CHAPTER I

PRELIMINARIES

In this chapter we collect the basic definitions and theorems on graphs and digraphs which are needed for the subsequent chapters. For graph theoretic terminology, we refer to Harary [28], Parthasarathy [38], Chartrand and Lesniak [20] and Bondy and Murty [15].

1.1. Definition: A graph $G$ is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of $G$ called edges. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$ respectively. $|V(G)| = p$ is called the order of $G$ and $|E(G)| = q$ is called the size of $G$. A graph of order $p$ and size $q$ is called a $(p, q)$-graph. If $e = (uv)$ is an edge of $G$, we say that $u$ and $v$ are adjacent and that each is incident with $e$. If every two vertices of $G$ are adjacent, then $G$ is called a complete graph. The complete graph on $p$ vertices is denoted by $K_p$.

1.2. Definition: The degree of a vertex $v$ in a graph $G$ is defined to be the number of edges incident to $v$ and is denoted by $\deg(v)$. A graph $G$ is called $r$-regular if $\deg(v) = r$ for each $v \in V(G)$. The minimum of $\{\deg(v) : v \in V(G)\}$ is denoted by $\delta$ and the maximum of $\{\deg(v) : v \in V(G)\}$ is denoted by $\Delta$. A vertex of degree 0 in $G$ is called an isolated vertex, a vertex of degree 1 is called a pendant vertex or an end vertex.
1.3. **Definition**: A *path* $P$ of length $n$ in a graph $G$ is a sequence of distinct vertices $(u_0, u_1, u_2, ..., u_n)$ where $e_i = u_iu_{i+1}$ for $i=0, 1, ..., n-1$. We also represent the path $P$ by $(e_0, e_1, e_2, ..., e_{n-1})$. $u_0$ and $u_n$ are called *end vertices* and $u_1, u_2, ..., u_{n-1}$ are called *internal vertices* of $P$. A collection $\{P_1, P_2, ..., P_n\}$ of paths is said to be *internally disjoint* if no vertex is an internal vertex of more than one path in the collection. A path on $n$ vertices is denoted by $P_n$.

1.4. **Definition**: A *cycle* in a graph $G$ is a sequence of vertices $(u_0, u_1, ..., u_n, u_0)$ where $u_i$ and $u_{i+1}$ are adjacent for all $i = 0, 1, 2, ..., n-2$, $u_{n-1}$ and $u_0$ are adjacent, and $u_1, u_2, ..., u_{n-1}$ are distinct. A cycle on $n$ vertices is denoted by $C_n$.

1.5. **Definition**: A *bipartite graph* $G$ is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and the other end in $V_2$; $(V_1, V_2)$ is called a bipartition of $G$. If further, every vertex of $V_1$ is joined to all the vertices of $V_2$, then $G$ is called a *complete bipartite graph*. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

1.6. **Definition**: The *complement* $\overline{G}$ of a graph $G$ has $V(G)$ as a vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

1.7. **Definition**: The *distance* $d(u,v)$ between any two vertices $u, v$ of a connected graph $G$ is defined to be the length of any shortest path joining $u$ and $v$. 
1.8. **Definition**: Two graphs $G_1$ and $G_2$ are said to be *isomorphic* if there exists a bijection $\phi: V(G_1) \rightarrow V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$; such a function $\phi$ is said to be an isomorphism from $G_1$ to $G_2$. An isomorphism of a graph $G$ onto itself is called an *automorphism* of $G$.

1.9. **Definition**: A graph $H$ is called a *subgraph* of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$ then $H$ is called a *spanning subgraph* of $G$.

1.10. **Definition**: For any subset $S$ of $V(G)$, the subgraph $< S >$ of $G$ induced by $S$ is the maximal subgraph of $G$ with vertex set $S$. Similarly for any subset $E'$ of $E(G)$, the subgraph $< E' >$ of $G$ induced by $E'$ is the subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and edge set is $E'$.

Let $v \in V(G)$ and $|V(G)| \geq 2$. Then $< V(G) - \{ v \} >$ is denoted by $G - v$. If $e \in E(G)$, then the spanning subgraph of $G$ with edge set $E(G) - \{ e \}$ is denoted by $G - e$. The graph obtained from $G$ by adding an edge $e$ is denoted by $G + e$.

1.11. **Definition**: A graph $G$ is said to be *connected* if every pair of vertices are joined by a path. A maximal connected subgraph of $G$ is called a *component* of $G$.

1.12. **Definition**: A vertex $v$ of a graph $G$ is called a *cut-vertex* of graph $G$ if the removal of $v$ increases the number of components. An edge $e$ of a graph $G$ is called a *cut edge* or *bridge* if the removal of $e$ increases the number of
components. A set of edges $S$ is called an *edge cut* of $G$ if the number of components of $G - S$ is greater than that of $G$. A *block* of a graph is a maximal connected, non-trival subgraph without cut-vertices.

1.13. **Definition**: A graph $G$ is called *hamiltonian* if it has a spanning cycle. A spanning cycle of $G$ is called a *hamiltonian cycle*. A spanning path of $G$ is called a *hamilton path*.

1.14. **Definition**: A graph is *acyclic* or a *forest* if it has no cycles. A *tree* is a connected acyclic graph. A forest in which every component is a path is called a *linear forest*.

1.15. **Definition**: If $e = uv$ is an edge of $G$ and $w$ is not a vertex of $G$, we say that the edge $e$ is subdivided if it is replaced by the edges $uw$ and $wv$.

1.16. **Definition**: The *connectivity* $\kappa = \kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or $K_1$, the trivial graph. The *line connectivity* or *edge connectivity* $\kappa' = \kappa'(G)$ of a graph $G$ is the minimum number of edges whose removal results in a disconnected graph.

1.17. **Definition**: A set of vertices in $G$ is said to be *independent* if no two of them are adjacent. The largest number of vertices in any independent set of $G$ is called the *independence number* of $G$ and is denoted by $\beta$. 
1.18. **Definition**: A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph $G$ is called a *vertex cover* of $G$. The smallest number of vertices in any vertex cover is called the *vertex covering number* and is denoted by $\alpha_0$.

1.19. **Definition**: The *union* $G_1 \cup G_2$ of $G_1$ and $G_2$ is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

1.20. **Definition**: The *join* of $G_1$ and $G_2$ is the graph $G = G_1 \vee G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

1.21. **Definition**: For two graphs $G$ and $H$ their *cartesian product* $G \times H$ has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)$ is adjacent to $(g_2, h_2)$ if $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$. Similarly $G \circ H$, the *weak product* of the graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$.

1.22. **Theorem** ([28], page 30) A graph $H$ is the *block graph of some graph* if and only if every block of $H$ is complete.

Let $G$ be a graph with $E(G) \neq \emptyset$. Then the intersection graph $\Omega(E)$ is called the *line graph* of $G$ and is denoted by $L(G)$.
1.23. **Theorem** ([28], page 74) A graph $G$ is a line graph if and only if the lines of $G$ can be partitioned into complete subgraphs in such a way that no point lies in more than two of the subgraphs.

In [21] F.R.K.Chung introduces the concept of tree covering number.

1.24. **Definition** [21] The *tree covering number* $\tau(G)$ of a graph $G$ is defined as the minimum number of trees covering all the edges of $G$ exactly once.

1.25. **Theorem** [21] $\tau(K_n) = \left\lceil \frac{n}{2} \right\rceil$.

1.26. **Theorem** [21] Let $G$ be a graph having $n$ vertices; $e$ edges, and $k$ connected components $k < n$. Then $k - 1 + \left\lceil \frac{e}{n-k} \right\rceil \leq \tau(G) \leq \left\lceil \frac{n+k-1}{2} \right\rceil$.

1.27. **Theorem** [14] If $G$ is a 2-connected cubic graph, $p \geq 8$, then $\tau(G) \leq \left\lceil \frac{p}{4} \right\rceil$.

1.28. **Theorem** [14] If $G$ is a 3-connected cubic graph, $p \geq 12$, then $\tau(G) \leq \left\lceil \frac{p}{6} \right\rceil$.

1.29. **Definition**: A *cyclically 4-edge connected graph* is one in which the removal of no three edges will disconnect the graph into two components such that each component contains a cycle.
1.30. **Theorem** [14] If G is a cyclically 4 – edge connected cubic graph with 
p vertices, \(8 \leq p \leq 16\), then \(\tau(G) = 2\).

In [24], M.F. Foregger and T.H. Foregger introduced the parameter 
\(\tau'(G)\).

1.31. **Definition** \(\tau'(G)\) is defined as the minimum number of subsets into 
which the vertex set of G can be partitioned so that each subset induces a 
tree.

In [34], C. St. J.A. Nash – Williams defined *arboricity* of a graph.

1.32. **Definition** [34] The *arboricity* of G, denoted by \(\gamma(G)\), is defined as the 
minimum number of forests in a forest decomposition of G.

1.33. **Theorem** [12] For the complete graph \(K_n\), \(\gamma(K_n) = \left\lfloor \frac{n}{2} \right\rfloor\) where \(\{ x \}\) is the 
least integer not less than \(x\).

1.34. **Theorem** [27] If G is n-regular \((n \geq 1)\) graph then we have \(\gamma(G) = 
\left\lfloor \frac{n+1}{2} \right\rfloor\).

1.35. **Theorem** [27] \(\gamma(L(K_n)) = n-1\) \((n \geq 2)\) and \(\gamma(L(K_{m,n})) = \left\lfloor \frac{m+n-1}{2} \right\rfloor\).

1.36. **Theorem** [28, page 68] If for all points \(v\) of G, \(\deg(v) \geq \frac{p}{2}\), where \(p \geq 3\), 
then G is hamiltonian.
Harary [29] introduced the concept of path cover (path partition) of a graph G.

1.37. **Definition**: A *path cover (path partition)* of a graph G is a collection \( \mathcal{P} \) of paths in G such that every edge of G lies in exactly one path in \( \mathcal{P} \).

The minimum cardinality of a path cover of G is called *path covering number* of G and is denoted by \( P_n(G) \).

1.38. **Theorem** [45] For any 3 - regular graph, \( P_n(G) = \frac{p}{2} \).

1.39. **Theorem** [45] \( P_n(K_{2n}) = n \).

Acharya and Sampathkumar introduced the concept of *graphoidal cover* in [1].

1.40. **Definition** [1]: A *graphoidal cover* of a graph G is a collection \( \Psi \) of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in \( \Psi \) has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in \( \Psi \).

(iii) Every edge of G is in exactly one path in \( \Psi \).

The minimum cardinality of a graphoidal cover of G is called the *graphoidal covering number* of G and is denoted by \( \eta(G) \).

Pakkiam and Arumugam [36] determined the graphoidal covering number of several families of graphs.
Arumugam and Suresh Suseela [8] introduced the concept of *acyclic graphoidal cover*.

1.41. **Definition**: An acyclic graphoidal cover of $G$ is a graphoidal cover $\Psi$ of $G$ such that every element of $\Psi$ is a path in $G$. The minimum cardinality of an acyclic graphoidal cover of $G$ is called the *acyclic graphoidal covering number* of $G$ and is denoted by $\eta_a(G)$ or $\eta_a$.

Throughout the thesis we use the term *graphoidal path cover* instead of *acyclic graphoidal cover* and $\gamma_T^{(2)}(G)$ (See 2.4.1) instead of $\eta_a(G)$.

1.42. **Theorem** [8] For any graph $G$, $\eta_a(G) \geq q - p$. Moreover, the following statements are equivalent.

(i) $\eta_a(G) = q - p$

(ii) there exists an acyclic graphoidal cover $\Psi$ without any exterior points.

Let $\Psi$ be a path cover of a graph $G$. Then the intersection graph $\Omega(\Psi)$ to be the graph whose vertices are the members of $\Psi$ and two vertices in $\Omega(\Psi)$ are adjacent if and only if the corresponding paths share a common vertex.

1.43. **Definition** [1] A graph $G$ is said to be *graphoidal* if there exists a graph $H$ and a graphoidal cover $\Psi$ of $H$ such that $G \cong \Omega(\Psi)$.

1.44. **Definition** A graph $G$ is said to be a path graph if $G$ is the intersection graph of a path cover of a graph $H$. 
Let $\Psi$ be a collection of non-trivial paths in a graph $H$. We define its edge intersection graph $\Gamma(\Psi)$ to be the graph whose vertices are the members of $\Psi$ and where two vertices are joined by an edge if and only if the corresponding paths share an edge.

**1.45. Definition [26]** An undirected graph $G$ is called an edge intersection graph of paths in a tree or EPT graph, if $G \cong \Gamma(\mathcal{P})$ for some tree $T$ and a path cover $\mathcal{P}$ of $T$.

**1.46. Definition [13]** A decomposition of a graph $G$ is a collection of subgraphs of $G$ whose edge sets partition the edge set of $G$. The subgraphs of the decomposition are called the parts of the decomposition.

**1.47. Definition:** A graph $G$ is said to be $F-$ decomposable or $F$-packable if $G$ has a decomposition in which all of its parts are isomorphic to the graph $F$.

Given a graph $G$ which is $F$-packable, the task of actually performing a packing of copies of $F$ into $G$ will be easier if $G$ has the property that every collection of edge-disjoint copies of $F$ in $G$ can be extended to a $F$-packing of $G$. This motivated Ruiz [42] to introduce the concept of randomly $F$-packable graph.

**1.48. Definition:** A graph $G$ is said to be randomly $F$-packable if for every proper $F$-packable subgraph $H$ of $G$, $G - E(H)$ is also $F$-packable.
1.49. **Theorem** [38] If $G$ is a 2-connected graph with $\delta \geq \frac{n+2}{3}$ and $\delta \geq \alpha_0$ then $G$ is hamiltonian.

1.50. **Theorem** [Page 102, 17] A $P_k$ - decomposition of the graph $\lambda K_p$ (where $p \geq 2$ and $k \geq 2$) exists if and only if $\lambda p(p-1) \equiv 0 \pmod{2k-2}$, $p \geq k$, where $P_k$ is a path of length $k-1$.

1.51. **Theorem** [Page 130, 17]: Let $m \geq 3$ and $n \geq 3$. The graph $C_m \times C_n$ can be decomposed into two hamiltonian cycles if and only if at least one of the numbers $m$, $n$ is odd.

1.52. **Theorem** [38] For any graph $G$, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

If $X$ is a subset of the vertex set $V(G)$ of $G$, then $d_G(v,X) = \min \{d_G(v,u) : u \in X\}$.

The $k$-packing number and $k$-covering number were first introduced by Meir and Moon in [33]

1.53. **Definition** [33] A subset $I \subseteq V(G)$ is a $k$-packing of $G$ if $d_G(v,u) > k$ for every pair $v$ and $u$ of distinct vertices from $I$. The $k$-packing number of $G$ is the number $\alpha_k(G)$ of vertices in any maximum $k$-packing of $G$.

1.54. **Definition** [33] A subset $C \subseteq V(G)$ is a $k$-covering of $G$ if $d_G(v,C) \leq k$ for every vertex $v \in V(G) - C$. The $k$-covering number of $G$, denoted by $\gamma_k(G)$, is the number of vertices in any minimum $k$-covering of $G$. 
1.55. **Theorem** [22] For any block graph $G$, $\alpha_{2k}(G) = \gamma_{k}(G)$.

1.56. **Definition**: A one-to-one mapping $f$ from $V(G)$ into $\{0,1,2,\ldots,q\}$ is called a $\beta$-valuation if the induced function $\tilde{f}$ on $E(G)$ given by $\tilde{f}(uv) = |f(u) - f(v)|$ is one-to-one.

A $\beta$-valuation $f$ is called an $\alpha$-valuation if there exists a non-negative integer $\lambda$ such that for every $uv \in E(G)$ with $f(u) < f(v)$, $f(u) \leq \lambda < f(v)$.

1.57. **Definition** [17] A decomposition $\mathcal{R}$ of graph $H$ into subgraphs is said to be cyclic if there exists an automorphism $f$ of $H$ which induces a cyclic permutation $f_v$ of the set $V = V(H)$ and satisfies the following implication: if $G \in \mathcal{R}$ then $fG \in \mathcal{R}$. (Here $f(G)$ is the subgraph of $H$ with vertex set $\{ f(u): u \in V(G)\}$ and edge set $\{ f(e): e \in E(H) \}$).

Balakrishnan and Sampathkumar [9] have proved the following theorem.

1.58. **Theorem** [9] For every positive integer $n$, there exists an $\alpha$-valuation of $Q_n(K_{3,3})$, where $Q_n(K_{3,3}) = K_{3,3} \times K_2 \times \cdots \times K_2$ $n$-times.

Bloom and Ruiz introduced the concept of common-weight decomposition in [13].

1.59. **Definition** Let $G = (V, E)$ be a graph. A difference labelling of $G$ is an injection $f$ from $V$ to the set of non-negative integers together with the weight function $f^*$ on $E$ given by $f^*(uv) = |f(u) - f(v)|$ for every edge $uv$ in $G$. 
1.60. **Definition**: A decomposition of a labelled graph into parts each part containing the edges having a common weight is called a common-weight decomposition.

Bloom and Ruiz in [13] have proved the following theorems

1.61. **Theorem** [13] Every part in a common-weight decomposition is a linear forest. Further the vertices of minimum and maximum labels are not internal vertices in any path of a part containing it.

1.62. **Theorem** [13] There is a labelling of a cycle C realizing a decomposition of C into parts having $m_1$ and $m_2$ edges respectively if and only if $m_1$ and $m_2$ are relatively prime.

1.63. **Theorem** [13] A labelling exists for every cycle with $2s$ edges ($s \geq 4$) which decomposes it into two perfect matchings.

1.64. **Theorem** [13] Let $C$ be a cycle having $(m_1 + m_2 + \ldots + m_k)$ edges with $k > 2$. There is a labelling that will produce a common-weight decomposition of $C$ into paths $P_{m_1+1}, \ldots, P_{m_k+1}$.

1.65. **Theorem** [32] Any minimum graphoidal path cover $\Psi$ of a tree $T$ is a common-weight decomposition of a suitable labelling of $T$.

1.66. **Theorem** [32] Let $G$ be a unicyclic graph with cycle $C = (v_1, v_2, \ldots, v_t, v_1)$. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_m}$ be the vertices of $C$ with degree greater than 2 where $1 \leq i_1 < i_2 < i_3 < \ldots < i_m = t$. If either $m = 0$ or $1$ or if the length of $(v_{i_j}, v_{i_{j+1}})$
-section of $C$ relatively prime to $t$ for atleast one $j$, then there exists a minimum graphoidal path cover $\Psi$ of $G$ such that $\Psi$ is a common-weight decomposition of a suitable labelling of the vertices of $G$.

The concept of path double cover was introduced by Bondy [16].

1.67. Definition : A path double cover of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that each edge of $G$ belongs to exactly two paths in $\mathcal{P}$.

1.68. Definition [32] The minimum cardinality of a path double cover of a graph $G$ is called path double covering number of $G$ and is denoted by $\eta_d$.

Throughout the thesis we use $\gamma_2$ instead of $\eta_d$.

1.69. Theorem [32] Let $G$ be a unicyclic graph with $n$ pendant vertices. Let $C = (v_1, v_2, \ldots, v_t, v_1)$ be the unique cycle in $G$. Let $m$ be the number of vertices of degree greater than 2 on $C$. Then

$$\eta_d(G) = \begin{cases} 
3 & \text{if } m=0 \\
n+2 & \text{if } m=1 \\
n+1 & \text{if } m=2 \\
n & \text{otherwise}
\end{cases}$$

1.70. Theorem [40] $Pn(K_{r(n)}) = \frac{1}{2} n(r-1)+1$, unless $r$ is even and $n$ odd, where $K_{r(n)}$ is the complete $r$-partite graph with each partite set containing $n$ vertices.

1.71. Definition : A directed graph or digraph $D$ is a finite non-empty set of objects called vertices together with a set of ordered pairs of distinct vertices
of $D$ called arcs or directed edges. The vertex set and the arc set of $D$ are denoted by $V(D)$ and $E(D)$ respectively. $|V(D)| = p$ is called the order of $D$ and $|E(D)| = q$ is called the size of $D$. If $e = (u,v) \in E(D)$, then $e$ is said to join $u$ and $v$. Also we say $e$ is incident from $u$ and is incident to $v$. Further we say that $u$ is adjacent to $v$ and $v$ is adjacent from $u$.

1.72. Definition: The in degree $id(v)$ of a vertex $v$ is the number of vertices which are adjacent to $v$. The out degree $od(v)$ is the number of vertices of $D$ that are adjacent from $v$.

The concepts of path and cycle for digraphs can be analogously defined as for graphs. The main difference is that the directions of the arcs must be followed in a path or a cycle.

1.73. Definition: A digraph $D$ is said to be symmetric if $(v,u)$ is an arc of $D$ whenever $(u,v)$ is an arc of $D$. If $G$ is a graph, then the symmetric digraph obtained by replacing each edge of $G$ by a symmetric pair of arcs is denoted by $G^*$. The symmetric digraph $K_p^*$ is called the complete symmetric digraph of order $p$. $D$ is called an anti-symmetric digraph or oriented digraph if $(v,u)$ is not an arc of $D$ whenever $(u,v)$ is an arc of $D$.

1.74. Definition: If $G = (V, E)$ is a graph, the oriented graph obtained by replacing each edge $uv$ of $G$ by an arc $uv$ or $vu$ (but not both) is called an orientation of $G$. 

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1.75. **Definition:** The underlying graph of a digraph $D$ is the graph $G$ obtained from $D$ by deleting all directions from the arcs of $D$ and deleting an edge from each pair of multiple edges if multiple edges are produced.

1.76. **Definition:** A directed tree is an anti-symmetric digraph whose underlying graph is a tree.

1.77. **Definition:** A vertex $v$ of a digraph $D$ is said to be reachable from a vertex $u$ if $D$ contains a $u$-$v$ path. $D$ is strongly connected or strong if for any two vertices of $D$ each vertex is reachable from the other.

1.78. **Definition** [38] Let $D = (V, E)$ be a graph. If for every $v \in V(D)$, $id(v) = od(v)$, the digraph is said to be an isograph.

1.79. **Definition:** A complete anti-symmetric digraph is called a tournament.

1.80. **Definition** [4] Define $dg(v) = od(v) - id(v)$, $x(v) = \max(0, dg(v))$ and $x(D) = \sum_{v \in V(D)} x(v)$. A digraph $D$ is said to be consistent if $x(D) = Pn(D)$.

1.81. **Theorem** [4] Suppose $D$ is a consistent digraph, $(v, w)$ is an arc of $D$, $dg_D(v) < 0$ and $dg_D(w) > 0$. If $H$ is the digraph obtained from $D$ by reversing $(v, w)$ then $H$ is consistent and $Pn(H) = 2 + Pn(D)$. 