1.1 With the appearance of Cauchy's *Analyse Algébrique* in 1821 and Abel's \(^1\) researches on the Binomial series in 1826, the old hazy notion of convergence of infinite series was put on sound foundation. It was, however, noticed that there were certain non-convergent series which were quite useful, and that operations performed on them uncritically often led to important results which could be verified independently. After persistent efforts in which a number of celebrated leading mathematicians took part, it was only in the closing decade of the last century and in the early years of the present century that satisfactory methods were devised so as to associate with them by processes closely connected with Cauchy's concept of convergence, certain values which may be called their "sums" in a reasonable way. Such processes of summation of series which were formerly tabooed being divergent, have given rise to the modern rigorous theory of Summability. Just as the concept of convergence gave rise to that of summability, so also, in more recent times, the notion of absolute convergence led to the formulation of another process called Absolute Summability.

The present thesis is based on certain investigations of the author into the theory and applications of absolute summability of infinite series. Before giving a résumé of the earlier researches in the light of which various new results have been obtained by the author, it seems desirable to state here the definitions and notations which will be required in the sequel.

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1) Abel, N.H. (1).
1.2 The most well-known processes of summability of divergent series, with which we are concerned in the present thesis, are either $T$-processes or $\varphi$-processes. As we know, the $T$-processes are based on the formation of the sequence $\{t_m\}$ of auxiliary means, defined by the sequence-to-sequence transformation:

\[(1.2.1) \quad t_m = \sum_{n} C_n t m \quad s_n \quad m = 1, 2, 3, \ldots,\]

where $s_n$ is the $n$-th partial sum of a given series $\sum a_n$. The matrix $|T| = (C_{m,n})$ in which $C_{m,n}$ is the element in the $m$-th row and $n$-th column, is usually a Toeplitz matrix.

The $\varphi$-processes are based upon the formation of a functional transform $t(x)$, defined either by a sequence-to-function transformation

\[(1.2.2) \quad t(x) = \sum_{n} \varphi_n(x) s_n,\]

or, more generally, by the transformation

\[(1.2.3) \quad t(x) = \int \varphi(x, y) s(y) dy,\]

where $x$ is a continuous parameter, and the function $\varphi_n(x)$ (or $\varphi(x,y)$) is defined over an appropriate interval of $x$ (or $x$ and $y$).

We know that a series $\sum a_n$ or the corresponding sequence $\{s_n\}$ is said to be convergent to the sum $s$, if $\lim_{n \to \infty} a_n = s$. By analogy, a series $\sum a_n$ is said to be summable by $T$-method (or $\varphi$-method) to the sum $s$ if $\lim_{m \to \infty} t_m = s \quad (\lim_{x \to a} t(x) = s)$, where $s$ is a suitable number.

A series $\sum a_n$ or the sequence $\{a_n\}$ is said to be absolutely convergent if the sequence $\{s_n\}$ is of bounded variation i.e.,

\[\sum |s_n - s_{n-1}| < \infty.\]

Defining similarly the absolute summability of an infinite series, we say that a series $\sum a_n$ is absolutely summable by a $T$-method or
simply summable $|T|$, if the corresponding auxiliary sequence $\{t_m\}$ is of bounded variation i.e.,

$$\Sigma |t_m - t_{m-1}| < \infty.$$  

Since $t_m \in BV \Rightarrow \lim_{m \to \infty} t_m = s$, it follows that if a series $\Sigma a_n$ is summable $|T|$ then it is also summable $(T)$.

Absolute summability by a $\phi$-method, or summability $|\phi|$ is defined in the same way with the obvious difference that in this case the corresponding function $t(x)$ should be a function of bounded variation in an interval of continuous parameter $x$.

1.3 The sequence - to - sequence transformation $T$ is said to be regular if $\lim_{n \to \infty} a_n = s \Rightarrow \lim_{m \to \infty} t_m = s$. In order that $T$-method may be regular, it is necessary and sufficient that

(i) $\Sigma |C_{m,n}| < H$, \hspace{1cm} $H$ is independent of $m$,

(ii) $\lim_{m \to \infty} C_{m,n} = 0$; \hspace{1cm} for each $n$ and

(iii) $\lim_{m \to \infty} \Sigma_{n=1}^{\infty} C_{m,n} = 1$.

Toeplitz proved this result for triangular matrix while extension to the general case is due to Steinhaus.

Similarly, the sequence - to - function transformation $\phi$ is said to be regular if $\lim_{n \to \infty} a_n = s \Rightarrow \lim_{x \to a} t(x) = s$. Regularity condition for this transformation are analogous to those of the $T$-transformation.

Thus if $x$ is a continuous parameter which tends to infinity and $t(x) = \Sigma \phi_n(x) a_n$, then the necessary and sufficient conditions that $t(x)$ should be defined by the above relation for $x \geq 0$ and

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1) see Zygmund, A. (161), p.74, or Hardy, G.H. (47), p.43.
2) Toeplitz, O. (151).
3) Steinhaus, H. (143).
lim t(x) = s whenever lim a_n = s, are
n→∞            n→∞

(i) \[ \sum_{n=1}^{\infty} |\varphi_n(x)| \text{ is convergent for } x \geq 0 \text{ and } \sum_{n=1}^{\infty} |\varphi_n(x)| < H, \]
where H is independent of x for x \geq x_0,

(ii) \[ \lim_{x \to \infty} \varphi_n(x) = 0 \text{ for every } n, \]

(iii) \[ \lim_{x \to \infty} \sum_{n=1}^{\infty} \varphi_n(x) = 1. \]

A real transformation T is said to be totally regular if \[ a_n \to s \Rightarrow t_n \to s \]
for all finite or infinite s.

The transformation T is said to be absolutely regular if T
is regular and the bounded variation of \( \{a_n\} \) implies the bounded
variation of \( \{t_n\} \).

The problem of absolute regularity of T-processes was for the
first time studied by Mears\(^2\) in 1937 and later on by Knopp and
Lorentz\(^3\), and Sunouchi\(^4\). Thus Mears showed that :

The necessary and sufficient conditions for the absolute
regularity of T-method defined by (1.2.1) are

(a) \[ \sum_{n=1}^{\infty} C_{n,k} \text{ converges for all values of } n, \]

(b) \[ \sum_{n=1}^{\infty} \sum_{p=k}^{\infty} (C_{n,p} - C_{n-1,p}) \leq C, \]
for all k, where C is an absolute constant.

One process of summability Q is said to be absolutely inclusive
of another process P symbolically, \( |P| \subset |Q| \) if absolute summability
by the P-process implies absolute summability by the Q-process. If
any two processes P and Q are absolutely inclusive of each other
they are said to be absolutely equivalent.

2) Mears, P.M. (110).
3) Knopp, K. and Lorentz, G.G. (64).
4) Sunouchi, G. (144).
1.4 ABSOLUTE ABEL SUMMABILITY. A series $\sum a_n$ is said to be absolutely summable (A) or summable $|A|$ if the series $\sum a_n x^n$ is convergent for $0 \leq x < 1$ and its sum-function $\Phi(x)$ is of bounded variation in $(0,1)$, that is to say, if

(1.4.1) $\int_0^1 |\Phi'(x)| \, dx < \infty$.

Analogous to Abel's classical theorem, we have the result due to Whittaker that absolute convergence implies summability $|A|$, i.e., summability $|A|$ is absolutely regular.

It was shown by Whittaker with the help of an example, suggested by Littlewood, that a Fourier series may converge at a point without being summable $|A|$ at that point, while Prasad constructed an example of a series which is summable $|A|$ at a point without being convergent at that point. This shows that convergence and summability $|A|$ are in general, mutually exclusive.

The definition of summability $|A|$ was later extended by Zygmund. According to him a series $\sum a_n$ is said to be summable $|A, \lambda|$ if the series $\sum a_n e^{-\lambda x}$ is convergent for all positive $x$ and the sum-function $f(x) = \sum a_n e^{-\lambda x}$ is of bounded variation in $(0, \infty)$, where $\{\lambda_n\}$ is a positive monotonic increasing sequence tending to infinity with $n$.

It is evident that summability $|A, n|$ is the same as the summability $|A|$.

Quite recently the concept of summability $|A|$ has been further extended by Flett in the following form:

3) Zygmund, A. (160).
4) Flett, T.N. (43).
A series \( \sum a_n \) is said to be summable \( |A|_k \) if \( k \geq 1 \), \( \gamma \) a real number, if the series \( \sum a_n x^k \) is convergent for \( 0 \leq x < 1 \) and its sum-function \( \phi(x) \) satisfies the condition

\[
\int_0^1 \frac{1}{(1-x)^k} \left| \phi'(x) \right| \, dx < \infty.
\]

(1.4.2)

For \( \gamma = 0 \) we get summability \( |A|_1 \) defined earlier by the same author.\(^1\) It is evident that summability \( |A|_1 \) is the same as the summability \( |A| \). However, so far as the inclusion relations are concerned, the summability \( |A|_k \) and the summability \( |A|_r \), \( k \neq r \), are independent of each other. It has been shown by Flett\(^2\) that for a fixed \( \gamma \), \( |A, \gamma|_r \not\subset |A, \gamma|_k \) for \( r > k \). The problem whether \( |A, \gamma|_k \subset |A, \gamma|_r \) remains still unanswered.

1.5 ABSOLUTE CESARO SUMMABILITY

The earliest definition of any special method of absolute summability was that of absolute Cesaro summability. Although the method was introduced by Fekete\(^3\) for integral orders in 1911, it was Kogbetliantz\(^4\) who defined this method for the general case, namely \( \alpha > -1 \), and investigated its properties in considerable details.

For any real \( \alpha \) and integers \( n \geq 0 \) we write

\[
\begin{align*}
A_n &= \frac{\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}, \quad n \geq 1, \\
A_0 &= 1, \\
A_{-n} &= 0, \quad n > 0.
\end{align*}
\]

The following are some of the important properties of the binomial coefficients \( A_n \) which will be required in the thesis:

\[
\begin{align*}
\Sigma \alpha A_n x^n &= (1-x)^{-\alpha-1}, \quad (|x| < 1) \\
\Sigma A_n \beta A_k &= A_{n+k}^\alpha.
\end{align*}
\]

(1.5.1) \hspace{1cm} (1.5.2)

\(\)\(^1\) Flett, T.M. (42) \hspace{1cm} \(\)\(^4\) Kogbetliantz, E.G. (65); (66)
\(\)\(^2\) Flett, T.M. (43) \hspace{1cm} \(\)\(^3\) Fekete, M. (38); (39) \hspace{1cm} \(\)
(1.5.3) \[ |a_n| \leq C(\alpha) n^\alpha, \]
(1.5.4) \[ a_n \geq C(\alpha) n^\alpha, \quad (\alpha > -1), \]
(1.5.5) \[ \sum_0^\infty |a_n| < \infty, \quad \alpha \leq -1. \]

Let \[ s_n^\alpha = \sum_{k=0}^n a_n - k s_k \quad \text{and} \quad \sigma_n^\alpha = \frac{s_n^\alpha}{a_n^\alpha}. \]

The sequence \( \{\sigma_n^\alpha\} \) is called the n-th Cesaro mean of order \( \alpha \) of the sequence \( \{a_n\} \). If \( \sigma_n^\alpha \to s \), as \( n \to \infty \), we say that the series \( \sum a_n \) or the sequence of its partial sums \( \{s_n\} \) is summable \( (C, \alpha) \) to \( s \).

The series \( \sum a_n \) is said to be absolutely summable by Cesaro means of order \( \alpha \) or simply summable \( (C, \alpha) \), \( \alpha > -1 \) if

\[ \sum |\sigma_n^\alpha - s - 1_n^\alpha| < \infty. \]

It is clear that summability \( (C, 0) \) is the same as absolute convergence and that \( (C, \alpha) \subset (C, \beta) \). It was however, demonstrated by Kogbetliants, with the help of an example, that, in general, absolute Cesaro summability of any order does not necessarily imply ordinary Cesaro summability of a lower order. 1) Concerning the consistency theorem for absolute Cesaro summability he proved that \( (C, \alpha) \subset (C, \beta) \), \( \beta > \alpha > -1 \). He also proved that if \( \beta > \alpha > 0 \) and \( \sum a_n \) is summable \( (C, \beta) \), then the \( (C, \alpha) \) transformed series of \( \sum a_n \) is summable \( (C, \beta - \alpha) \) and conversely.

In 1933 Pekete 2) proved that, for integral values of \( \alpha \), \( (C, \alpha) \subset |A| \). By means of a negative example he also demonstrated that summability \( |A| \) does not necessarily imply summability \( (C, r) \) for any positive \( r \) whatsoever. In conjunction with the consistency theorem for \( |C| \) summability his result leads to the conclusion that \( (C, \alpha) \subset |A| \) for every \( \alpha > -1 \).

1) Kogbetliants, E.G. (65).
2) Pekete, M. (41).
The definition of absolute Cesàro summability was recently extended by Flett. According to him a series \( \sum a_n \) is said to be summable \( |C, \alpha|_k, k \geq 1, \alpha > -1 \), if
\[
\sum_{n=1}^{k-1} \left| \frac{\alpha}{n} - \frac{\alpha}{n-1} \right| < \infty.
\]

Obviously, summability \( |C, \alpha|_1 \) is the same as the summability \( |C, \alpha|_k \). It is known that \( |C, \alpha|_k \subset |C, \beta|_r, k > 1, r \geq k \), \( \beta > \alpha + \frac{1}{k} - \frac{1}{r} \), \( \alpha > -1 \); for \( k = 1 \), the result is true whenever \( \alpha > -1 \) and \( \beta > \alpha + 1 - \frac{1}{r} \). However, if \( \beta < \alpha + \frac{1}{k} - \frac{1}{r} \), \( r > k \), then the above result breaks down. Also the converse of the above consistency theorem is not true, that is \( |C, \beta|_r \not\subset |C, \alpha|_k \) when \( r > k \) for any \( \alpha > -1 \).

Analogous to Fekete's result, Flett proved that a series \( \sum a_n \) is summable \( |A|_k \) whenever it is summable \( |C, \alpha|_k, k \geq 1, \alpha > -1 \).

The definition of the summability \( |C, \alpha|_k \) was subsequently further extended by Flett. Thus \( \sum a_n \) is said to be summable \( |C, \alpha|_k \), \( \alpha > -1, k \geq 1, \gamma \) a real number, if
\[
\sum_{n=1}^{k+\gamma-1} \left| \frac{\alpha}{n} - \frac{\alpha}{n-1} \right| < \infty.
\]
It is evident that \( |C, \alpha, 0|_k = |C, \alpha|_k \).

The Tauberian problems for the summability \( |C, \alpha|_k \) have been studied by the present author and Flett. Thus with a view to generalize the well known Tauberian theorem of Hyslop the author has recently proved that if \( \sum a_n \) is summable \( |A|_k, k \geq 1 \) and \( \sum \{ n a_n \} \) is summable \( |C, \alpha+1|_k, \alpha > -1 \), then \( \sum a_n \) is summable \( |C, \alpha|_k \). It may be observed that this result not only includes the

1) Flett, T.M. (42).
2) Flett, T.M. (42).
3) Flett, T.M. (43).
4) Hyslop, J.M. (57).
5) Waghar, S.M. (73). The same result was also almost simultaneously obtained by Flett (44). However, his method of proof is different from that of the author.
previous result of Hyslop for \( k = 1 \) but also, extends the range of order from \( \alpha > 0 \) to \( \alpha > -1 \).

### 1.6 ABSOLUTE RIESZ SUMMABILITY

Let \( \{\lambda_n\} \) be a positive and monotonic increasing sequence \( \alpha \to \infty \) with \( n \) and let \( \sum a_n \) be a given infinite series. We write \( A_n = a_1 + \ldots + a_n \) and if \( w > 0, \lambda_n < w \leq \lambda_{n+1} \), then \( A_\lambda(w) = A_n = A_\lambda^0(w) = \sum a_\lambda \). Also for \( \alpha > 0 \) we write

\[
A_\lambda^\alpha(w) = \sum_{\lambda_n < w} (w - \lambda_n)^\alpha a_\lambda = \int_0^w (w-t)^\alpha dA_\lambda(t).
\]

Let

\[
C_\lambda^\alpha(w) = \frac{A_\lambda^\alpha(w)}{w^\alpha}.
\]

Then \( C_\lambda^\alpha(w) \) is called the Riesz mean of 'order \( \alpha \)' and 'type \( \lambda_n \)' of the series \( \sum a_n \). If \( C_\lambda^\alpha(w) \) is a function of bounded variation in \( (h, \infty) \), that is to say

\[
\int_h^\infty |dC_\lambda^\alpha(w)| < \infty,
\]

the series \( \sum a_n \) is said to be absolutely summable \( (R, \lambda, \alpha) \), \( \alpha > 0 \) or simply summable \( |R, \lambda, \alpha| \), where \( h \) is a convenient positive number.

It is well known \(^2\) that summability \( |C, \alpha| \) is equivalent to the summability \( |R, n, \alpha| \), \( \alpha > 0 \).

The summability \( |R, \lambda, \alpha| \) was recently generalized by the author. According to him a series \( \sum a_n \) is said to be summable \( |R, \lambda, \alpha|_k, k \geq 1 \), \( \alpha > 0 \) and \( ak > 1 \), if the integral

\[
\int_h^{\infty} |w^{k-1}| \frac{d}{dw} C_\lambda^\alpha(w)\Big|_{w=h}^w k < \infty, \quad \frac{1}{k} + \frac{1}{k'} = 1.
\]

It has been shown by him that the additional restriction, namely

\[\]

---

1) Ubrechhoff, N. (128); (129).
2) Hyslop, J. M. (56).
3) Mashar, S. M. (74).
ak' > 1, is a necessary condition for the validity of the definition.

It is obvious that summability $|R, \lambda, \alpha|_k$ and the summability $|R, \lambda, \alpha|$ are the same for $\alpha > 0$. The author has obtained a number of interesting results concerning this method of summability. He has shown that the summability $|C, \alpha|_k$ is equivalent to the summability $|R, n, \alpha|_k$ for $\alpha > 0$, $ak' > 1$. He has also established the first and second theorems of consistency for the summability $|R, \lambda, \alpha|_k$. His results include for $\alpha > 0$, $ak' > 1$, a result of Flett\(^3\) and for $k = r = 1$, $\alpha > 0$, a theorem of Obrechkoff\(^4\), and also a result of Guha\(^5\) concerning the second theorem of consistency. In a recent paper he\(^6\) has also obtained a generalization of a theorem of Tatchell\(^7\) for integral values of $\alpha$.

The definition of summability $|R, \lambda, \alpha|_k$ was further extended by the author\(^8\). A series $\sum a_n$ is said to be summable $|R, \lambda, \alpha, \gamma|_k$, $k \geq 1$, $\alpha > 0$, $ak' > 1$, if the integral

$$\int_{\gamma} \left( \frac{d}{dw} C^{\alpha}(w) \right)^k dw$$

is convergence, $\gamma$ being a real number.

For $\gamma = 0$ this method of summability reduces to the summability $|R, \lambda, \alpha|_k$ defined earlier by the author. As in the case of summability $|R, \lambda, \alpha|_k$ the author has proved that if $\alpha > 0$, $ak' > 1$, $k \geq 1$ and $\alpha \gamma - 1/k$, then the summability $|R, n, \alpha, \gamma|_k$ and the summability $|C, \alpha, \gamma|_k$ are

1) Maghar, S. M. (100).
2) Maghar, S. M. (74); (78).
3) Flett, T. M. (142).
4) Obrechkoff, N. (129).
5) Guha, U. (46).
6) Maghar, S. M. (81).
7) Tatchell, J. B. (150).
8) Maghar, S. M. (76).
equivalent\(^1\). He has also proved the first theorem of consistency for such a method of summability which includes as a special case a result of Flett\(^2\) when \(\alpha > 0\), \(\alpha k' > 1\) and \(\alpha > \gamma > 0\).

1.7 **ABSOLUTE NÖRLUND SUMMABILITY**

Nörlund summability, though named after S.E. Nörlund, is essentially due to G.F. Woronoi, who in the year 1902 gave his definition of the summability of divergent infinite series and of divergent infinite integrals. Woronoi's result having been published in a rare Russian journal, did not attract attention for a long time. Except for a short review in *Jahrbuch über die Fortschritte der Mathematik* and a brief account in Kogbetliantz's monograph\(^4\) his work remained practically obscure till it was noticed and its English translation published by Tamarkin\(^5\) in 1932. Several years after the publication of Woronoi's result, Nörlund\(^6\) independently rediscovered the same method and proved several theorems on it.

Let \(\{p_n\}\) be a sequence of constants, real or complex and let us write \(p_n = p_0 + \ldots + p_n\), \(p_{-1} = p_{-1} = 0\). The sequence-to-sequence transformation

\[
t_n = \sum_{m=0}^{n} \frac{p_{n-m}s_m}{p_n}, \quad p_n \neq 0,
\]

defines the sequence \(\{t_n\}\) of Nörlund means of a sequence \(\{s_n\}\) generated by the sequence of coefficients \(\{p_n\}\). The series \(\sum a_n\) is said to be summable \((N, p_n)\) to \(s\) if \(\lim_{n \to \infty} t_n = s\) and is said to be absolutely summable \((N, p_n)\) or summable \(|N, p_n|\) if \(\sum |t_n - t_{n-1}| < \infty\). Necessary and

---

1) Mashar, S.M. (100). The same result was also obtained independently by D. Borwein (12).
2) Flett, T.M. (43).
4) Kogbetliantz, E.G. (66).
5) Tamarkin, J.D. (149).
7) Mears, F.M. (109).
sufficient conditions for the absolute regularity of a method are

\[(i) \frac{p_n}{n} \to 0, \quad n \to \infty, \quad (ii) \sum_{n=m}^{\infty} \left| \frac{p_{n+m} - p_{n+m-1}}{p_n - p_{n-1}} \right| < \infty,\]

for all positive integral values of \(m\).

In the special case in which \(p_n = \left( \frac{n+\alpha-1}{\alpha-1} \right), \alpha > 0\), the Nörlund mean is the familiar \((C, \alpha)\) mean, and the summability \(|N, p_n|\) is the same as the summability \(|C, \alpha|\). On the other hand, if \(p_n = \frac{1}{n+1}\), the summability \(|N, p_n|\) is equivalent to the method known as absolute harmonic summability.

If \(p_n > 0\) and \(p_n - p_{n+1} > 0\) the above conditions of absolute regularity are satisfied. Hence, in particular, summability \(|N, \frac{1}{n+1}|\) is absolutely regular. Also, it is known\(^2\) that \(|N, \frac{1}{n+1}| \subset |C, \delta|, \delta > 0\) but the converse need not be true.

1.8 **SUMMABILITY** \(|N, p_n|\). Let \(p_n > 0, p_0 > 0\) and \(p_n = \sum_{k=0}^{n} p_k\) tend to infinity with \(n\). The sequence-to-sequence transformation

\[T_n = \frac{1}{p_n} \sum_{m=0}^{n} p_m s_m,\]

where \(s_m\) is the \(m\)-th partial sum of a given infinite series \(\sum a_n\), defines the \((N, p_n)\) means of the sequence \(\{s_n\}\). The series \(\sum a_n\) is said to be summable \((N, p_n)\) to \(s\) if \(\lim_{n \to \infty} T_n = s\). It is easy to see that this method is regular\(^3\).

The series \(\sum a_n\) is said to be summable \(|N, p_n|\) if \(\{T_n\}\) is a sequence of bounded variation, that is \(\sum|T_n - T_{n-1}| < \infty\). It follows from a known result of Mohanty\(^4\) that summability \(|N, p_n|\) is absolutely regular. If \(p_n = \frac{1}{n+1}\), the method of summability \(|N, p_n|\) is equivalent

1) Mears, F.M. (110); Peyerimhoff, A. (136).
2) Mc Fadden, L. (108).
4) Mohanty, R. (116).
to the well known summability method $|R, \log n, 1|$. It is known $^2$ that summability $|\overline{N}, p_n|$ is equivalent to the Riesz summability $|R, p_n, 1|$. 

1.9 $|L|$ SUMMABILITY

A sequence $\{a_n\}$ is said to be summable (L) if

$$F(x) = \frac{-1}{\log(1-x)} \sum_{n=1}^{\infty} \frac{a_n x^n}{n}$$

tends to a finite limit as $x \to 1$ in the open interval $(0, 1)$. 

If $F(x) \in BV(\delta, 1)$ for some $\delta$, $0 < \delta < 1$, then we say that the sequence $\{a_n\}$ is summable $|L|^3$.

Concerning inclusion relation between the summability $|A|$ and the summability $|L|$, Mohanty and Patnai have proved that $|A| \subseteq |L|$. Thus it follows that $|C, \delta| \subseteq |A| \subseteq |L|$, $\delta > 0$.

There are similarly other processes of absolute summability but, since in what follows, we do not require their use, we do not propose to deal with them here.

1.10 ABSOLUTE SUMMABILITY FACTORS.

Given the series $\sum a_n$, if a series $\sum a_n c_n$ is summable in some sense while, in general, $\sum a_n$ itself not so summable, then $\{c_n\}$ is said to be a summability factor of the series $\sum a_n$. If the summability in question is absolute, the factors are naturally called absolute summability factors.

1) A sequence $\{a_n\}$ is said to be summable $R, \log n, 1$ if $\frac{\sum_{n=1}^{\infty} \frac{a_n c_n}{k+1}}{\log(n+1)} \sim \frac{1}{R, \lambda, 1}$ for $\lambda_n = \log n$.

2) See Mohanty, R. (118).


It was Fekete who initiated the work on absolute summability factors in 1917. He obtained necessary and sufficient conditions for \( \sum a_n e_n \) to be summable \(|C, \alpha|\) whenever \( \sum a_n \) is summable \(|C, \alpha|\), \( \alpha \) being a positive integer. In 1925, Kogbetliants proved that if \( \sum a_n \) is summable \(|C, \delta|\), \( \delta > 0 \), then the series \( \sum a_n (n+1)^{-\gamma} \), \( 0 < \gamma \leq \delta \) is summable \(|C, \delta-\gamma|\). Sunouchi extended the scope of this theorem by relaxing the condition \( \gamma \leq \delta \). Peyerimhoff supplied simpler proof of the above theorem of Kogbetliants. The work of Fekete and Kogbetliants was carried further by Bosanquet who extended the results of Fekete and established, among others, the following theorem.

**Theorem A.** If \( \alpha \) and \( \beta \) are integers and \( 0 \leq \alpha \leq \beta \), necessary and sufficient conditions that \( \sum a_n e_n \) should be summable \(|C, \alpha|\) whenever \( \sum a_n \) is summable \(|C, \beta|\) are

\[
(1.10.1) \quad e_n = O\left( n^{\alpha-\beta} \right),
\]

\[
(1.10.2) \quad \Delta^\beta e_n = O\left( n^{-\beta} \right).
\]

If \( \alpha > \beta \geq 0 \) the conditions are (1.10.2) and

\[
(1.10.3) \quad e_n = O(1).
\]

Peyerimhoff extended the above theorem for all non-negative values of \( \alpha \) and \( \beta \) by establishing a theorem, the sufficiency part of

---

1) Fekete, M. (40).
2) Kogbetliants, E. G. (65), Theorem VI.
3) Sunouchi, G. (145).
4) Peyerimhoff, A. (132).
5) Bosanquet, L. S. (17).
6) For real \( \alpha \) and integers \( n \geq 0 \) we write

\[
\Delta^\alpha u_n = \sum_{m=n}^{\infty} a_{m-n} u_m
\]

whenever the series is convergent.
7) Peyerimhoff, A. (133).
which was stated by Andersen 1). Chow 2) generalized the theorem of Peyerimhoff by proving the following:

Theorem B. If $0 < \beta < \alpha$ and $\left\{ \frac{\lambda_n}{n} \right\}$ is a sequence of positive numbers such that $\left\{ \frac{\lambda_n}{n} \right\}$ is non-increasing, necessary and sufficient conditions that $\sum a_n e_n$ should be summable $|C, \beta|$ whenever $\sum \frac{\lambda_n}{n} \left| t_n^\alpha \right| < \infty$ are

\[(1.10.4) \quad \Delta \left( \frac{e_n}{n^k} \right) = O \left( n^{\alpha-1} \lambda_n \right), \]

\[(1.10.5) \quad e_n = O \left( n^{\beta-\alpha} \lambda_n \right). \]

If $\beta > \alpha > 0$ and $\left\{ \frac{\lambda_n}{n^\delta} \right\}$ is a sequence of positive numbers such that $\left\{ \frac{\lambda_n}{n^\delta} \right\}$ is non-increasing for some $\delta$ such that $0 < \delta < 1$, then the conditions are (1.10.4) and

\[(1.10.6) \quad e_n = O \left( \lambda_n \right). \]

If we consider the summability $|C, \alpha|_k$ it is natural to enquire whether it is possible to obtain a generalization of the above theorem of Chow for such a method of summability. We answer this question in the affirmative in Section B of Chapter III of the present thesis. Our main result is as follows.

Theorem 1. Let $\alpha > 0$, $\beta > 0$ and $\left\{ \frac{\lambda_n}{n} \right\}$ be a sequence of positive numbers such that $\left\{ \frac{\lambda_n}{n} \right\}$ is non-increasing. The necessary and sufficient conditions that $\sum a_n e_n$ be summable $|C, \beta|$ whenever $\sum \frac{\lambda_n}{n} \left| t_n^\alpha \right| < \infty$, $k > 1$ are

\[(1.10.7) \quad \left\{ n^{\alpha+1-1/k'} \Delta \left( \frac{e_n}{n^k} \right) \right\} \epsilon \left( \frac{k'}{1/k} \right)^k, 1/k + 1/k' = 1, 3 \]

1) Andersen, A.F. (3).
2) Chow, H.C. (31).
3) We write $\left\{ a_n \right\} \epsilon^k$ if $\sum |a_n|^k < \infty$, $1 \leq k < \infty$ and $\left\{ a_n \right\} \epsilon$ if $a_n = O(1)$. 
For $k = 1$ and $\beta \leq \alpha$ the conditions are (1.10.7) and (1.10.8).

If $\beta > \alpha$, $k = 1$ and $\left\{ \lambda_n \right\}$ is a positive sequence such that $\left\{ \lambda_n / n^\delta \right\}$ is non-increasing for some $\delta$, $0 < \delta < 1$, then the necessary and sufficient conditions are (1.10.7) and (1.10.9).

It may be remarked that our theorem includes, as a special case, for $k = 1$ the above theorem of Chow. On the other hand, if we take $\lambda_n = 1$ we deduce a theorem of Mehdi 1) for $k > 1$.

It may be further observed that the conditions for the summability $\left| C, \alpha \right|_k$, $k > 1$ (take $\lambda_n = 1$) and the summability $\left| C, \alpha \right|$ (i.e. for $k = 1$) are altogether different. However there is an underlying unity between them. The two sets of conditions can be expressed in a unified manner as stated above.

There is another significant difference between the conditions on $\left\{ \lambda_n \right\}$ for $k = 1$ and $k > 1$. In the case of $k = 1$ two different conditions on $\left\{ \lambda_n \right\}$ have to be assumed while for $k > 1$ we need only assume that $\left\{ \lambda_n / n^\delta \right\}$ is non-increasing. Thus we notice that our theorem for $k > 1$ is of more compact type.

In an attempt to extend a theorem of Cheng 2) on absolute Cesàro summability factors of Fourier Series, Sunouchi 3) proved the following theorem for absolute Cesàro summability factors of infinite series.

---

2) Cheng, M.T. (26).
3) Sunouchi, G. (1146).
Theorem C. If
\[ \sum_{n=1}^{m} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right| = O(\log m), \quad m \to \infty, \]
then \( \sum a_n \left\{ \log (n+1) \right\}^{1-\frac{1}{\alpha}} (\varepsilon > 0) \) is summable \( \mathbb{C}, \alpha \), \( \alpha \geq 0 \).

This result was subsequently extended by Dikshit \(^1\) who replaced the term "log m" by a wider class of sequences \( \{\lambda_n\} \). His result is as follows:

\textbf{Theorem D.} Let \( \alpha > 0 \). If
\[ \sum_{n=1}^{m} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right| = O(\lambda_n), \quad m \to \infty, \]
where \( \{\lambda_n\} \) is a positive monotonic non-decreasing sequence, then \( \sum a_n \varepsilon_n \) is summable \( \mathbb{C}, \alpha \), provided
\[ (1.10.10) \quad \sum \lambda_n^h \varepsilon_n \left\{ \Delta \varepsilon_n \right\} < \infty, \]
\[ (1.10.11) \quad \varepsilon_n \lambda_n = O(1), \quad n \Delta \lambda_n = O(\lambda_n), \]
where \( h \) is the least integer not less than \( \alpha \), and when \( \alpha \) is non-integral,
\[ (1.10.12) \quad \left| \Delta \varepsilon_n \right| \text{ is monotonic non-increasing.} \]

Quite recently these results have been generalized for integral values of \( \alpha \) by Srivastava. \(^2\) He has shown that if \( \beta > -1 \) then under the hypotheses of Theorem D with the condition (1.10.11) replaced by
\[ (1.10.13) \quad \sum \lambda_n^\alpha \left| \Delta \varepsilon_n \right| < \infty, \quad \varepsilon_n = o(1), \]
the series \( \sum a_n \varepsilon_n \) is summable \( \mathbb{C}, \beta \).

---

2) Srivastava, V.P. (142).
In Section B of Chapter V, we have studied the corresponding problem for the summability \( |C, \beta|_k \) which is the converse of the problem \( (\lambda_n = 1) \) discussed in Section B of Chapter III. For \( k = 1 \) our result includes the above theorem of Srivastava. On the other hand when \( k > 1 \), it may be pointed out that in our theorem conditions for integral and non-integral values of \( \beta \) are not the same. In the latter case one of the conditions contains an additional term \( (\log n)^{1/k} \). We have shown by means of an example that such a restriction is essential for the validity of our theorem. Thus we get one of the very few known instances of summability factor theorems which demonstrate that the conditions for integral and non-integral values of the order of absolute Cesaro summability need not always be the same.

Following the lines of proof of our theorem we have obtained necessary and sufficient conditions in order that \( \Sigma a_n x_n \) should be summable \( |C, \beta|_k, \beta > -1 \) whenever \( \Sigma a_n \) is summable \( |C, \alpha| \). This result extends a theorem of Mehdi \(^1\) who proved the result only for \( \beta > 0 \).

We have also studied its application to Fourier series. By applying a result of Chen \(^2\) we have deduced a criterion for the \( |C, \beta|_k \) summability factors of Fourier series.

Taking inspirations from Theorem C of Sunouchi, Pati \(^3\) has recently established the following summability factor theorem for an infinite series:

\(^1\) Mehdi, M.R. (111).
\(^2\) Chen, K.K. (23).
\(^3\) Pati, T. (131).
Theorem 1. If \( \{ \varepsilon_n \} \) is a convex \(^1\) sequence such that
\[
\sum \frac{\varepsilon_n}{n} < \infty
\]
then the series \( \sum a_n \varepsilon_n \) is summable \(|0, 1|\).

This result was subsequently generalized by the author\(^2\) who proved:

Theorem 2. If
\[
\sum \varepsilon_n \geq 0, \quad \sum \frac{\varepsilon_n}{n} < \infty,
\]
and
\[
\sum \frac{|s_v|}{v} = O(\log m), \quad m \to \infty, \quad k \geq 1,
\]
then the series \( \sum a_n \varepsilon_n \) is summable \(|0, 1|_k\).

In Section 3 of Chapter V we have further generalized the above theorem by proving the following more general result.

Theorem 3. If
\[
\begin{cases}
(i) \varepsilon_n = o(1), \\
(ii) \sum \frac{n \log n \varepsilon_n}{1} = O(1),
\end{cases}
\]
and
\[
\sum \frac{|t_n|}{n} = O(\log m), \quad m \to \infty, \quad k \geq 1,
\]
1) \( \{ \varepsilon_n \} \) is said to be convex if \( \Delta \varepsilon_n \geq 0 \).
2) Mishah, S.M. (85); see also Mishra, B.P. (114).
then \( \sum a_n c_n \) is summable \( |C, 1| \), where \( \frac{1}{t_n} \) is the n-th \( (C, 1) \) mean of \( \{a_n\} \).

We have shown that the conditions (1.10.16) and (1.10.17) are lighter than the corresponding conditions (1.10.15)(a) and (1.10.15)(b). It has also been demonstrated by means of an example, that our result is the best possible in the sense that summability \( |C, 1| \) cannot be replaced by the summability \( |C, \alpha| \) for any \( \alpha \) such that \( 0 < \alpha < 1 \). (It is known that \( |C, \alpha| \subset |C, \beta| \), \( \beta > \alpha \), \( k > 1 \), see \( \S 1.5 \)).

Quite recently, work has been started in investigating summability factors for absolute Horlund summability. Concerning the special methods, namely that of absolute harmonic summability, Lal \(^1\) proved the following theorem:

**Theorem G.** If \( \{c_n\} \) is a convex sequence such that

\[
\frac{c_n}{n} < \infty \quad \text{and} \quad \sum_{l}^{m} \frac{c_v}{l} = O(\log m), \quad m \to \infty,
\]

then \( \sum \frac{a_n c_n \log(n+1)}{n} \) is summable \( |N, \frac{1}{n+1}| \).

Later on, Singh \(^2\) established the following more general result:

**Theorem H.** If \( \sum a_n \) is summable \( |C, 1| \), then

\[
\sum \frac{a_n \log(n+1)}{n} \text{ is summable } |N, \frac{1}{n+1}|.
\]

It may be remarked that Theorem G is a corollary of Theorem H if we appeal to Theorem E of Pati.

\(^1\) Lal, S.N. (66).
\(^2\) Singh, T. (141).
Recently, Nand Kishore 1) has obtained a theorem for absolute Nörlund summability factors which includes Theorem H as a special case for $p_n = \frac{1}{n+1}$. His result has been subsequently generalized by the author whose results are of necessary and sufficient type. In one paper 2) he has obtained necessary and sufficient conditions for $\sum a_n e_n$ to be summable $|N, p_n|$ whenever $\sum a_n$ is summable $|C, 1|$, while in another paper he has considered the summability $|N, p_n|$ of $\sum a_n e_n$ whenever $\sum a_n$ is summable $|N, q_n|$. 3) A number of known as well as new results can be deduced from his latter result.

Theorem H of Singh has been recently generalized by Das, Srivastava and Mohapatra 4) in the following form:

**Theorem I.** Necessary and sufficient conditions that $\sum a_n e_n$ should be summable $|N, \frac{1}{n+1}|$ whenever $\sum a_n$ is summable $|C, 1|$ are

(i) $e_n = O\left(\frac{\log n}{n}\right)$, (ii) $\Delta e_n = O\left(\frac{1}{n}\right)$. 5)

In Chapter IV we consider a more general problem. Instead of absolute harmonic summability we consider absolute Nörlund summability and replace summability $|C, 1|$ by the summability $|C, \alpha|_k$, $\alpha \geq 0$ (integers) $k \geq 1$. We first obtain necessary and sufficient conditions such that $\sum a_n e_n$ is summable $|N, p_n|$.

1) Nand Kishore (125).
2) Mashar, S.M. (102).
4) Das, G., Srivastava, V.P. and Mohapatra, R.N. (35).
5) In a joint note with Dr. G. Das, Professor L.S. Bosanquet has succeeded in obtaining the result for the summability $|C, \alpha|$. Their note is yet to appear.
whenever $\sum a_n$ is summable $[C, \alpha]_k$, $\alpha \geq 0$, $k > 1$. From this result we deduce the following:

**Theorem 3.** Suppose that $\alpha > 0$ (integer) and $\{p_n\}$ is non-negative and non-increasing. Necessary and sufficient conditions that $\sum a_n \epsilon_n$ should be summable $[N, p_n]$ whenever $\sum a_n$ is summable $[C, \alpha]_k$, $k > 1$ are

\[
\begin{align*}
(1.10.18) \quad & \left\{ \frac{\alpha - 1}{k'} \epsilon_n \right\} \epsilon [k'] \\
(1.10.19) \quad & \left\{ \frac{\alpha + 1}{k}, \alpha \epsilon \left( \frac{\epsilon_n}{n} \right) \right\} \epsilon [k'].
\end{align*}
\]

By taking $p_n = \frac{1}{n+1}$ we obtain a criterion for $[N, \frac{1}{n+1}]$ summability factors. On the other hand, if we choose $p_0 = 1$, $p_n = 0$, $n \geq 1$ we obtain necessary and sufficient conditions in order that $\sum a_n \epsilon_n$ be absolutely convergent whenever $\sum a_n$ is summable $[C, \alpha]_k$. It may be further remarked that our theorem of Chapter IV is the only known result of its kind.

It is easy to see that if a sequence $\{s_n\}$ is bounded, then it need not be summable $[R, \log n, 1]$. In this connection Bhatt \(^1\) obtained the following summability factor theorem.

**Theorem J.** If $\{\epsilon_n\}$ is a convex sequence such that

$\sum \frac{\epsilon_n}{n} < -$ and the sequence $\{s_n\}$ is bounded then the series $\sum a_n \epsilon_n \log n$ is summable $[R, \log n, 1]$.

Later on he \(^2\) succeeded in generalizing his theorem in the following form:

\[^1\] Bhatt, S.N. (8)
\[^2\] Bhatt, S.N. (9)
Theorem K. If \( \{ e_n \} \) is a convex sequence such that
\[
\sum \frac{e_n}{n} < \infty \quad \text{and the sequence } \{ k_n \}, \text{the } (R, \log n, 1) \text{ mean of the sequence } \{ n a_n \log (n+1) \}, \text{satisfies the condition}
\]
(1.10.20) \( |k_n| = o\left( \left( \frac{\log(n+1)}{k} \right)^{(c,1)} \right) \) \( k \geq 0 \),

then the series \( \sum a_n e_n \left( \log(n+1) \right)^{1-k} \) is summable \( (R, \log n, 1) \).

In Chapter II we prove a general theorem for the summability \( \{ n, p_n \} \) which includes Theorem K as a special case for \( p_n = \frac{1}{n+1} \). Our theorem also generalizes a theorem of Prasad and Bhatt,\(^1\) and a theorem of Chen.\(^2\) A recent result of Tripathi \(^3\) can also be deduced as a corollary from our result.

It will be of interest to mention that the proof of our theorem depends mainly upon the following lemma of the author.

"If \( \{ e_n \} \) is a convex sequence such that \( \sum p_n e_n < \infty \), where \( p_n > 0, p_n \rightarrow \infty \), then \( \{ e_n \} \) is a non-negative monotonic \( \text{decreasing} \)
sequence tending to zero and \( e_n p_n = o(1), n \rightarrow \infty \)."

This result, which includes as a special case a result of Chow,\(^4\) has been found quite useful in the theory of summability factors.

It is well known that, in general, summability \( \{ n, p_n \} \)
and the summability \( \{ C, a \}, k > 1 \) are independent of each other. This suggests the consideration of two types of
summability factor problems. Type I. To find suitable
summability factors \( \{ e_n \} \) such that \( \sum a_n e_n \) is summable \( \{ n, p_n \} \).

---

1) Prasad, B.N. and Bhatt, S.N. (138).
3) Tripathi, L.M. (153).
4) Chow, H.C. (29).
whenever $\Sigma a_n$ is summable $|C, \alpha|k$. Type II (converse).

To obtain suitable summability factors $\{E_n\}$ such that $\Sigma a_n E_n$ is summable $|C, \alpha|k$ whenever $\Sigma a_n$ is summable $|N, p_n|$.

In Section A of Chapter III we consider the problem of type I. Our main result is as follows:

Theorem 4. The necessary and sufficient conditions for the series $\Sigma a_n E_n$ to be summable $|N, p_n|$ whenever $\Sigma a_n$ is summable $|C, \alpha|k$, $\alpha \geq 0$, $k \geq 1$, are

(i) $\left\{ a + \frac{1}{k} - \frac{1}{k'} \right\} \frac{a}{\triangle (\frac{E_n}{n})} \in \ell$, 

(ii) (a) $\left\{ n - \frac{1}{k'} \right\} E_n \in \ell$, $0 \leq \alpha \leq 1$,

(ii) (b) $\left\{ n - \frac{1}{k'} \frac{E_n}{p_n} \right\} \frac{p_n}{p_n} \in \ell$, $\alpha > 1$.

where (a) $p_n = \circ (p_n + 1)$, (b) $(n+1) p_n = \circ (P_n)$, and (c) $P_n = \circ (n^\alpha p_n)$, $\alpha > 1$.

It may be remarked that Theorem 4 includes as a special case for $k = 1$ a theorem of Mohapatra 1) while for $p_n = 1$, a theorem of Mehdi. 2) On taking $p_n = \frac{1}{n+1}$, we deduce an interesting theorem concerning $|R, \log n, 1|$ summability factors of an infinite series.

1) Mohapatra, R.N. (123).
The problem of type II has been discussed in Section A of Chapter V. Our main result includes a theorem of Mohapatra \(^1\) and also a particular case of a theorem of Mehdi. \(^2\) A number of new results can also be obtained as corollaries from our theorem.

1.11 ABSOLUTE SUMMABILITY OF INFINITE SERIES

Concerning absolute Cesàro summability of an infinite series, Chow \(^3\) obtained the following theorem which is an analogue of a theorem of Hardy and Littlewood \(^4\) for summability \((C, \alpha)\).

Theorem L. In order that \(\sum a_n\) should be summable \(\mid C, \alpha+1\mid\), \(\alpha \geq 0\) it is necessary and sufficient that the series \(\sum b_n\), where

\[
b_n = \sum_{v=n}^{\infty} \frac{a_v}{v+1} (C, \alpha)
\]

should be summable \(\mid C, \alpha\mid\).

We devote Chapter VI of the present thesis to the study of a generalization of the above theorem. Our result not only includes the above theorem of Chow but also extends the range of \(\alpha\) from \(\alpha \geq 0\) to \(\alpha > -1\).

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1) Mohapatra, R.N. (122).  
3) Chow, H.C. (28).  
4) Hardy, G.H. and Littlewood, J.L. (48).
1.12 ABSOLUTE SUMMABILITY OF FOURIER SERIES AND ASSOCIATED SERIES

Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the Fourier series of $f(t)$ be defined by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

Then the conjugate series of the Fourier series of $f(t)$ is given by

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).$$

In 1914 Bernstein\(^1\) proved that if $f(x) \in \text{Lip} \alpha$ in $(0, 2\pi)$, then the Fourier series of $f(x)$ converges absolutely for $\alpha > 1/2$, but not necessarily when $\alpha \leq 1/2$. Later on, in 1928, Zygmund\(^2\) established that if $f(x) \in BV(0, \pi)$ and $f(x) \in \text{Lip} \alpha$ $(0 < \alpha < 1)$, then the Fourier series of $f(x)$ is absolutely convergent. He also demonstrated, by means of an example, that bounded variation of $f(x)$ alone is not sufficient to ensure the absolute convergence of the Fourier series of $f(x)$. These theorems of Bernstein and Zygmund were subsequently generalized and improved in various ways by a number of workers like Szasz\(^3\), Hardy and Littlewood\(^4\), Tomic\(^5\) and others. Noble\(^6\) applied these ideas to the study of the absolute convergence of lacunary Fourier series, some of whose results were, later on, generalized by the author\(^7\).

Izumi\(^8\) has recently obtained a number of theorems on the absolute convergence of Fourier series in which he imposes certain conditions on the behaviour of the Fourier coefficients. A typical result of his is as follows.

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1) Bernstein, S. (4); (5).
2) Zygmund, A. (159).
3) Szasz, O. (147); (148).
4) Hardy, G.H. and Littlewood, J.E. (51).
5) Tomic, M. (152).
6) Noble, M.E. (126).
7) Mazhar, S.M. (75).
8) Izumi, S. (60).
Theorem N. Let \( f(t) \sim \sum_{n} a_n \cos nt \). If

\[
\int_0^\infty \log \frac{2\pi}{t} |f(t)| < \infty \quad \text{and} \quad \sum_{n} \frac{\delta}{n \Delta (n a_n)} \in \mathcal{BV}, \quad \text{for some } \delta > 0,
\]

then \( \sum |a_n| < \infty \).

In chapter XII we have proved from theorems on the absolute convergence of Fourier series which generalize a number of results of Izumi. It may be remarked that our new conditions are not only lighter than the corresponding conditions of Izumi but also necessary for the absolute convergence of the corresponding series. Thus, for example in the above theorem (Theorem N) we have succeeded in replacing the condition (1.12.2) by a lighter condition, namely

\[
\left\{ e^{-\alpha x} \sum_{n} e^{\alpha n} a_n \right\} \in \mathcal{BV}, \quad 0 < \alpha < 1,
\]

which is also necessary for the convergence of the series \( \sum |a_n| \).

It may be further remarked that our technique of proof is entirely different from that of Izumi. We first study absolute Riesz summability of certain type and order 1 and then apply a well known Tauberian theorem to deduce our results.

The earliest applications of absolute summability to Fourier series are due to Whittaker \(^2\) and Prasad. \(^3\) Prasad proved inter alia, that Jordan's sufficient condition for the convergence of a Fourier series at a point also suffices to ensure its summability \( |A| \). His results have been further improved by a number of workers.

\(^{1)}\) Mashar, S.M. (91).
\(^{2)}\) Whittaker, J.M. (155).
\(^{3)}\) Prasad, B.N. (137).
Quite recently Mohanty and Patnaik \(^3\) introduced the definition of summability \(|L|\) and studied its application. Their theorem on \(|L|\) summability of Fourier series is as follows:

**Theorem 1.** If \(X(t) \in L(0, \pi)\) then \(\sum A_n(x)\) is summable 

\[
\frac{1}{t \log \frac{k}{t}}
\]

where \(\phi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) - 2s \right\},\)

\[\chi(t) = \int \phi(u) \cot \frac{u}{L} \, du\]

and \(k\) is a constant greater than \(\pi\).

Chapter XI of this thesis deals with the generalized \(|L|\) summability of Fourier series.

4) We shall say that a sequence \(\{s_n\}\) is summable \(|L|_k, k \geq 1\) if

\[
1 \int_0^1 (1-x)^{k-1} |\frac{d}{dx} F(x)|^k \, dx < \infty, \quad k \geq 1,
\]

where

\[F(x) = \frac{-1}{\log(1-x)} = \sum_{1}^{\infty} \frac{s_n x^n}{n}\]

and \(\delta\) is some positive number such that \(0 < \delta < 1\).

---

2) Bosanquet, L.S. (13).
4) Mazhar, S.M. (103).
It is clear that summability $|L|_1$ is the same as the summability $|L|$. Concerning inclusion relations between summability $|L|_k$ and summability $|L|_r$ we have proved (Theorem 2) that they are mutually exclusive for $k \neq r$. Thus, in particular, it follows that $|L|_k$, $k > 1$ is independent of the summability $|L|$. Regarding inclusion relation between the summability $|L|_k$ and the summability $|A|_k$ we have shown (Theorem 1) that $|A|_k \subseteq |L|_k$, $k \geq 1$. This generalizes a result of Mohanty and Patnaik. Using a result of Flett we deduce that $|C, \alpha|_k \subseteq |L|_k$, $k \geq 1$, $\alpha > -1$. Also, if $k \neq r$ it follows from our result that $|L|_k \cap |A|_r$.

Finally, we have studied an application of summability $|L|_k$ to Fourier series. It may be remarked that our theorem (Theorem 3) generalizes the above theorem of Mohanty and Patnaik (Theorem N).

In 1936 Bosanquet 1) carried further the work of Prasad, Misra and his own and established general theorems for $|C|$, summability of Fourier series analogous to known results of Hardy and Littlewood concerning ordinary Cesàro summability. He proved that if $\phi_\alpha(t) \in BV, (0, \pi), \alpha \geq 0$ then $\Sigma A_n(x)$ is summable $|C, \beta|, \beta > \alpha$ and conversely, if $\Sigma A_n(x)$ is summable $|C, \alpha|$, the $\phi \beta(t) \in BV (0, \pi)$ for every $\beta > \alpha+1$, where $\phi_\alpha(t) = \alpha t^{-\alpha} \int_0^t (t-u)^{\alpha-1} \phi (u) du$, $\alpha > 0$, $\phi_0(t) = \phi (t) = \frac{1}{2} \{f(x+t) + f(x-t)\}$. Since a Lebesgue integral is absolutely continuous and, therefore, for $\alpha \geq 1$, $\phi_\alpha(t) \in BV$ in every range $(\eta, \pi)$, $\eta > 0$, it is evident from Bosanquet's result that the summability $|C, \beta|, \beta > 1$ of a Fourier series at a point depends only upon the behaviour of the generating function.

1) Bosanquet, L.S. (14); (15).
2) Hardy, G.H. and Littlewood, J.E. (48); (50).
in the immediate neighbourhood of the point under consideration. On the other hand, it was later found \(1)\) that the summability of the Fourier series is not a local property in that it depends upon the behaviour of the generating function in the entire interval of the definition. Mohanty \(2)\), Bhatt \(3)\), and Jurkat and Peyerimhoff \(4)\) obtained conditions in terms of \(A_n(x)\) in order that the \(|C|\) summability of Fourier series may be ensured by a local condition.

From a result of Hyslop \(5)\) it follows that the summability of the \(r\)th derived series of a Fourier series is a local property. Considering the case \(r = 1\), Lal \(6)\) has recently shown that summability \(|C, 2|\) of the first derived series of a Fourier series is not a local property. He has also obtained a condition in terms of \(A_n(x)\) in order that it may become of local character. His results were quite recently extended to \(r\)-th derived series by Bhatt \(7)\) who proved that summability \(|C, r+1|\) of the \(r\)-th derived series of a Fourier series is not a local property. and that on assuming suitable conditions in terms of \(A_n(x)\) and \(B_n(x)\) the following theorem holds.

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3) Bhatt, S.N. (7).
6) Lal, S.N. (67)
7) Bhatt, S.N. (10).
Theorem 0. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over $(0, 2\pi)$. Then the summability $|C, r+l|$ of the $r$-th derived series of a Fourier series depends only on the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point $t = x$, if, when $r$ is even, \( \sum \frac{A_n(x)}{n} < \infty \) and when $r$ is odd, \( \sum \frac{B_n(x)}{n} < \infty \).

Concerning summability $|C, a|_k$ of Fourier series Flett has proved that summability $|C, a|_k$, $a > 1$, $k \geq 1$ of a Fourier series of Lebesgue integrable function is a local property whereas summability $|C, 1|_k$ is not so. His results generalize the previous results of Bosanquet, and Bosanquet and Kestelman.

Chapter IX of the present thesis is devoted to the study of localization problem for summability $|C, a|_k$ of $r$-th derived series of a Fourier series. In that chapter we have proved that summability $|C, r+l|_k$ of the $r$-th derived series of Fourier series is not necessarily a local property of the generating function. We have also obtained sufficient conditions which ensure the local character of the summability $|C, r+l|_k$, $k \geq 1$ of the $r$-th derived series of Fourier series. We may remark that in view of a result of Flett our conditions are also necessary for the summability $|C, r+l|_k$ of the series concerned. We may also note that our results include, as special cases, the results of Lal and Bhatt, and also for $r=0$ a result of Flett.

1) Flett, T.M. (43).
2) Mazhar, S.M. (95).
Extending a well known theorem of Wiener 1) on absolute convergence of Fourier series, Randels 2) established the following:

Theorem P. If \( f(x) \) is such that at every point \( y \) on the closed interval \([ -\pi, \pi] \) there are a function \( g_y(x) \) and a \( \delta > 0 \) such that (i) \( g_y(x) = f(x) \) for \( |x-y| < \delta \) and (ii) the Fourier series of \( g_y(x) \) is summable \(|C, 1|\), then the Fourier series of \( f(x) \) is summable \(|C, 1|\).

This result was extended to the summability \(|C, \alpha|, \alpha \geq 0\) by Magarik 3) whose result is as follows:

Theorem Q. If \( f(x) \) is such that at every point \( y \) on the closed interval \([ -\pi, \pi] \) there are a function \( g_y(x) \) and a \( \delta > 0 \) such that (i) \( g_y(x) = f(x) \) for \( |x-y| < \delta \) and (ii) both the Fourier series of \( g_y(x) \) and its conjugate series are summable \(|C, \alpha|, \alpha \geq 0\), then the Fourier series of \( f(x) \) is summable \(|C, \alpha|\).

Later on, Kiyohara 4) improved Theorem Q by showing that the condition that the conjugate series of Fourier series of \( g_y(x) \) should be summable \(|C, \alpha|\) is superfluous. In Chapter VIII we examine the corresponding problem for the summability \(|C, \alpha|_k, \alpha \geq 0, k \geq 1\). Our result includes, as special cases, all the previously known results stated above.

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2) Randels, W.C. (139).
3) Magarik, V.A. (70).
4) Kiyohara, M. (63).
The behaviour of Fourier series regarding $|C|$ summability when $f(t) \in \text{Lip } \alpha$, was investigated by Hyslop and Chow. Hyslop proved that if $f(x) \in \text{Lip } \alpha, 0 < \alpha \leq \frac{1}{2}$, then $\sum A_n(x)$ is summable $|C, \frac{1}{2} - \alpha + \varepsilon|, \varepsilon > 0$ for every value of $x$. Bosanquet remarked that this result is the best possible inasmuch as that $\varepsilon$ cannot be replaced by zero. Chow using the concept of Lip $(\alpha, p)$ class obtained a generalization of the above theorem. Later on, inspired by a result of Hardy and Littlewood on absolute convergence, he obtained a further extension of his result, which is applicable even to negative order summability. With the help of this theorem and one due to Hardy and Littlewood Chow obtained a generalization of Zygmund's result in the form that if $f(t) \in \text{BV } (0, n)$ and $f(t) \in \text{Lip } \alpha$, then the Fourier series of $f(t)$ is summable $|C, \beta|, \beta > -\frac{\alpha}{2}$ for all values of $t$. Also Chen studied the negative order Cesaro summability of Fourier series and its conjugate series for functions of bounded variation and Lipschitz class. Matsuyama has also obtained a number of theorems concerning Lip$(\alpha, p)$ class which generalize certain results of Hyslop and Bosanquet. Concerning summability $|C, \delta|, (\delta > -1)$ of Fourier series of $f(t) \in \mathcal{L}$

1) Hyslop, J.M. (68).
2) Chow, H.C. (27).
3) Hardy, G.H. and Littlewood, J.E. (53).
4) Chow, H.C. (30).
5) Hardy, G.H. and Littlewood, J.E. (51).
6) Chen, K.K. (21); (22). For further details see (24).
9) Bosanquet, L.S. (14).
he has obtained certain interesting results which have been
inspired mainly by the earlier researches of Bernstein and
Szasz on the absolute convergence of Fourier series and
Hyslop's result referred to in the foregoing discussion.

Let \( w(\theta, t) = f(t+\theta) - f(\theta), \quad w_o(\theta, t) = w(\theta, t), \)
\[ w_a(\theta, t) = \frac{1}{(2\pi)^{\alpha-1}} \int_0^t (t-u)^{\alpha-1} w(\theta, u) \, du, \quad \alpha > 0, \]
\[ \Omega_p^o(t) = \left( \frac{1}{2\pi} \int_{-\pi}^\pi |w_a(\theta, t)|^p \, d\theta \right)^{1/p}, \]
and
\[ \Omega_p^o(t) = \left( \frac{1}{2\pi} \int_{-\pi}^\pi |w(\theta, t)|^p \, d\theta \right)^{1/p}. \]

In 1955, Chow \(^1\) by an appeal to a theorem of Wang deduced
that if \( \Omega_p^o(t) = O\left( (\log \frac{1}{t})^{-1-\delta} \right), \) for some \( \delta > 0, \) then the
Fourier series of \( f(t) \) and its conjugate series are summable
\(|C, \beta| \) almost everywhere for every \( \beta > \frac{1}{2} \). Generalizing this
theorem Chow proved \(^2\)

Theorem R. Let \( 1 \leq p \leq 2. \) If
\[ \Omega_p^o(t) = O\left( (\log \frac{1}{t})^{-1-\delta} \right), \quad \delta > 0, \quad t \to 0 \]
or, more generally, if
\[ \int_{-\pi}^\pi \frac{\Omega_p^o(t)}{|t|} \, dt < \infty, \]
then the Fourier series of \( f(t) \) and its conjugate series are
both summable \(|C, \beta| \) almost everywhere for \( \beta > \frac{1}{p} \).

\(^1\) Chow, H.C. (33).
\(^3\) Chow, H.C. (33).
Yano obtained some interesting results in this line of work specially concerning the summability $|C, \frac{1}{p}|$ of Fourier series.

Recently Hsiang has succeeded in effecting further improvement of the theorem of Chow given above. He proved:

Theorem S. Let $1 \leq p \leq 2$. If

\begin{align*}
(1) & \quad \int_{-\pi}^{\pi} \frac{\Omega_{p}^{0}(t)}{t} \, dt < \infty, \\
(2) & \quad \int_{-\pi}^{\pi} \frac{\Omega_{p}^{1}(t)}{t^{2}} \, dt < \infty,
\end{align*}

then the Fourier series of $f(t)$ and its conjugate series are summable $|C, \beta|$, almost everywhere for $\beta > \frac{1}{p}$.

Chapter VII of this thesis deals with a generalization of the above theorem of Hsiang. We have shown that condition (ii) of Theorem S can be replaced by a lighter condition, namely

\[ \int_{-\pi}^{\pi} \frac{\Omega_{p}^{\alpha}(t)}{|t|^{\alpha+1}} \, dt < \infty, \quad \alpha \geq 1. \]

In 1926 Hardy and Littlewood considered the Cesaro summability of the series $\sum \frac{s_{n}(x) - s}{n}$, where $s_{n}(x)$ denotes the $n$-th partial sum of a Fourier series and $s$ an appropriate number. Necessary and sufficient condition for its convergence was given by Zygmund. Recently Mohanty and Mohapatra have investigated the

1) Yano, K. (158).
2) Hsiang, F.C. (55).
3) Mazhar, S.M. (92).
4) Hardy, G.H. and Littlewood, J.E. (49).
absolute convergence and summability $|C, \delta|$ of such a series. They have proved the following theorems.

**Theorem T.** If (i) $\varphi_1(t) \log \frac{t}{t} \in BV(O,\pi)$,

(ii) $\int_0^\pi \frac{|\varphi_1(t)|}{t} dt < \infty$,

(iii) $\{n A_n\} \in BV$ for some $\delta > 0$,

then the series $(\ast)$ is absolutely convergent, where

$\varphi(t) = \frac{1}{2} \left\{ f(x + t) + f(x - t) - 2s \right\}$ and

$\varphi_{\alpha}(t) = x t \int_0^t (t-u) \alpha-1 \varphi(u) du$ with $\varphi_0(t) = \varphi(t)$.

**Theorem U.** If $\int_0^\pi \frac{|\varphi(t)|}{t} dt < \infty$,

then the series $(\ast)$ is summable $|C, \delta|$, $\delta > 0$.

**Theorem V.** If the series $(\ast)$ is absolutely convergent,

then

$\int_0^\pi \frac{|\varphi_{1+\delta}(t)|}{t} dt < \infty$, $\delta > 0$.

The author 1) generalized Theorems T and U in the following manner:

**Theorem W.** Necessary and sufficient conditions that the series $(\ast)$ be absolutely convergent whenever $\varphi_1(t) \log \frac{t}{t} \in BV(O,\pi)$ are

(i) $\int_0^\pi \frac{|\varphi_2(t)|}{t} dt < \infty$,

(ii) $\left\{ e^{-n\alpha} \frac{n}{m} \left\{ \frac{e^{\alpha s}}{m} \right\} \right\} \in BV$,

where $0 < \alpha < 1$.

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1) Mazhar, S.M. (100); (87).
Theorem X. If \( \int_0^x \frac{|g(t)|}{t} \, dt < \infty, \alpha > 0 \), then the series \( \sum \) is summable \( |C, \beta| \) for \( \beta > \alpha \).

Generalization of Theorem V is considered in Section A of Chapter X of the present thesis. The result proved is as follows.

Theorem 5. **If the series \( (*) \) is summable \( |C, \alpha|, \alpha > 0 \), then**
\[
\int_0^a \frac{|g(t)|}{t} \, dt < \infty,
\]
where \( \beta > \alpha + 1 \).

It is clear that our theorem includes, for \( \alpha = 0 \), Theorem V of Mohanty and Mohapatra.

The Section B of Chapter X is devoted to the study of summability \( |N, p_n| \) of the series \( (*) \). Very recently Dikshit\(^1\) has obtained a theorem on \( |N, p_n| \) summability which includes Theorem U as a special case. Since Theorem U is a particular case of our general theorem (Theorem X) we have obtained a number of theorems concerning the summability \( |N, p_n| \) of \( (*) \) so as to generalize Theorem X. Our main results are as follows:

**Theorem 6. Let \( \{p_n\} \) be any monotonic sequence of non-negative numbers such that**

a) \[
\left\{ \frac{(n+1)p_n}{p_n} \right\} \in BV,
\]

b) \[
\sum_{k=n}^{n+k} \frac{p_k}{k^\alpha} \frac{n^\alpha}{(n+1)p_n} < \infty, \quad k = 1, 2, \ldots, \quad 0 \leq \alpha < 1.
\]

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1) Dikshit, H.P. (37)
and \[ \int_0^\infty \frac{|\varphi_n(t)|}{t} \, dt < \infty, \]

then the series \((\star)\) is summable \(|N, p_n|\).

**Theorem 7.** If \[ \int_0^\infty \frac{|\varphi_n(t)|}{t} \, dt < \infty, \]

then the series \((\star)\) is summable \(|N, p_n|\), where \(\{p_n\}\) is a non-decreasing sequence of numbers such that

(a) **holds** and

(b) \[ \sum_{k=1}^\infty \frac{1}{p_n} < \infty, \quad k = 1, 2, \ldots \]

(c) \[ \left\{ p_{n+1} - p_n \right\} \text{ is ultimately monotonic.} \]

Section C of Chapter X deals with the negative order Cesaro summability of the series \((\star)\) and other allied problems. Our result for negative order summability may be considered as an analogue of a theorem of Chen\(^1\) on Fourier series.

It is known that absolute convergence of \(\sum \lambda_n(x)\) does not necessarily imply the absolute convergence of the series \((\star)\). We have shown by means of an example that even summability \(|C, a|\) \(a \geq 0\) of the series \(\sum \lambda_n(x)\) does not necessarily imply the summability \(|C, a|\) of the series \((\star)\).

In another theorem we have established the following result which supplements Theorem V of Mohanty and Mohapatra.

**Theorem 8.** If the series \((\star)\) is absolutely convergent then

\[ \int_0^\infty \frac{p(t) \varphi_1(t)}{t} \, dt < \infty. \]

\(^1\) Chen, K.K. (24).
Finally we have extended our theorem on negative order absolute Cesaro summability to the summability \(|C, \beta|_k\).

The author has obtained a number of other interesting results. However, on account of lack of space, it is not possible to discuss them here in the present thesis.

\[ \int_0^\infty \frac{|\rho(t)|}{t} \, dt < \infty. \]

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1) For details see the references given in the bibliography.