CHAPTER X

ON THE ABSOLUTE SUMMABILITY OF A SERIES ASSOCIATED WITH A FOURIER SERIES

10.1 Let \( f(t) \) be integrable in the sense of Lebesgue over \((-\pi, \pi)\) and periodic with period \(2\pi\) and let

\[
(10.1.1) \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} a_n(t).
\]

Numbers \(x\) and \(a\) being fixed, we write

\[
\varphi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) - 2f(x) \right\},
\]

\[
\Phi(t) = \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} \varphi(u) \, du, \quad a > 0,
\]

\[
\varphi(t) = \left[ (x+1) t - x \right]^{-a} \Phi(t), \quad \varphi(t) = \varphi(t),
\]

and

\[
\varphi_n = \sum_{n=0}^{\infty} A_n(x).
\]

In this chapter we are concerned with the series

\[
(10.1.2) \quad \sum_{n=1}^{\infty} \frac{\varphi_n}{n^{\alpha}}.
\]

Section A of this Chapter is devoted to the study of absolute Cesàro summability of non-negative orders, while Section B deals with its absolute Nörlund summability. In Section C we examine absolute Cesàro summability of negative order \(\alpha\), and other associated problems.

† Section A was published in *Mathematica Scandinavica*, 21(1967), 90-104, while Section B has been accepted for publication in *Rivista di Matematica della Università di Parma*, 12 (1971).
10.2 Cesàro summability of the series (10.1.2) was first investigated by Hardy and Littlewood, whereas a necessary and sufficient condition for its convergence was given by Zygmund. Recently Mohanty and Mohapatra have investigated the absolute convergence and summability \(|C, \delta|\) of this series. In this connection they proved the following theorems:

Theorem A. If

(i) \(\varphi_1(t) \log \frac{k}{t} \in BV (0, n)\),

(ii) \(\int_0^\infty \frac{\varphi_n(t)}{t} dt < \infty\),

(iii) \(\delta\left\{ n A_n(x) \right\} \in BV \) for some \(\delta > 0\),

then the series (10.1.2) is absolutely convergent.

Theorem B. If \(\int_0^\infty \frac{\varphi(t)}{t} dt < \infty\),

then the series (10.1.2) is summable \(|C, \delta|\), \(\delta > 0\).

Theorem C. If the series (10.1.2) is absolutely convergent,
then

\(\int_0^\infty \frac{\varphi_{1+\delta}(t)}{t} dt < \infty\), \(\delta > 0\).

1) Hardy, G.H. and Littlewood, J.E. (49).
1) The author generalized the first two theorems in the following manner.

Theorem D. Necessary and sufficient conditions that the series (10.1.2) be absolutely convergent whenever \( \mathcal{g}_1(t) \log \frac{k}{t} \in BV (0, \infty) \), are

\[
(1) \quad \int_0^\infty \frac{\mathcal{g}_2(t)}{t} \ dt < \infty,
\]

\[
(2) \quad \left\{ \frac{1}{e^{n^\alpha}} \left( \sum_{m=1}^{\infty} \frac{s_m^\alpha s_m^\beta}{\alpha m} \right) \right\} \in BV,
\]

where \( 0 < \alpha < 1 \).

Theorem E. If \( \int_0^\infty \frac{\mathcal{g}_\alpha(t)}{t} \ dt < \infty, \quad \alpha > 0 \),

then the series (10.1.2) is summable \(|C, \beta|\), where \( \beta > \alpha \).

Theorem D gives necessary and sufficient conditions for the absolute convergence while Theorem E includes, as a special case for \( \alpha = 0 \), Theorem B cited above. The Theorem E is directly related to a theorem of Bosanquet on Fourier series which states that if \( \mathcal{g}_\alpha(t) \in BV (0, \infty) \), then \( \Sigma A_n(x) \) is summable \(|C, \beta|\), \( \beta > \alpha \). Since in the case of Fourier series the converse problem is also known, the question arises whether it is possible to obtain converse of Theorem E for the corresponding series (10.1.2). We answer this question in affirmative in this Section.

1) Mashar, S.M. (190). The Theorem D stated here is an improved version.

2) Bosanquet, L.S. (195).
10.3 In what follows, we prove the following theorem.

**Theorem 1.** If the series (10.1.2) is summable \((C, \alpha)\), \(\alpha > 0\), then

\[
\sum_{t=0}^{\infty} \frac{|g(t)|}{t} \lt \infty,
\]

(10.3.1)

where \(\beta > \alpha + 1\).

It may be remarked that this theorem generalizes Theorem C of Mohanty and Mohapatra.

10.4 The following lemmas are pertinent for the proof of this theorem.

1) **Lemma 1.** Let

\[
\gamma(x) = \int_{0}^{1} (1-u)^{\alpha-1} \cos xu \, du,
\]

then for \(\delta > 0\), \(t > 0\), \(n > 0\), we have

\[
\left| \Delta \left( \frac{d}{dt} \right)^{\lambda} \gamma_{\delta}(nt) \right| \leq \left\{ \begin{array}{ll}
A \frac{\lambda t^{p}}{\lambda > 0, \ p > 0}, \\
A \frac{-p - 2}{\lambda - 2}, \ p + \lambda \leq \delta - 2,
\end{array} \right.
\]

2) **Lemma 2.** If \(\alpha > -1\) and \(\sum a_n\) is summable \((C, \alpha)\), then \(S_n = o\ (n^\alpha)\) where \(\alpha' < \alpha\) and \(S_n\) is the \(n\)th Cesàro sum of order \(\alpha\) of the series \(\sum a_n\).

1) Bosanquet, L.S. (15).
Lemma 3. Let \( A_n = (n) \) and

\[
J_m(t) = \sum_{n=m}^{\infty} A_{n-m}^{h-\alpha} h+1 \left\{ (n+1) \Delta \gamma_\beta(nt) \right\},
\]

then

\[
J_m(t) = \begin{cases} 
O(t^\alpha), \\
O(t^{\alpha+\beta-\min(1, (mt)^{1-\beta})}), & 0 < \alpha < \beta - 1 < K+1, \ h = [\alpha] 
\end{cases}
\]

Proof. We write \( J_m \) as follows:

\[
J_m(t) = \sum_{m}^{m+p} + \sum_{m+p+1}^{\infty} = \sum_{1}^{\infty} + \sum_{2}, \text{ say},
\]

where \( p = \left\lfloor \frac{1}{t} \right\rfloor \).

Now

\[
|\sum_1| = O\left( \sum_{m}^{m+p} (n-m+1)^{h-\alpha} t \ \min(1, (mt)^{1-\beta}) \right)
\]

\[= O\left\{ t^\alpha \ \min(1, (mt)^{1-\beta}) \right\}.
\]

by Lemma 1. Further

\[
|\sum_2| \leq \sum_{n=m+p+1}^{\infty} \sum_{m+p+1}^{N} (n+1) \Delta \gamma_\beta(nt)
\]

\[= O\left\{ t^{\alpha-h} t^h \min(1, (m+p) t^{1-\beta}) \right\}
\]

\[= O\left\{ t^\alpha \ \min(1, (mt)^{1-\beta}) \right\}.
\]

This completes the proof of Lemma 3.
10.5 Proof of Theorem 1. Without loss of any generality we can assume that \( 0 < \alpha < \beta - 1 < h + 1 \), \( h = [\alpha] \). Let \( S_n \) denote the \( n \)-th Cesaro sum of order \( h \) of the series \( \sum_{m=1}^{\infty} \frac{(S_m - s)}{m+1} \)

Then applying Lemmas 1 and 2, we have

\[
\beta^{-1} \phi_\beta(t) = -\beta^\alpha + \sum_{n=0}^{\infty} A_n(x) \gamma_\beta(nt)
\]

\[= \sum_{n=0}^{\infty} \frac{S_{n+1}}{n+1} (n+1) \triangle \gamma_\beta(nt)\]

\[= \sum_{n=0}^{\infty} \frac{S_n}{\triangle} \{ (n+1) \triangle \gamma_\beta(nt) \}\]

\[= \sum_{n=0}^{\infty} \frac{S_n}{\triangle} \{ (n+1) \triangle \gamma_\beta(nt) \} = \sum_{m=0}^{\infty} A_{n-m} S_m\]

\[= \sum_{m=0}^{\infty} S_m \triangle J_m(t)\]

\[= \sum_{m=0}^{\infty} \frac{\alpha}{\alpha - 1} \triangle J_m(t), \text{ by Lemma 3,}\]

(10.5.1) \[= \sum_{m=0}^{\infty} \sigma_m A_m \triangle J_m(t),\]

where \( \sigma_m = \frac{a_m}{A_m} \).

Since \( S_m = S_m - S_{m-1} \)

\[= A_m \begin{bmatrix} \sigma_m & 0 \\ \frac{m}{\alpha + m} & \sigma_{m-1} \end{bmatrix}\]

\[= \Theta \left( \frac{m^\alpha}{\sigma_m - \sigma_m^\alpha \frac{m}{m-1}} \right) + \Theta (m^{\alpha - 1})\]

and by Lemma 1,
$$h \triangle \left\{ (n+1) \gamma_{\beta}(nt) \right\} = o(n^{1-\beta}),$$

we have

$$\left\{ \begin{array}{l}
\sum_{m=0}^{p} \sum_{n=p+1}^{h+1} \frac{\alpha-1}{A_{p+1-m}} \frac{h-\alpha}{A_{p-m}} \left\{ (n+1) \gamma_{\beta}(nt) \right\} \\
\sum_{m=0}^{\max p'} \sum_{n=p+1}^{h+1} \frac{\alpha-1}{A_{p+1-m}} \left\{ (n+1) \gamma_{\beta}(nt) \right\}
\end{array} \right.$$ 

$$= \circ \left\{ \begin{array}{l}
\sum_{m=0}^{p} \frac{\alpha-1}{A_{p-m}} \left( \frac{h-\alpha}{A_{p-m}} \right) \\
\sum_{m=0}^{p} \frac{\alpha-1}{A_{p-m}} \left( \frac{h-\alpha}{A_{p-m}} \right)
\end{array} \right.$$ 

by virtue of the fact that $\beta > \alpha + 1$ and $\sum_{n} \frac{\alpha}{\sigma_{n} - \sigma_{n-1}} < \infty$.

Thus we get

$$\lim_{p \to \infty} \sum_{m=0}^{p} \sum_{n=p+1}^{h+1} \frac{\alpha-1}{A_{p+1-m}} \frac{h-\alpha}{A_{p-m}} \left\{ (n+1) \gamma_{\beta}(nt) \right\} = 0$$

which justifies the change of order of summation.

Let

$$V_m(t) = \sum_{\nu=m}^{\infty} A_{\nu \gamma} \triangle J_{\nu}(t).$$
We shall now show that

\[ v_m(t) = \begin{cases} \mathcal{O}(m^\alpha t^\alpha) + \mathcal{O}(t), \\ \mathcal{O}(m^{1+\alpha-\beta} t^{1+\alpha-\beta}). \end{cases} \]

We have

\[
v_m(t) = \sum_{v=m}^{\infty} A_v \triangle J_v(t) = \sum_{v=m}^{\infty} A_v J_v(t) + A_{m-1} J_m(t)
\]

\[
= \mathcal{O}\left( \sum_{v=m}^{\infty} v \cdot t^{1-\beta} \right) + \mathcal{O}\left( m^{1+\alpha-\beta} t^{1+\alpha-\beta} \right)
\]

\[ = \mathcal{O}(m^{1+\alpha-\beta} t^{1+\alpha-\beta}). \]

This proves the second part of (10.5.2). In the expression

\[ S_{n-s} = \frac{2}{\pi} \int_0^\pi \varnothing(t) \sin(n+\frac{1}{2})t \frac{dt}{2 \sin t/2}. \]

Let \( \varnothing(t) = \sin^2 t/2 \) for all \( t \), then

\[ S_{0-s} = \frac{1}{2} \text{ and } S_{n-s} = 0 \text{ for } n > 0. \]

Therefore \( \sigma_n = \frac{1}{2} \) for every \( n \). Also

\[ \beta^{-1} \varnothing_\beta(t) = \frac{1}{2\beta t^{\beta}} \int_0^t (t-u)^\beta \sin u \, du. \]

Thus from (10.5.1) we get

\[ \sum_{v=m}^{\infty} A_v \triangle J_v(t) = \frac{1}{2\beta t^{\beta}} \int_0^t (t-u)^\beta \sin u \, du \]
so that

\[ v_m(t) = \sum_{n=0}^{\infty} A_n \triangle J_n(t) - \sum_{n=0}^{\infty} \frac{m}{n} A_n \triangle J_n(t) \]

\[ = \int_{\beta}^{t} \frac{1}{(t-u)^{\beta}} \sin u \, du - \sum_{n=0}^{\infty} \frac{m}{n} A_n \triangle J_n(t) \]

\[ + A_m J_m(t) \]

\[ = O(t) + O(\sum_{n=0}^{m} \frac{m}{n} A_n \triangle J_n(t)) + O(m^\alpha t^\alpha) \]

\[ = O(t) + O(t^m), \]

by Lemma 3. This completes the proof of (10.5.2).

From (10.5.1) we observe that

\[ \beta^{-1} g_\beta(t) = \sum_{m=0}^{\infty} \frac{\alpha}{\sigma_m} \triangle v_m(t) = \sum_{m=0}^{\infty} \left( \frac{\alpha}{\sigma_m} - \frac{\alpha}{\sigma_{m-1}} \right) v_m(t). \]

Now

\[ \int_{0}^{\pi} \left| g_\beta(t) \right| \frac{dt}{t} \leq \beta \int_{0}^{\pi} \left| \frac{\alpha}{\sigma_m} - \frac{\alpha}{\sigma_{m-1}} \right| \left| \frac{v_m(t)}{t} \right| \frac{dt}{t} \]

\[ = \beta \sum_{m=0}^{\infty} \left| \frac{\alpha}{\sigma_m} - \frac{\alpha}{\sigma_{m-1}} \right| \int_{0}^{\pi} \left| \frac{v_m(t)}{t} \right| \frac{dt}{t}. \]

Since

\[ \sum_{m=0}^{\infty} \left| \frac{\alpha}{\sigma_m} - \frac{\alpha}{\sigma_{m-1}} \right| < \infty, \]

by hypothesis it is sufficient to prove that

\[ \int_{0}^{\pi} \left| \frac{v_m(t)}{t} \right| \frac{dt}{t} < \infty. \]
uniformly in $m$.

Now

$$\int_0^\pi \frac{|V_m(t)|}{t} \, dt = \int_0^{1/m} + \int_{1/m}^\pi$$

$$= \bigcirc \left( \int_0^{1/m} \left( m^\alpha t^{-1} + 1 \right) \, dt \right) + \bigcirc \left( \int_{1/m}^\pi t^{\alpha-\beta} m^{l+\alpha-\beta} \, dt \right)$$

$$= \bigcirc (1).$$

This completes the proof of Theorem 1.
10.6 It is known\footnote{1) Mazhar, S.M. (87).} as mentioned in Section A, (see Theorem B), that if

\begin{equation}
\int_{0}^{\infty} \frac{f(t)}{t} \, dt < \infty, \quad \alpha > 0,
\end{equation}

then the series (10.1.2) is summable \(|C, \beta|\), \(\beta > \alpha\).

The object of this Section is to study the corresponding problem for summability \(|N, p_n|\).

We prove the following theorems. In the sequel we assume that \(p_n \geq 0\).

**Theorem 2.** Let \(\{p_n\}\) be a non-increasing sequence of numbers such that

\begin{equation}
\left\{ R_n \right\} = \left\{ \frac{(n+1)p_n}{p_n} \right\} \in BV, \quad p_k = p_0 + p_1 + \ldots + p_k,
\end{equation}

\begin{equation}
\frac{p_k}{k^\alpha} \leq \frac{n^\alpha}{n=1} \frac{1}{(n+1)p_n} \leq C, \quad k = 1, 2, \ldots, 0 \leq \alpha < 1.
\end{equation}

If the condition (10.6.1) holds for \(0 < \alpha < 1\), then the series (10.1.2) is summable \(|N, p_n|\).

**Theorem 3.** Suppose that (10.6.2) holds and that

\begin{equation}
\frac{1}{m=k} \frac{1}{(n+1)p_n} \leq C, \quad k = 1, 2, \ldots
\end{equation}

\begin{equation}
\frac{n}{p_n} \sum_{k=0}^{n-1} \Delta p_k \leq C.
\end{equation}

\footnote{2) In a recent paper Dikshit (37) has obtained a result which includes Theorem B.}

\footnote{3) Where \(C\) is a constant not necessarily the same at each occurrence.}
If (10.6.1) holds for \(0 < \alpha < 1\), then the series (10.1.2) is summable \(N, p_n\).

Theorem 4. If

\[
\int_0^\infty \frac{|g(t)|}{t} \, dt < \infty,
\]

then the series (10.1.2) is summable \(N, p_n\), where \(\{p_n\}\) is a non-decreasing sequence of numbers such that (10.6.2) holds and

\[
(10.6.7) \quad \frac{p_k}{n} \sum_{n=1}^{\infty} \frac{1}{p_n} \leq c, \quad k = 1, 2, \ldots
\]

(10.6.8) \(\{p_{n+1} - p_n\}\) is ultimately monotonic.

It is evident that if \(\{p_n\}\) is a non-decreasing sequence satisfying (10.6.2), then (10.6.6) holds. Also (10.6.3) \(\Rightarrow\) (10.6.4). Thus we deduce the following:

Theorem 5. Let \(\{p_n\}\) be any monotonic sequence of non-negative numbers such that (10.6.2) and (10.6.3) hold. If (10.6.1) holds, then the series (10.1.2) is summable \(N, p_n\).

10.7 The following lemmas will be required to prove the theorems of this Section.

Lemma 4. If \(\int_0^\infty \frac{|g(t)|}{t} \, dt < \infty\), then

\[
\int_0^\infty \frac{|g(t)|}{t} \, dt < \infty,
\]

where \(\beta > \alpha \geq 0\).

---

1) Bosanquet, L.S. (13).
Lemma 5. If (10.6.6) holds, then a necessary and sufficient condition for the series (10.1.2) to be summable \(| N, p_n |\) is that

\[(10.7.1) \quad \sum_{n=1}^{\infty} \frac{1}{n+1} \left| \frac{j}{n} \right| < \infty, \]

where \[\frac{j}{n} = \sum_{k=1}^{n} p_{n-k} (s_k - s)\]

and the sequence \(\{p_n\}\) satisfies the conditions (10.6.2) and (10.6.4).

Proof. Let

\[t_n = \frac{1}{p_n} \sum_{k=1}^{n} p_{n-k} \left( \frac{s_k - s}{k} \right),\]

then

\[t_n - t_{n-1} = \frac{\sigma_n(x)}{n+1} + \frac{1}{(n+1)p_{n-1}} \sum_{k=0}^{n-1} \frac{(R_k - R_n)}{n-k} \cdot p_k (s_{n-k} - s).\]

Thus to establish the lemma it is sufficient to show that under our assumptions

\[(10.7.2) \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \sum_{k=0}^{n-1} \frac{(R_k - R_n)}{n-k} \cdot p_k (s_{n-k} - s) < \infty.\]

Now

\[
\sum_{k=0}^{n-1} \frac{(R_k - R_n)}{n-k} \cdot p_k (s_{n-k} - s)
\]

\[= \frac{2}{\pi} \int_0^\pi \sum_{k=0}^{n-1} \frac{(R_k - R_n)}{n-k} \cdot p_k (s_{n-k} - s) \cdot dt, \quad D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2 \sin \frac{t}{2}} \]

\[= \frac{2}{\pi} \sum_{k=0}^{n-1} \frac{(R_k - R_n)}{n-k} \cdot p_k (s_{n-k} - s) \cdot dt, \quad D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2 \sin \frac{t}{2}} \]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} \frac{R_k - R_n}{P_k(-1)} \sum_{k=0}^{n-k} \int_{0}^{\pi} \frac{1}{t} \sum_{k=0}^{n-k} \frac{R_k - R_n}{P_k} \left\{ t^2 \frac{d}{dt} D_{n-k}(t) \right\} dt.
\]

It is therefore, sufficient to prove that

\[(10.7.3) \quad \sum_{l=0}^{\infty} \frac{1}{l(n+1)} \sum_{k=0}^{n-l} \frac{R_k - R_n}{P_k} (-1)^k < \infty,\]

\[(10.7.4) \quad \sum_{l=0}^{\infty} \frac{1}{l(n+1)} \sum_{k=0}^{n-l} \frac{R_k - R_n}{P_k} \left\{ t^2 \frac{d}{dt} D_{n-k}(t) \right\} < \infty,\]

uniformly in \(0 < t < \pi\).

**Proof of (10.7.3).** We have on applying Abel's transformation

\[
= \sum_{l=0}^{\infty} \frac{1}{l(n+1)} \sum_{k=0}^{n-l} \frac{R_k - R_n}{P_k} \left\{ t^2 \frac{d}{dt} D_{n-k}(t) \right\} < \infty,
\]

by virtue of the conditions (10.6.2) and (10.6.4).

**Proof of (10.7.4).** Let \(n = \left\lfloor \frac{\pi}{2} \right\rfloor\) and \(T = \left[ \frac{\pi}{2} \right]\).
Then the left hand side expression of (10.7.4) is

\[ \leq \sum_{n \leq T} \frac{1}{n(n+1)^{P_n-1}} \left| \sum_{k=0}^{m-1} \right| + \sum_{n> T} \frac{1}{n(n+1)^{P_n-1}} \left| \sum_{k=0}^{m-1} \right| \]

\[ + \sum_{n \leq T} \frac{1}{n(n+1)^{P_n-1}} \left| \sum_{k=m}^{n-1} \right| = \sum_1 + \sum_2 + \sum_3, \text{ say.} \]

Now \[ \sum_1 \leq C \sum_{n \leq T} \left( \frac{1}{n^{P_n-1}} \sum_{k=0}^{m-1} t^{P_k} = O(1). \right) \]

Applying Abel's transformation to the inner sum of \[ \sum_2 \]

and using the facts that \( \left\{ \frac{P_k}{n-k} \right\} \) is monotonic non-decreasing with respect to \( k \) for \( k < n \) and

\[ t^2 \sum_{v=0}^{k \frac{d}{dt} D_{v-T}(t) = O(n) + O\left(\frac{1}{t}\right), \]

we have

\[ \sum_2 \leq \sum_{n> T} \left( \frac{1}{n(n+1)^{P_n-1}} \sum_{k=0}^{m-2} \right| \Delta R_k \right| \frac{P_k}{n-k} \left\{ O(n) + O\left(\frac{1}{t}\right) \right\} \]

\[ + \sum_{n> T} \left( \frac{1}{n(n+1)^{P_n-1}} \sum_{k=m+1}^{m-2} \right| R_{m-1} - R_n \right| \frac{P_{m-1}}{n-m+1} \left\{ O(n) + O\left(\frac{1}{t}\right) \right\} \]

\[ = O \left( \sum_{n> T} \frac{1}{n(n+1)^{P_n-1}} \sum_{k=0}^{m-1} P_k |\Delta R_k| \right) + O \left( \sum_{n> T} \frac{1}{n(n+1)^{P_n-1}} \sum_{v=m-1}^{n-1} \Delta R_v \right) \]

\[ = O(1) + \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{v=m-1}^{n-1} |\Delta R_v| \]

\[ = O(1) + \sum_{v=1}^{\infty} |\Delta R_v| = O(1). \]
Again
\[ E_3 \leq \sum_{l=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \sum_{k=m}^{n-1} |\Delta R_k|. \]
\[ \leq c \sum_{l=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \sum_{k=m}^{n-1} P_k |\Delta R_k| \]
\[ \leq c \sum_{l=1}^{\infty} \frac{1}{n+1} \sum_{k=m}^{n-1} |\Delta R_k| = O(l), \text{ as in } E_2. \]

This completes the proof of Lemma 5.

**Lemma 6.** Let
\[ g(n, t) = \frac{2}{n p_{n-1}} \sum_{k=1}^{n} p_{n-k} D_k(t). \]

Then for all \( t \) and all non-negative sequence \( \{ p_n \} \)
\[ g(n, t) = O(n), \]
\[ \frac{d}{dt} g(n, t) = O(n^2). \]

(A) **If** \( \{ p_n \} \) **is a non-increasing sequence of non-negative numbers, then for** \( nt > 1 \)
\[ g(n, t) = O\left(\frac{n p_{\lfloor \frac{1}{t} \rfloor}}{t p_{n-1}}\right), \]
\[ \frac{d}{dt} g(n, t) = O\left(\frac{n p_{\lfloor \frac{1}{t} \rfloor}}{t p_{n-1}}\right). \]
(B) If \( \{p_n\} \) is a non-negative sequence satisfying (10.6.2) and (10.6.5), then
\[
g(n, t) = \Theta \left( \frac{1}{nt^2} \right), \quad \frac{d}{dt} g(n, t) = \Theta \left( \frac{1}{t^2} \right).
\]

Since the proof is quite easy we omit the same.

**Lemma 7.** Let
\[
F(n, u) = \int_{u}^{n} (t-u)^{a} \frac{d}{dt} g(n, t) \, dt, \quad 0 \leq a < 1.
\]

If the condition (A) of Lemma 6 is satisfied, then
\[
F(n, u) = \begin{cases} 
\Theta \left( n^{a+1} \right), & 0 < u \leq n^{-1}, \\
\Theta \left( \frac{n^{a} \left[ \frac{1}{n} \right]}{u \left( \frac{1}{n} \right)^{\alpha}} \right), & n^{-1} < u < n^{\alpha}, 
\end{cases}
\]

while if condition B of Lemma 6 is satisfied
\[
F(n, u) = \begin{cases} 
\Theta \left( n^{a+1} \right), & 0 < u \leq n^{-1}, \\
\Theta \left( \frac{n^{a-1}}{u^{\alpha}} \right), & n^{-1} < u < n. 
\end{cases}
\]

**Proof.** Let
\[
F(n, u) = \int_{u}^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{n}.
\]

Applying Mean value theorem and Lemma 6, the result follows.

\[\text{\footnotesize \[†\ It is sufficient to assume that } \frac{(n+1)p_d}{p_n} \in B.\]
10.8 Proof of Theorem 2. By virtue of Lemmas 4 and 5 and the fact that (10.6.3) $\Rightarrow$ (10.6.4) it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} |\sigma_n(x)| < \infty.$$  

Now

$$\sigma_n(x) = \int_0^n \rho(t) g(n,t) dt$$

$$= \left[ \Phi_1(t) g(n,t) \right]_0^n - \int_0^n \Phi_1(t) \frac{d}{dt} g(n,t) dt$$

$$= \frac{c}{P_{n-1}} \sum_{k=1}^{n} p_{n-k} (-1)^k \int_0^n \Phi_{a}(u) du \int_0^u \frac{d}{dt} g(n,t) dt.$$  

Thus in view of (10.6.1) it is sufficient to prove that

(10.8.1) $\sum_{n=1}^{\infty} \frac{1}{(n+1)} \frac{1}{P_{n-1}} \left| \sum_{k=1}^{n} p_{n-k} (-1)^k \right| < \infty,$

(10.8.2) $\sum_{n=1}^{\infty} \frac{u^{a+1}}{(n+1)} |P(n,u)| < \infty$ uniformly in $0 < u < \pi.$

Proof of (10.8.1). We have

$$\sum_{k=1}^{n} p_{n-k} (-1)^k = O \left( \sum_{v=0}^{n-2} |\Delta p_v| \right) + o(p_{n-1})$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \left| \sum_{k=1}^{n} p_{n-k} (-1)^k \right| = O \left( \sum_{v=0}^{n-2} \frac{1}{(n+1)p_{n-1}} \sum_{v=0}^{n-2} |\Delta p_v| \right) +$$
by virtue of the condition (10.6.2).

Proof of (10.8.2). We have

\[
\sum_{n=1}^{\infty} \frac{u^{n+1}}{n+1} \left| \frac{F(n,u)}{P(n,u)} \right| \leq C \sum_{n=1}^{\infty} \frac{u^{n+1} p^{n+1}}{n! u^{n+1} n+1}
\]

\[
= C \sum_{n=1}^{\infty} \frac{u^{n+1}}{n+1} \frac{n! p^{\frac{1}{u}}}{u^{n+1} n+1} + 1
\]

\[
\leq C + C u^{n} p^{\frac{1}{u}} \sum_{n=1}^{\infty} \frac{n!}{u^{n+1} n+1} + 1
\]

\[
= O(1), \text{ uniformly in } 0 < u < n, \text{ by virtue of (10.6.3)}.
\]

This completes the proof of Theorem 2.

10.9 Proof of Theorem 3. In the proof of Theorem 2 up to (10.8.1) we have used only two conditions namely, (10.6.2) and (10.6.4) of Theorem 3. It is therefore, sufficient to prove (10.8.2). Applying Lemma 7, we have
\[
\sum_{n=1}^{\infty} \frac{u^{n+1}}{(n+1) \ln u} \left| \frac{F(n,u)}{\ln u} \right| \leq C \sum_{n=1}^{\infty} \frac{u^{n+1}}{(n+1) \ln u} + C \sum_{n=1}^{\infty} \frac{u^{n-1}}{n+1} \frac{n^{a-2}}{u^2}
\]

\[
= O(1) + O\left( \sum_{n=1}^{\infty} \frac{u^{n-1} n^{a-2}}{n+1} \right) = O(1),
\]

uniformly in \(0 < u < \pi\). This proves Theorem 3.

10.10 Proof of Theorem 4. It is obvious that (10.6.7) implies (10.6.4) and hence by virtue of Lemma 5 it is sufficient to prove that

\[
\sum_{n=1}^{\infty} \frac{1}{(n+1) \ln u} \left| \frac{\sigma_n(x)}{\ln u} \right| < \infty.
\]

As in the proof of Theorem 2

\[
\sigma_n(x) = \frac{C}{\ln n} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{0}^{\pi} \Phi_1(t) \frac{d}{dt} g(n,t) dt.
\]

Since (10.8.1) is true under the conditions (10.6.2) and (10.6.7) it is sufficient by virtue of the hypotheses to prove that

\[
(10.10.1) \sum_{n=1}^{\infty} \frac{t^2}{(n+1) \ln u} \left| \frac{d}{dt} g(n,t) \right| < \infty \text{ uniformly in } 0 < t < \pi.
\]

We have by partial summation

\[
\frac{d}{dt} g(n,t) = \frac{p}{\ln n} \sum_{k=1}^{n} (p_{n-k} - p_{n-k-1}) \sum_{v=0}^{k} \frac{d}{dt} D_v(t).
\]
Writing the sum in (10.10.1) as

\[ I = I_1 + I_2, \quad n \leq \frac{1}{t}, \quad n > \frac{1}{t} \]

we have

\[ I_1 \leq C \sum_{n \leq \frac{1}{t}} \frac{t^2}{(n+1)^2} \sum_{k=1}^{n} (p_{n-k} - p_{n-k-1}) \frac{n^2}{t}\]

\[ \leq C \sum_{n \leq \frac{1}{t}} \frac{n}{t} \frac{p_{n-1}}{p_{n-1}} \]

\[ \leq C, \text{ uniformly in } 0 < t < \infty. \]

Let \( m_0 \) be a constant such that \( \{p_n - p_{n-1}\} \) is monotonic for \( n > m_0 \). Then

\[ I_2 \leq C \sum_{n > \frac{1}{t}} \frac{t^2}{(n+1)^2} \sum_{k=1}^{n-m_0} (p_{n-k} - p_{n-k-1}) \frac{k}{v=0} \frac{d}{dt} D_v(t) \]

\[ + C \sum_{n > \frac{1}{t}} \frac{t^2}{(n+1)^2} \sum_{k=n-m_0}^{n} (p_{n-k} - p_{n-k-1}) \frac{k}{v=0} \frac{d}{dt} D_v(t) \]

\[ = I_{21} + I_{22}, \text{ say.} \]

By virtue of the fact that

\[ \sum_{n \leq \frac{1}{t}} \frac{1}{p_n} < \infty, \]

we observe that
We now proceed to show that $L_{21} \leq C$.

**Case (i).** Let $\{p_n - p_{n-1}\}$ be monotonic non-decreasing for $n > m_0$. Since

\[(10.10.2) \quad \sum_{k=1}^{n-m_0-1} E \left( p_{n-k} - p_{n-k-1} \right) \sum_{v=0}^{k} \frac{4}{v!} \frac{d^v p_v(t)}{dt^v},\]

we have

\[
L_{21} = O(1) \sum_{n \geq \frac{1}{t}} \frac{\frac{t^2}{(n+1)p_{n-1}}}{p_{n-1}} \frac{n}{t^3} (p_{n-1} - p_{n-2})
\]

\[= O\left( \frac{1}{t} \sum_{n \geq \frac{1}{t}} \frac{p_{n-1} - p_{n-2}}{p_{n-1}} \right) = O\left( \frac{1}{t} \sum_{n \geq \frac{1}{t}} \frac{1}{n} \left| \frac{(n-1)p_{n-2}}{p_{n-2}} \right| \right)
\]

\[+ O\left( \frac{1}{t} \sum_{n \geq \frac{1}{t}} \frac{1}{n^2} \right) = O(1).
\]

**Case (ii).** Suppose $\{p_n - p_{n-1}\}$ is monotonic non-increasing. In this case the expression in (10.10.2) is $O\left( \frac{n}{t^2} \left[ \frac{1}{p_n} \right] \right)$.

Hence

\[
L_{21} = O\left( \sum_{n \geq \frac{1}{t}} \frac{\frac{t^2}{(n+1)p_{n-1}}}{p_{n-1}} \frac{n}{t^2} \left[ \frac{1}{p_{n-1}} \right] \right)
\]
\[ = O \left( \frac{P_{\left[ \frac{1}{t} \right] n + \frac{1}{t} P_{n-1}}}{P_{\left[ \frac{1}{t} \right]}} \right) = O \left( \frac{[\frac{1}{t}] P_{\left[ \frac{1}{t} \right]}}{P_{\left[ \frac{1}{t} \right]}} \right) \]

= \Theta(1), uniformly in 0 < t < \pi.

This proves Theorem 4.

SECTION C.

10.11 For the sake of convenience we shall write

\[
\begin{align*}
\left( \varphi(t) \right)_\alpha &= \Phi(t), \quad \alpha > 0, \\
\left[ \varphi(t) \right]_\alpha &= \varphi(t), \quad \alpha > -1, \\
\left[ \varphi(t) \right]_0 &= \varphi(t),
\end{align*}
\]

and

\[
\frac{d}{dt} \left( \varphi(t) \right)_{1-\alpha} = \left( \varphi(t) \right)_{-\alpha}, \quad 0 < \alpha < 1,
\]

if the derivative exists.

It is well known that \( |C, \alpha| \Rightarrow |C, \beta|, \beta > \alpha > -1. \) Thus absolute Cesaro summability of negative order is stronger than the absolute Cesaro summability of positive order.

Results concerning summability \( |C, \delta|, \delta > 0 \) of the series (10.1.2) have been obtained by Mohanty and Mohapatra and the present author. However, no such result
exists for negative order summability. The object of this Section is to provide such a result and also obtain some other allied results of interest. We first prove the following theorem.

**Theorem 6.** If

$$\int_0^\infty \frac{|\chi(t)|}{t} \, dt < \infty,$$

where

$$\chi(t) = \frac{t^{\alpha-q}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u^q \phi(u) \frac{du}{(t-u)^\alpha},$$

$$\alpha-q < -1, \quad \beta > 1,$$

then the series (10.1.2) is summable.

10.12 The following lemma is required for the proof of our theorem.

**Lemma 8.** Suppose that \(0 < \alpha < 1\) and that the function \((h(t))_{-\alpha}\) exists as a summable function on \((a,b)\), then the relation

$$h(t) = \left( (h(t))_{-\alpha} \right)^\alpha$$

holds good almost everywhere in \((a,b)\).

10.13 **Proof of Theorem 6.** Without loss of any generality we can assume that \(-\alpha < \beta < 0\). We have

$$n \frac{\Delta_n - s}{n} = \frac{2}{\pi} \int_0^n \phi(t) \frac{\sin(n+\frac{1}{2})t}{2 \sin \frac{t}{2}} \, dt.$$

---

1) The corresponding problem for Fourier series has been studied by Chen, K.K. (22).

2) Chen K.K. (24, p.142).
Writing $T_n^\beta$ for the $n$-th Cesaro mean of order $\beta$ of the sequence \( \{ n \frac{S_n - S}{n} \} \) and $K_n(t)$ for that of the sequence $\{ D_n(t) \}$, where
\[
D_n(t) = \frac{\sin(n+\frac{1}{2})}{2 \sin t/2} t,
\]
we have
\[
T_n^\beta = \frac{2}{\pi} \int_0^\pi \varphi(t) K_n(t) \, dt.
\]
By virtue of the hypothesis,
\[
\int_0^\pi \left\{ t^q \varphi(t) \right\}^{-\alpha} \, dt < \infty
\]
and hence applying Lemma 8 we observe that
\[
t^q \varphi(t) = \left( t^q \varphi(t) \right)^{-\alpha} = \left( \chi(t) \right)^{q-\alpha}
\]
almost everywhere. Thus
\[
T_n^\beta = \frac{2}{\pi} \int_0^\pi \left( t^{-\alpha} \chi(t) \right)^{q-\alpha} t^{-q} K_n(t) \, dt
\]
\[
= \frac{2}{\pi} \frac{1}{\Gamma(q)} \int_0^\pi t^{-q} K_n(t) \, dt \int_0^t \chi^{\alpha-1}(u) u^{q-\alpha} \chi(u) \, du
\]
\[
= \frac{2}{\pi} \frac{1}{\Gamma(q)} \int_0^\pi \chi^{\alpha-1}(u) u^{q-\alpha} \, du \int_0^\pi t^{-q} (t-u)^{\alpha-1} K_n(t) \, dt
\]
\[
= \int_0^\pi \chi(u) u^{q-\alpha} P(n,u) \, du,
\]
where
\[
P(n,u) = \frac{2}{\pi} \frac{1}{\Gamma(q)} \int_0^\pi t^{-q} (t-u)^{\alpha-1} K_n(t) \, dt.
\]
Now the series (10.1.2) will be summable \(|C, \beta|\) if
\[
\sum_{n=1}^{\infty} \frac{T_n^\beta}{n} \left| \int_1^{\infty} \chi(u) u^{-\alpha} P(n,u) \, du \right| \\
\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi} \chi(u) u^{-\alpha} = \frac{P(n,u)}{n} \, du < \infty.
\]

By virtue of the hypothesis it is sufficient to prove that
\[
\sum_{n=1}^{\infty} \frac{|P(n,u)|}{n} = O(u^{\alpha-q-1})
\]
uniformly in \(0 < u < \pi\).

Now it can be easily shown that
\[
K_n^\beta(t) = \begin{cases} 
O(n), \\
O(n^{-\beta} t^{-\beta-1}),
\end{cases}
\]
and
\[
\int_0^t K_n^\beta(u) \, du = O(1), \quad 0 < t < \pi.
\]

Writing
\[
P(n,u) = \int_{u+\frac{1}{n}}^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{\pi} = P_1(n,u) + P_2(n,u),
\]
and using the above estimate, we have
\[
P_1(n,u) = O\left(\frac{1}{n} u^{-q}\right)
\]
and
\[
P_2(n,u) = (u+\frac{1}{n})^{-q} \int_{u+\frac{1}{n}}^{\pi} K_n^\beta(t) \, dt, \quad u+\frac{1}{n} \eta < x,
\]
\[
= O\left(u^{-q} \frac{1}{n}\right).
\]
Thus
\[ P(n,u) = O\left( \frac{1}{n^{u^{q}}} \right). \]

Again
\[ P_1(n,u) = O\left( \frac{-\beta - \alpha - q - \beta - 1}{n} \right), \]
and employing a refined expression for \( K_n(t) \), we obtain
\[
P_2(n,u) = \left( \frac{1}{n} + u \right)^{-q} n^{-\alpha} \sum_{u} \left( \frac{\sin \left[ (n+\frac{1}{2} + \frac{\beta}{2})t \right]}{n^{-\beta} A_n(2\sin t/2)^{\beta+1}} \right)
= O\left( \frac{-q - \beta - 1}{n} \right) + O\left( \frac{-q - 1}{n} \right)
= O\left( \frac{-\alpha - \beta}{n} \right), \text{ if } nu > 1.
\]

Thus we get
\[
P(n,u) = \begin{cases} 
O\left( \frac{1}{n^{u^{q}}} \right), & u \leq n, \\
O\left( \frac{-\alpha - \beta - q - \beta - 1}{n} \right), & n < u \leq n.
\end{cases}
\]

Hence
\[
\sum_{n} \frac{|P(n,u)|}{n^{u^{q}}} = \sum_{n\leq u} n^{-u^{q} - 1} + \sum_{n > u} n^{-u^{q} - 1}
= O\left( \frac{1}{n^{u^{q}}} \right) + O\left( \frac{1}{n^{u^{q} - 1} u^{q} - 1} \right)
\]

$$\bigcirc(u^{\alpha-l-q}) + \bigcirc(u^{\alpha-l-q}) = \bigcirc(u^{\alpha-l-q})$$

uniformly in $0_t < u < \pi$.

This completes the proof of Theorem 6.

10.14 If we take $q = 0$ in Theorem 6 we get the following theorem analogous to Theorem B of the author.

Theorem 7. If
$$\int_0^\pi \left[ \frac{\varphi(t)}{t} \right]^{-a} \, dt < \infty, \quad 0 < a < 1,$$

then the series (10.1.2) is summable $|C, \beta|, \beta > -a$.

10.15 Let
$$\varphi(t) = \left( \log \frac{k}{|t|} \right)^{-1}$$
in $(0, \pi)$ and defined elsewhere by periodicity.

Then
$$\varphi(t) = \sum_{n=0}^{\infty} A_n(x) \cos nt$$

with
$$A_n(x) = \bigcirc \left( n^{-1} (\log n)^{-2} \right),$$

so that $\sum |A_n(x)| < \infty$ and consequently $\sum A_n(x)$ is

is summable $|C, \alpha|, \alpha > 0$. But

$$\int_0^\pi \left[ \frac{\varphi(t)}{t} \right]^\gamma \, dt$$

is not convergent for any $\gamma$ and hence by virtue of Theorem

1, the series (10.1.2) is not summable $|C, \alpha|$. Thus we

1) Mohanty, R. (116).
conclude that summability $|C, \alpha|$, $\alpha \geq 0$ of $\sum A_n(x)$ does not necessarily imply the summability $|C, \alpha|$ of (10.1.2).

10.16 Taking $\alpha=0$ we observe from Theorem 1 that if the series (10.1.2) is absolutely convergent, then

$$\int_0^\infty \frac{|\varphi(t)|}{t} \, dt < \infty, \quad \delta > 0.$$ 

The question arises as to what can be said when $\delta = 0$. In this case we show that it is possible to find a suitable function $\rho(t)$ such that the absolute convergence of the series (10.1.2) implies the convergence of the integral

$$\int_0^\infty \frac{\rho(t) \varphi(t)}{t} \, dt.$$ 

**Theorem 8.** If the series (10.1.2) is absolutely convergent, then

$$\int_0^\infty \frac{\rho(t) \varphi(t)}{t} \, dt < \infty,$$

where

$$\int_0^\infty \frac{\rho(t)}{t} \, dt < \infty.$$ 

**Proof of Theorem 8.** We have

$$[\varphi(t)]_1 = \frac{1}{t} \int_0^t \varphi(u) \, du = \frac{1}{t} \int_1^\infty \frac{\pi}{n} A_n(x) \sin \frac{n \pi t}{2} + \frac{a_n}{2} - s$$
\[
\frac{1}{t} \sum_{n=1}^{\infty} \left\{ \frac{n}{s} \nabla (\sin nt) \right\} + \frac{s \sin t}{t} + \left( \frac{1}{2} a_0 - s \right).
\]

Hence
\[
\int_0^\pi \frac{p(t) |\phi(t)|}{t} \, dt
\]
\[
\leq C \sum_{n=1}^\infty \frac{s_n}{n} \int_0^\pi \frac{|p(t)|}{t^2} \left| \nabla (\sin nt) \right| \, dt
\]
\[
+ C \int_0^\pi \frac{|p(t)|}{t} \, dt
\]
\[
\leq C \sum_{n=1}^\infty \frac{|s_n - s|}{n} \left( \int_0^\pi \frac{|p(t)|}{t} \, dt + \int_0^\pi \frac{|p(t)| \sin(nt)}{t^2} \frac{1}{n+1} \, dt \right) + C
\]
\[
\leq C \sum_{n=1}^\infty \frac{|s_n - s|}{n} + C < \infty.
\]

This proves Theorem 8.

10.17 Finally we extend Theorem 6 to the summability \(| C, \beta |_{k, k \geq 1}\). Our result is as follows:

**Theorem 9.** Let \(0 < \alpha < \min (1, q + \frac{1}{k})\), \(k \geq 1, q \geq 0\).

If
\[
\int_0^\pi \frac{|\chi(t)|^k}{t} \, dt < \infty,
\]
then the series (10.1.2) is summable \(| C, \beta |_{k, \beta > -\alpha}\).

**Proof of Theorem 9.** As in Theorem 6 by virtue of the condition \(\alpha < \min (1, q + \frac{1}{k})\), we have
\[
\sum_{1}^{\infty} \frac{1}{n} \leq \frac{1}{n} \left( \int_{0}^{\infty} \chi(u) u^{q-\alpha} |P(n,u)| \, du \right)^{k} \\
\leq \frac{1}{n} \left( \int_{0}^{\infty} \chi(u) u^{q-\alpha} |P(n,u)| \, du \right)^{k-1} \\
\leq C \frac{1}{n} \left( \int_{0}^{\infty} \frac{\chi(u)}{u} u^{l+q-\alpha} |P(n,u)| \, du \right)^{k} \\
= C \int_{0}^{\infty} \frac{\chi(u)}{u} u^{l+q-\alpha} \left( \sum_{1}^{\infty} \frac{1}{n} |P(n,u)| \right) \, du.
\]

Therefore, by virtue of the hypothesis it is sufficient to show that

\[
\sum_{1}^{\infty} \frac{1}{n} |P(n,u)| = O(u^{\alpha-q-1})
\]

uniformly in \(0 < u < \pi\). Since this result has already been established in Theorem 6, the proof of Theorem 9 is completed.