{3} \right\rceil \) and \( n \geq 3 \).

Proof: By 2.2.11, \( \left\lceil \frac{2n}{3} \right\rceil = \gamma_r(K_{n,n}) \) and \( \gamma_r(K_{n,n}) \leq \gamma_r^{(d)}(K_{n,n}) \), it follows that

\[ \gamma_r^{(d)}(K_{n,n}) \geq \left\lceil \frac{2n}{3} \right\rceil \] for any \( n \). By 2.4.10, the result is true for \( n \equiv 0 \) (mod 3).

Let \( n \equiv 1 \) (mod 3) so that \( n=3r+1 \) for some \( r \). Let \( \mathcal{J}_1 = \{ T_1', T_2', \ldots, T_{2r}' \} \) be a minimum graphoidal tree \( d \)-cover for \( K_{3r,3r} \) as in 2.4.10. For \( 1 \leq i \leq r \), we define

\[ T_i = T_i' \cup \{(x_i, y_{3r+1})\}, \]

\[ T_{r+i} = T_{r+i}' \cup \{(y_i, x_{3r+1})\} \] and

\[ T_{2r+i} = \{(x_{3r+i}, y_{r+i}): 1 \leq i \leq 2r+1\} \cup \{(y_{3r+i}, x_{r+i}): 1 \leq i \leq 2r\}. \]

Clearly \( \mathcal{J}_2 = \{ T_1, T_2, \ldots, T_{2r+1} \} \) is a graphoidal tree \( d \)-cover for \( K_{3r+1,3r+1} \), as

\[ \Delta(T_i) \leq 2r+1 = \left\lceil \frac{2n}{3} \right\rceil - d \] for each \( i \). Hence \( \gamma_r^{(d)}(K_{n,n}) = \gamma_r^{(d)}(K_{3r+1,3r+1}) \leq 2r+1 = \left\lceil \frac{2n}{3} \right\rceil \).

Let \( n \equiv 2 \) (mod 3) and \( n=3r+2 \) for some \( r \). Let \( \mathcal{J}_3 \) be a minimum graphoidal tree \( d \)-cover for \( K_{3r+1,3r+1} \) as in the previous case. Let \( \mathcal{J}_3 = \{ T_1, T_2, \ldots, T_{2r+1} \} \). For \( 1 \leq i \leq r \), we define

\[ T_i' = T_i \cup \{(x_i, y_{3r+2})\} \]
\[ T_{r+i} = T_{r+i} \cup \{ (y_i, x_{3r+2}) \} \]
\[ T_{2r+1} = T_{2r+1} \]
\[ T_{2r+2} = \{ (x_{3r+2}, x_{r+i}) : 1 \leq i \leq 2r+2 \} \cup \{ (y_{3r+2}, x_{r+i}) : 1 \leq i \leq 2r+1 \}. \]

Clearly, \( S_4 = \{ T_1', T_2', \ldots, T_{2r+2}' \} \) is a graphoidal tree, d-cover for \( K_{3r+2, 3r+2} \), as
\[ \Delta(T_i') \leq 2r+2 = \left\lceil \frac{2n}{3} \right\rceil \leq d \text{ for each } i. \]
Hence \( \gamma_T^{(d)}(K_{n,n}) = \gamma_T^{(d)}(K_{3r+2, 3r+2}) \leq 2r+2=\left\lceil \frac{2n}{3} \right\rceil. \) Therefore, \( \gamma_T^{(d)}(K_{n,n}) = \left\lceil \frac{2n}{3} \right\rceil \) for every \( n. \)

2.4.12. Remark: In 2.4.6, 2.4.8 and 2.4.11, \( \gamma_T^{(d)}(K_{n,n}) \) was found except for the case \( \frac{3d}{2} < n < 2d-1 \) .......... (1)

However, we settle the problem upto \( n=12 \) (or) \( d=7 \) completely. From (1), it follows that \( d \geq 5. \)

Consider \( d = 5. \) By (1), we get \( n=8 \) only
\[ \gamma_T^{(5)}(K_{8,8}) \geq \gamma_T(K_{8,8}) = \left\lceil \frac{16}{3} \right\rceil = 6 \]

Let \( V(K_{8,8}) = X \cup Y \) where \( X = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \} \) and \( Y = \{ y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \} \).
Here $\gamma_T^{(5)}(K_{8,8}) = \left\lceil \frac{16}{3} \right\rceil = 6$

Consider $d=6$. The only value of $n$ satisfying (1) is 10.

$$\gamma_T^{(6)}(K_{10,10}) \geq \gamma_T(K_{10,10}) = \left\lceil \frac{20}{3} \right\rceil = 7$$
Hence $\gamma_T^{(6)}(K_{10,10}) = 7 = \left\lceil \frac{20}{3} \right\rceil$.

Consider $d=7$. The only possible values of $n$ are 11 and 12.

For $n=11$, $\gamma_T^{(7)}(K_{11,11}) \geq \gamma_T(K_{11,11}) = \left\lceil \frac{22}{3} \right\rceil = 8$
Hence \( \gamma_T^{(7)}(K_{11,11}) = 8 = \left\lfloor \frac{22}{3} \right\rfloor \).

For \( n=12 \), \( \gamma_T^{(7)}(K_{12,12}) \geq \gamma_T(K_{12,12}) = \left\lfloor \frac{24}{3} \right\rfloor = 8. \)

Let \( V(K_{12,12}) = X \cup Y \) where \( X = \{x_1, x_2, \ldots, x_{12}\} \) and \( X = \{y_1, y_2, \ldots, y_{12}\} \)

Fig. 25

Hence \( \gamma_T^{(7)}(K_{12,12}) = 8 = \left\lfloor \frac{24}{3} \right\rfloor \)

Now we turn to the case of trees.
2.4.13. **Theorem** Let $G$ be a tree and let $U = \{v \in V(G): \deg(v) - d > 0\}$. Then

$$
\gamma_T^{(d)}(G) = \sum_{v \in V(G)} \chi_U(v)(\deg(v) - d) + 1
$$

where $d \geq 2$ and $\chi_U(v)$ is the characteristic function of $U$.

**Proof**: Proof is by induction on the number of vertices $m$ whose degrees are greater than $d$. If $m = 0$, then $\mathcal{I} = \{G\}$ is clearly a graphoidal tree $d$-cover. Hence the result is true in this case and $\gamma_T^{(d)}(G) = 1$. Let $m > 0$. Let $u \in V(G)$ with $\deg_G(u) = d + s$ ($s > 0$). Now decompose $G$ into $s+1$ trees $G_1, G_2, ..., G_s, G_{s+1}$ such that $\deg_{G_i}(u) = 1$ for $1 \leq i \leq s$, $\deg_{G_{s+1}}(u) = d$. By induction hypothesis,

$$
\gamma_T^{(d)}(G_i) = \sum_{\deg_{G_i}(v) > d} (\deg_{G_i}(v) - d) + 1 = k_i, \quad 1 \leq i \leq s + 1.
$$

Now $\mathcal{I}_i$ is the minimum graphoidal tree $d$-cover of $G_i$ and $|\mathcal{I}_i| = k_i$ for $1 \leq i \leq s + 1$. Let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup ... \cup \mathcal{I}_{s+1}$.

Clearly $\mathcal{I}$ is a graphoidal tree $d$-cover of $G$. By our choice of $u$, $u$ is internal in only one tree $T$ of $\mathcal{I}$. More over, $\deg_T(u) = d$ and $\deg_{G_i}(v) = \deg_G(v)$ for $v \neq u$ and $v \in V(G_i)$ for $1 \leq i \leq s + 1$.

Therefore, $\gamma_T^{(d)}(G) \leq |\mathcal{I}| = \sum_{i=1}^{s+1} k_i$.

$$
= \sum_{i=1}^{s+1} \left( \sum_{\deg_{G_i}(v) > d} (\deg_{G_i}(v) - d) + 1 \right)
$$

$$
= \sum_{i=1}^{s+1} \left( \sum_{\deg_{G_i}(v) > d} (\deg_{G_i}(v) - d) \right) + (s + 1)
$$

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\[
\begin{align*}
= & \sum_{\deg_G(v) > d} (\deg_G(v) - d) + (s + 1) \\
= & \sum_{\deg_G(v) > d} (\deg_G(v) - d) + (\deg_G(u) - d) + 1 \\
= & \sum_{\deg_G(v) > d} (\deg_G(v) - d) + 1 \\
= & \sum_{v \in V(G)} \chi_U(v)(\deg_G(v) - d) + 1
\end{align*}
\]

For each \(v \in V(G)\) and \(\deg_G(v) > d\) there are at least \((\deg_G(v) - d) + 1\) subtrees of \(G\) in any graphoidal tree \(d\)-cover of \(G\) and so \(\gamma_T^{(d)}(G) \geq \sum_{\deg_G(v) > d} (\deg_G(v) - d) + 1\).

Hence \(\gamma_T^{(d)}(G) = \sum_{v \in V(G)} \chi_U(v)(\deg_G(v) - d) + 1\).

2.4.14. **Corollary**: Let \(G\) be a tree in which degree of every vertex is either greater than or equal to \(d\) or equal to one. Then \(\gamma_T^{(d)}(G) = m(d - 1) - p(d - 2) - 1\) where \(m\) is the number of vertices of degree 1 and \(d \geq 2\).

**Proof**: Since all the vertices of \(G\) other than pendant vertices have degree \(\geq d\) we have,

\[
\gamma_T^{(d)}(G) = \sum_{v \in V(G)} (\deg_G(v) - d)\chi_U(v) + 1 = \sum_{v \in V(G)} (\deg(v) - d) + md - m + 1
\]

\[
= 2q - dp + md - m + 1 = 2p - 2 - dp + md - m + 1 \quad \text{(as } q = p - 1) \\
= m(d - 1) - p(d - 2) - 1
\]

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Recall that \( n_d = \min_{\mathcal{G}} n_3 \) and \( n = \min_{\mathcal{G}} n_3 \) where \( \mathcal{G} \) is the collection of all graphoidal tree \( d \)-covers of \( G \), \( \mathcal{G} \) is the collection of all graphoidal tree covers of \( G \) and \( n_3 \) is the number of vertices which are not internal vertices of any tree in \( \mathcal{G} \).

Clearly \( n_d = n \) if \( d \geq \Delta \). Now we prove this for any \( d \geq 2 \).

2.4.15. Lemma: For any graph \( G \), \( n_d = n \) (\( d \geq 2 \)).

**Proof:** Since every graphoidal tree \( d \)-cover is also a graphoidal tree cover for \( G \), we have \( n \leq n_d \). Let \( \mathcal{G} = \{ T_1, T_2, \ldots, T_m \} \) be any graphoidal tree cover of \( G \).

Let \( \Psi_i \) be a minimum graphoidal tree \( d \)-cover of \( T_i \) \((i = 1, 2, \ldots, m)\). Let \( \Psi = \bigcup_{i=1}^{m} \Psi_i \). Clearly \( \Psi \) is a graphoidal tree \( d \)-cover of \( G \). Let \( n_\Psi \) be the number of vertices which are not internal in any tree of \( \Psi \). Clearly \( n_\Psi = n_3 \).

Therefore, \( n_d \leq n_\Psi = n_3 \) for \( \mathcal{G} \in \mathcal{G} \), where \( \mathcal{G} \) is the collection of graphoidal tree covers of \( G \) and so \( n_d \leq n \). Hence \( n = n_d \).

We have the following result for graphoidal path cover. This theorem is proved by S. Arumugam and J. Suresh Suseela in [8]. We prove this, by deriving a minimum graphoidal path cover from a graphoidal tree cover of \( G \).

2.4.16. Theorem: \( \gamma_T^{(2)}(G) = q - p + n_2 \)

**Proof:** From 2.4.2 it follows that \( \gamma_T^{(2)}(G) \geq q - p + n_2 \). Let \( \mathcal{G} \) be any graphoidal tree cover of \( G \) and \( \mathcal{G} = \{ T_1, T_2, \ldots, T_k \} \). Let \( \Psi_i \) be a minimum
graphoidal tree d-cover of $T_i$ ($i=1,2, \ldots, k$). Let $m_i$ be the number of vertices of degree 1 in $T_i$ ($i=1,2,\ldots, k$). Then by 2.4.14 it follows that $\gamma_{T_i}^{(2)}(T_i) = m_i - 1$ for all $i=1, 2,\ldots, k$. Consider the graphoidal tree 2-cover $\Psi = \bigcup_{i=1}^{k} \Psi_i$ of $G$.

Now $|\Psi_3| = \sum_{i=1}^{k} |\Psi_i| = \sum_{i=1}^{k} (m_i - 1)$

$= \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} q_i - \sum_{i=1}^{k} p_i$

$= q - \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} m_i$

Consider $\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} (\text{Number of internal vertices of } T_i + \text{Number of pendant vertices of } T_i)$

$= p - n_3 + \sum_{i=1}^{k} m_i$

Therefore, $|\Psi_3| = q - p + n_3$.

Choose a graphoidal tree cover $\mathcal{S}$ of $G$ such that $n_3 = n$. Then for the corresponding $\Psi_3$ we have $|\Psi_3| = q - p + n = q - p + n_2$, as $n_2 = n$ by 2.4.15.

2.4.17. Corollary: If every vertex is an internal vertex of a graphoidal tree cover, then $\gamma_{T}^{(2)}(G) = q - p$.

Proof: Clearly $n = 0$ by definition. By 2.4.15, $n_2 = n$. So we have $n_2 = 0$. □
J. Suresh Suseela and S. Arumugam proved the following result in [8]. However, we prove the result using graphoidal tree cover.

2.4.18. Theorem: Let $G$ be a unicyclic graph with $r$ vertices of degree 1. Let $C$ be the unique cycle of $G$ and let $m$ denote the number of vertices of degree greater than 2 on $C$. Then

$$
\gamma^{(2)}_T(G) = \begin{cases} 
2 & \text{if } m = 0 \\
 r+1 & \text{if } m = 1 \text{ and } \deg(v) \geq 3 \text{ where } v \text{ is the unique vertex of degree } > 2 \text{ on } C \\
r & \text{otherwise}
\end{cases}
$$

Proof: By 2.4.15 and 2.4.16, we have $\gamma^{(2)}_T(G) = q - p + n$. We have $q(G) = p(G)$ for unicyclic graph. So we have $\gamma^{(2)}_T(G) = n$. If $m = 0$, then clearly $\gamma^{(2)}_T(G) = 2$.

Let $m = 1$. Let $v$ be the unique vertex of degree $> 2$ on $C$. Let $e = vw$ be an edge on $C$. Clearly $\mathcal{S} = \{G - e, e\}$ is a minimum graphoidal tree cover for $G$ and so $n = r + 1$. Since there is a vertex of $C$ which is not internal in a tree of a graphoidal tree cover, we have $n = r + 1$. When $m = 1$, $\gamma^{(2)}_T(G) = r + 1$. Let $m \geq 2$. Let $v$ and $w$ be vertices of degree greater than 2 on $C$ such that all vertices in a $(v, w)$ – section of $C$ other than $v$ and $w$ have degree 2. Let $P$ denote this $(v, w)$–section. If $P$ has length 1 then $P = (v, w)$. Clearly $\mathcal{S} = \{G - P, P\}$ is a graphoidal tree cover of $G$. Also $n = r$ and so $\gamma^{(2)}_T(G) = r$ when $m \geq 2$.

Hence the theorem.

2.4.19. Theorem: Let $G$ be a graph such that $\gamma_T(G) \leq \delta(G) - d + 1(\delta > d \geq 2)$.

Then $\gamma^{(d)}_T(G) = q - p(d - 1)$.
Proof: By 2.4.3, $\gamma_T^{(d)}(G) \geq q - p(d - 1)$. Let $\mathfrak{T}$ be a minimum graphoidal tree cover of $G$. Since $\delta > \gamma_T$, every vertex is an internal vertex of a tree in a graphoidal tree cover $\mathfrak{T}$. Moreover, since $\delta \geq d + \gamma_T - 1$ the degree of each internal vertex of a tree in $\mathfrak{T}$ is $\geq d$. Let $\Psi_i$ be a minimum graphoidal tree $d$–cover of $T_i$ ($i = 1, 2, \ldots, k$). Let $m_i$ be the number of vertices of degree 1 in $T_i$ ($i=1,2,\ldots, k$).

Then by 2.4.14, for $i = 1, 2, \ldots, k$ we have

$$\gamma_T^{(2)}(T_i) = -p_i(d - 2) + m_i(d - 1) - 1$$

Consider the graphoidal tree $d$–cover $\Psi_T = \bigcup_{i=1}^{k} \Psi_i$ of $G$.

$$|\Psi_T| = \sum_{i=1}^{k} |\Psi_i| = \sum_{i=1}^{k} \left( m_i(d - 1) - p_i(d - 2) - 1 \right)$$

$$= \sum_{i=1}^{k} \left[ (m_i - p_i)(d - 1) + q_i \right]$$

$$= (d - 1) \sum_{i=1}^{k} (m_i - p_i) + \sum_{i=1}^{k} q_i$$

Now, $\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} (\text{Number of internal vertices of } T_i + \text{Number of pendant vertices of } T_i)$

$$= p + \sum_{i=1}^{k} m_i$$
Therefore, \(|\Psi_T| = -(d-1)p+q\). In other words, \(\gamma_T^{(d)}(G) \leq q-p(d-1)\). Hence \(\gamma_T^{(d)}(G) = q-p(d-1)\).

**2.4.20. Corollary** Let \(G\) be a graph such that \(\delta = \left\lfloor \frac{p}{2} \right\rfloor + k\) where \(k \geq 1\). Then 
\[\gamma_T^{(d)}(G) = q-p(d-1)\] for \(d \leq k+1\).

**Proof:** \(\delta - d + 1 = \left\lfloor \frac{p}{2} \right\rfloor + k - d + 1 \geq \left\lfloor \frac{p}{2} \right\rfloor \geq \gamma_T(G)\) (by Corollary 2.2.26) By 2.4.19, \(\gamma_T^{(d)}(G) = q-p(d-1)\).

**2.4.21. Corollary** Let \(G\) be an \(r\)-regular graph where \(r > \left\lfloor \frac{p}{2} \right\rfloor\). Then \(\gamma_T^{(d)}(G) = q-p(d-1)\) for \(d \leq r+1 - \left\lfloor \frac{p}{2} \right\rfloor\).

**Proof:** Here \(\delta = r\) and so the result follows from 2.4.20

**2.4.22. Corollary** \(\gamma_T^{(d)}(K_{m,n}) = q-p(d-1)\) where \(2 \leq d \leq \frac{2m-n}{3}\) and \(6 \leq m \leq n \leq 2m-6\).

**Proof:** Consider \(\delta - d + 1 \geq m - \frac{2m-n}{3} + 1\)

\[
\begin{align*}
= & \frac{3m-2m+n}{3} + 1 \\
= & \frac{m+n}{3} + 1 \geq \left\lfloor \frac{m+n}{3} \right\rfloor = \gamma_T(K_{m,n})
\end{align*}
\]

Hence by 2.4.19, \(\gamma_T^{(d)}(K_{m,n}) = q-p(d-1)\).
Our first result on the graphoidal tree $d$-covering number of graph products is for grid graphs.

2.4.23. **Theorem:** For $m, n \geq 3$, $\gamma_T^{(d)}(P_m \times P_n) = 2$ ($d \geq 3$) and $\gamma_T^{(2)}(P_m \times P_n) = q - p$.

**Proof:** Clearly $\gamma_T^{(d)}(P_m \times P_n) \geq 2$ and by 2.2.13 and 2.2.14, we have $\gamma_T^{(d)}(P_m \times P_n) \leq 2$ for $d \geq 3$. Hence $\gamma_T^{(d)}(P_m \times P_n) = 2$ for $d \geq 3$. By 2.2.14, every vertex is an internal vertex of a tree in a minimum graphoidal tree cover. By 2.4.17, $\gamma_T^{(2)}(P_m \times P_n) = q - p$. ■

2.4.24. **Theorem**

For $m \geq 3, n \geq 2$, $\gamma_T^{(d)}(P_n \times C_m) = 2, d \geq 3$ and $\gamma_T^{(2)}(P_n \times C_m) = q - p$.

**Proof:** Clearly $\gamma_T^{(d)}(P_n \times C_m) \geq 2$ and by 2.2.15 and 2.2.16 we have $\gamma_T^{(d)}(P_n \times C_m) \leq 2$ for $d \geq 3$. Hence $\gamma_T^{(d)}(P_n \times C_m) = 2$ for $d \geq 3$. Since $\delta(P_n \times C_m) = 3$ and $\gamma_T(P_n \times C_m) = 2$, we have $\gamma_T(P_n \times C_m) = \delta - d + 1$ (when $d = 2$).

By 2.4.19, $\gamma_T^{(2)}(P_n \times C_m) = q - p$. ■

2.4.25. **Theorem** $\gamma_T^{(d)}(C_m \times C_n) = 3, d \geq 4$ and $\gamma_T^{(2)}(C_m \times C_n) = q - p$.

**Proof:** For $d \geq \Delta = 4$, $\gamma_T^{(d)}(C_m \times C_n) = \gamma_T(C_m \times C_n) = 3$, by 2.2.17. Since $\delta(C_m \times C_n) = 4$ and $\gamma_T(C_m \times C_n) = 3$, we have $\gamma_T(C_m \times C_n) = \delta - d + 1$ (when $d = 2$).

By 2.4.19, $\gamma_T^{(2)}(C_m \times C_n) = q - p$. ■