CHAPTER VII

ON THE ABSOLUTE CESARO SUMMABILITY
OF A FOURIER SERIES AND ITS CONJUGATE SERIES

7.1 Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable \( (L) \) over \(( -\pi, \pi ) \). Let

\[
\hat{f}(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right).
\]

Then the series conjugate to it is

\[
\sum_{n=1}^{\infty} \left( b_n \cos nt - a_n \sin nt \right).
\]

We write

\[
w(\theta,t) = f(t+\theta) - f(\theta),
\]

\[
w_{\alpha}(\theta,t)= \frac{1}{(\alpha)} \int_{0}^{t} (t-u)^{\alpha-1} w(\theta,u) \, du, \quad (\alpha > 0)
\]

\[
w_{0}(\theta,t) = w(\theta,t).
\]

7.2 The following theorem was obtained by Chow\(^1\) as a corollary of a theorem of Wang\(^2\).

**Theorem A.** If \( \Omega^{0}_p(t) = \Omega \left\{ \left( \log \frac{1}{|t|} \right)^{-1-\delta} \right\}, \quad (t \to 0) \)

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1) Chow, H.C. (33).

for some \( \delta > 0 \), then the Fourier series of \( f(t) \) and its conjugate series are summable \(| C, \beta |\) almost everywhere for every \( \beta > 1/2 \).

Generalizing Theorem A, Chow proved the following theorem.

**Theorem B.** Let \( 1 \leq p \leq 2 \). If

\[
\Omega_p^0(t) = O \left\{ \left( \log \frac{1}{|t|} \right)^{-1 - \delta} \right\}, \quad \delta > 0,
\]

or, more generally, if

\[
\int_{-\pi}^{\pi} \frac{\Omega_p^0(t)}{|t|} \, dt < \infty,
\]

then the Fourier series of \( f(t) \) and its conjugate series are both summable \(| C, \beta |\) almost everywhere for \( \beta > 1/p \).

Recently, Hsiang \(^1\) has succeeded in effecting further improvement of the theorem of Chow given above. He proved:

**Theorem C.** Let \( 1 \leq p \leq 2 \). If

(i) \[
\int_{-\pi}^{\pi} \frac{\Omega_p^0(t)}{|t|} \, dt < \infty,
\]

(ii) \[
\int_{-\pi}^{\pi} \frac{\Omega_p^1(t)}{t^2} \, dt < \infty,
\]

then the Fourier series of \( f(t) \) and its conjugate series are summable \(| C, \beta |\) almost everywhere for \( \beta > 1/p \).

It has been shown by Hsiang that condition (i) of Theorem C implies that

\(^1\) Hsiang, F.C. (56).
Although he stated his theorem with conditions (i) and (ii) he only used the condition (7.2.1) in place of (i). This could have been an oversight since condition (7.2.1) was not numbered by him.

The object of this chapter is to obtain a generalization of the above theorem of Hsiang (see Lemma 2 below).

7.3 In what follows we establish the following theorem.

**Theorem.** Let 1 ≤ p ≤ 2. If

(i)' \[ \sum_{n=1}^{\infty} \frac{1}{p} (u) \in B \left[ -\pi, \pi \right]. \]

and

(ii) \[ \int_{-\pi}^{\pi} \frac{\Omega_{1}^{\alpha}(t)}{|t|^{1+\alpha}} \, dt < \infty, \quad (\alpha > 1), \]

then the Fourier series of f(t) and its conjugate series are summable \( C, \beta \) almost everywhere for \( \beta > 1/p \).

7.4 The following lemmas are required for the proof of this theorem.

**Lemma 1.** Let \( \alpha > \beta > 0 \). If

\[ \sum_{n=1}^{\infty} \frac{\beta}{p} (u) \in B \left[ -\pi, \pi \right], \] then \[ \sum_{n=1}^{\infty} \frac{\beta}{p} (u) \in B \left[ -\pi, \pi \right]. \]

**Proof.** Let \( 0 < t < \pi \), then by Minkowski's inequality, we have

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1) cf. Hardy, G.H. and Littlewood, J.E. (52).
\[ \Omega^\alpha_p(t) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\left| (\alpha-\beta) \right|} \right) p \int_0^t \frac{(t-u)^{\alpha-\beta-1} w_\beta(\theta, u) du}{p \omega \theta} \right)^{1/p} \]

\[ \leq \int_0^t \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\left| (\alpha-\beta) \right|} \right) p \int_0^\pi (t-u)^{\alpha-\beta-1} w_\beta(\theta, u) \omega \theta \right)^{1/p} du \]

\[ = \int_0^t (t-u)^{\alpha-\beta-1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\left| (\alpha-\beta) \right|} \right) p \int_0^\pi w_\beta(\theta, u) \omega \theta \right)^{1/p} du \]

\[ = \frac{1}{\left| (\alpha-\beta) \right|} \int_0^t (t-u)^{\alpha-\beta-1} \Omega^\beta_p(u) du \]

\[ = \Omega(1). \]

Since \( \Omega^\beta_p(u) = \Omega(1) \).

Similarly if \(-\pi \leq t \leq 0\), we take

\[ \Omega^\alpha_p(t) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\left| (\alpha+\beta) \right|} \right) p \int_0^t \frac{(t+u)^{\alpha-\beta-1} w_\beta(\theta, u) du}{p \omega \theta} \right)^{1/p} \]

and the result continues to hold true.

**Lemma 2.** For \( \alpha > \beta > 0 \) we have

\[ \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} \int_{-\pi}^{\pi} \frac{\Omega^\alpha_p(t)}{t+\alpha} dt \leq \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)} \int_{-\pi}^{\pi} \frac{\Omega^\beta_p(t)}{t+\beta} dt. \]

**Proof.** Let \( 0 \leq t \leq \pi \), then we have

\[ \int_0^\pi \frac{\Omega^\alpha_p(t)}{t+\alpha} dt \leq \frac{1}{\left| (\alpha-\beta) \right|} \int_0^\pi \frac{dt}{t+\alpha} \int_0^t \frac{(t-u)^{\alpha-\beta-1} \Omega^\beta_p(u) du}{p \omega \theta} \]

\[ = \frac{1}{\left| (\alpha-\beta) \right|} \int_0^\pi \frac{\Omega^\beta_p(u) du}{p \omega \theta} \int_{t \alpha}^{\alpha+1} \frac{(t-u)^{\alpha-\beta-1} \Omega^\beta_p(u) du}{p \omega \theta} \]

\[ \leq \frac{1}{\left| (\alpha-\beta) \right|} \int_0^\pi \frac{\Omega^\beta_p(u) du}{p \omega \theta} \int_{t \alpha}^{\alpha+1} \frac{(t-u)^{\alpha-\beta-1} \Omega^\beta_p(u) du}{p \omega \theta} \]

\[ = \frac{1}{\left| (\alpha-\beta) \right|} \int_0^\pi \frac{\Omega^\beta_p(u) du}{p \omega \theta} \int_{t \alpha}^{\alpha+1} \frac{(t-u)^{\alpha-\beta-1} \Omega^\beta_p(u) du}{p \omega \theta} \]
\[
\frac{1}{|\alpha - \beta|} \left. \int_0^\pi \Omega_\beta (u) du \right|_{\frac{\Gamma (\beta+1)}{\Gamma (\alpha+1)}} \int \frac{t^{-1-\alpha} (t-u)^{\alpha-\beta} dt}{u^+} \\
= \frac{1}{|\alpha - \beta|} \int_0^\pi \Omega_\beta (u) \left. \frac{\Gamma (\beta+1)}{\Gamma (\alpha+1)} \right| \frac{\Gamma (\alpha-\beta)}{u^+} du \\
\leq \frac{\Gamma (\beta+1)}{\Gamma (\alpha+1)} \int_{-\pi}^\pi \Omega_\beta (u) \left. \frac{\Gamma (\alpha-\beta)}{|u|^1+} \right| du.
\]

In a similar manner we can prove the result for \(-\pi \leq t \leq 0\).

This completes the proof of Lemma 2.

**Lemma 3.1** Let \(c_0 = \frac{a_0}{2}, c_n = a_{n+1} b_n\) \((n \geq 1)\), and 

\[F(z) = \sum_{n=0}^\infty c_n z^n = \sum_{n=0}^\infty c_n z^n e^{ni\theta}.
\]

Then the Fourier series of \(f(t)\) and its conjugate series are both summable \(|C, \beta|\) almost everywhere on the unit circle for \(\beta > 1/p\), provided that 

\[
\int_0^\pi \left. \int_{-\pi}^\pi |F'(pe^{i\theta})|^p d\theta \right|^{1/p} dp
\]

is bounded as \(r \to 1-0\).

Since an implicit proof of this lemma is given in a journal which is not easily available, it seems desirable to include a direct proof for the convenience of the reader.

**Proof.** Let \(t_n^\beta (\theta)\) denote the \(n\)-th \((C, \beta)\) mean of the sequence \(\{n c_n e^{ni\theta}\}\), then it is sufficient to prove that 

\[
\sum_{n=0}^\infty \frac{|t_n^\beta (\theta)|}{n} < \infty
\]

almost everywhere.

Case (1). Let $1 < p \leq 2$. We have
\[ \varphi(z,0) = \lim_{\beta \to \infty} \beta \sum_{n=1}^{\infty} \left| A_n^\beta \right| \varphi_n(\beta \varphi) \left( \frac{z}{\beta \varphi} \right)^{1/p} \infty \left( 1-\beta \right)^{-\beta} \varphi(z,0), \]
where $A_n^\beta = \left( n+\beta \right)^\beta$.

By Hausdorff-Young's theorem it follows that
\[ \psi(r,\theta) = \left( \sum_{n=1}^{\infty} \left( A_n^\beta \right)^q \left| t_n^\beta(\theta) \right|^q r^{nq} \right)^{1/q} \quad \frac{1}{p} + \frac{1}{q} = 1. \]

Applying Hölder's inequality we obtain
\[ \int_0^{2\pi} \psi(r,\theta) d\theta \leq K \left( \int_0^{2\pi} \left| \psi(r,\theta) \right|^p d\theta \right)^{1/p} \]
\[ \leq K \left( \int_0^{2\pi} \left| \psi(r,\theta) \right|^p d\theta \right)^{1/p} \]

\[ = K \left( \int_0^{2\pi} \int_0^{2\pi} \left| 1-\beta \varphi \right|^p \left| rF'(re^{i\theta}) \right|^p d\xi d\theta \right)^{1/p} \]
\[ \leq K \left( \int_0^{2\pi} \left| 1-\beta \varphi \right|^p d\xi \int_0^{2\pi} \left| rF'(re^{i\theta}) \right|^p d\theta \right)^{1/p} \]
\[ \leq K \left( \int_0^{2\pi} \left| 1-\beta \varphi \right|^p d\xi \int_0^{2\pi} \left| rF'(re^{i\theta}) \right|^p d\theta \right)^{1/p} \]
\[ \leq K \left[ \left( \int_0^{1-r} \left( 1-r \right)^{2} \right)^{1/2} dx \right] \left( \int_0^{2\pi} \left| rF'(re^{i\theta}) \right|^p d\theta \right)^{1/p} \]
\[ \leq K \left( 1-r \right)^{1/2} \left( \int_0^{2\pi} \left| rF'(re^{i\theta}) \right|^p d\theta \right)^{1/p} \]

\[ \leq \left( \frac{2\pi}{r} \right)^{1/2} \left( \int_0^{2\pi} \left| rF'(re^{i\theta}) \right|^p d\theta \right)^{1/p} \]

1) cf. Hardy, G.H. and Littlewood, J.E. (52).
\[ T(r, \theta) = \sum_{n=1}^{\infty} A_n \frac{t_0^n(\theta)}{n} r^n \leq K (1-r)^{-\frac{\beta}{p}} \Psi(r, \theta) \]

so that
\[ \sum_{n=1}^{\infty} \left| \frac{t_0^n(\theta)}{n} \right| \leq K \int_0^1 (1-r)^{-\beta - \frac{1}{p}} \Psi(r, \theta) \, dr \]

Therefore
\[ \int_0^{2\pi} \left( \sum_{n=1}^{\infty} \left| \frac{t_0^n(\theta)}{n} \right| \right) \, d\theta \leq K \int_0^1 \left( \int_0^{2\pi} \left| r F'(re^{i\theta}) \right|^p \, d\theta \right)^{1/p} \left| \frac{1}{1-r^p} \right| ^{1/p} \]

by the hypothesis. Thus the series \( \sum_{n=1}^{\infty} \left| \frac{t_0^n(\theta)}{n} \right| \) converges almost everywhere.

**Case (ii).** Let \( p = 1 \). Since
\[ A_n t_0^n(\theta) = \frac{1}{2\pi i} \int_{|z|=r} z^{-n-1} (1-z)^{-\beta} e^{i\theta} \, dz \]

it follows that
\[ \sum_{n=1}^{\infty} \frac{A_n}{n} \left| t_0^n(\theta) \right| r^n \leq K (1-r)^{-1} \int_0^{2\pi} \frac{|r F'(re^{i(\theta+\xi)})|}{|1-re^{i\xi}|^{1+\beta}} \, d\xi. \]

Multiplying by \( (1-r)^{\beta} \) and integrating between \((0,1)\), we obtain
\[ \sum_{n=1}^{\infty} \frac{1}{n} \left| t_0^n(\theta) \right| \leq K \int_0^1 (1-r)^{-\beta - 1} \, dr \int_0^{2\pi} \frac{|r F'(re^{i(\theta+\xi)})|}{|1-re^{i\xi}|^{1+\beta}} \, d\xi \]

and therefore
This completes the proof of the lemma.

7.5 Proof of the Theorem. Without loss of any generality we can assume that $a$ is an integer.

Now

$$F'(re^{i\theta}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(t)e^{it}dt}{(e^{it} - r)^2} = \frac{e^{-i\theta}}{\pi} \int_{-\pi}^{\pi} \frac{f(t)e^{it}dt}{(e^{it} - r)^2}$$

$$= \frac{e^{-i\theta}}{\pi} \int_{-\pi}^{\pi} \frac{e^{it}w(\theta, t)dt}{(e^{it} - r)^2}$$

$$= \frac{e^{-i\theta}}{\pi} \left[ \frac{(-1)^s}{s!} \int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - r)^2} w_s(\theta, t) \right]^{n}$$

$$+ \frac{e^{-i\theta}}{\pi} \int_{-\pi}^{\pi} w(\theta, t) \left( \frac{d}{dt} \right)^a \frac{e^{it}}{(e^{it} - r)^2} dt$$

$$= L_1 + L_2, \text{ say.}$$

By Minkowski’s inequality we have

$$M_p(r, F') \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |L_1|^p \, d\theta \right)^{1/p} + \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |L_2|^p \, d\theta \right)^{1/p}$$

$$= M_1 + M_2, \text{ say.}$$

By virtue of Lemma 3 it is sufficient to prove that

(7.5.1) \( \int_{0}^{R} M_1 \, d\rho = O(1), \quad r \to 1 - 0, \)

(7.5.2) \( \int_{0}^{R} M_2 \, d\rho = O(1), \quad r \to 1 - 0. \)

1) see Chow, H.C. (33).
Now

\[-\frac{\partial}{\partial t} \frac{e^{it}}{(e^{it})^2} = A_0 g(t) + A_1 e^{it} g(t) + A_2 e^{2it} g(t) + \ldots + A_s e^{sit} g(t),\]

where \(A_i\)'s are constants, real or imaginary, depending upon \(s\) and

\[g(t) = \frac{e^{it}}{(e^{it})^2},\]

so that

\[M_1 = \sum_{s=1}^{\infty} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| w_s(e^{it}) \right|^p d\theta \right)^{1/p} \]

\[\quad + \sum_{s=1}^{\infty} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| w_s(e^{it}) \right|^p d\theta \right)^{1/p} \]

\[\quad = \sum_{s=1}^{\infty} \frac{1}{(1+p)^2} \Omega_p^s(-n) \quad + \sum_{s=1}^{\infty} \frac{1}{(1+p)^2} \Omega_p^s(-n) \]

\[= \sum_{s=1}^{\infty} \frac{1}{(1+p)^2} \Omega_p^s(-n) \quad \text{for} \quad s \geq 1.\]

Since \(\Omega_p^s(t) \in B[-\pi, \pi]\) for \(s \geq 1\).

Also

\[M_2 \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| w_\alpha (e^{it}) \frac{\partial}{\partial t} \right|^p g(t) \right)^{1/p} \]

\[\leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| w_\alpha (e^{it}) \frac{\partial}{\partial t} g(t) \right|^p d\theta \right)^{1/p} \]

\[\leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (-\frac{\partial}{\partial t})^\alpha g(t) \right|^p d\theta \right)^{1/p} \]

\[= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| w_\alpha (e^{it}) \frac{\partial}{\partial t} g(t) \right|^p d\theta \right)^{1/p} \]

\[= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (-\frac{\partial}{\partial t})^\alpha g(t) \right|^p d\theta \right)^{1/p} \]

\[= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (-\frac{\partial}{\partial t})^\alpha g(t) \right|^p d\theta \right)^{1/p} \]

\[= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (-\frac{\partial}{\partial t})^\alpha g(t) \right|^p d\theta \right)^{1/p} \]
\[
\leq K \int_{-\pi}^{\pi} \left( 1 - 2p \cos t + p^2 \right)^{-\frac{\alpha}{2}} \Omega_p^a(t) \, dt
\]

\[
= K \left( \int_{-\pi}^{0} + \int_{0}^{\pi} \right) = N_1 + N_2, \text{ say.}
\]

We write
\[
\int_0^\pi = \int_0^{\pi/2} + \int_{\pi/2}^\pi = P_1 + P_2, \text{ say.}
\]

Now
\[
\int_0^{P_1} dp = \int_0^{\pi/2} \left( \int_0^{\pi/2} \Omega_p^a(t) \left( 1 - 2p \cos t + p^2 \right)^{-\frac{\alpha}{2}} \, dt \right) \, dp
\]

\[
= O \left\{ \int_0^{\pi/2} \Omega_p^a(t) \, dt \int_0^1 \frac{dp}{(1 - 2p \cos t + p^2)^{1+\alpha/2}} \right\}
\]

\[
= O \left\{ \int_0^{\pi/2} \frac{\Omega_p^a(t) \, dt}{t^{1+\alpha}} \right\},
\]

Also
\[
\int_0^{P_2} dp = O \left\{ \int_0^\pi \Omega_p^a(t) \, dt \int_0^1 \frac{dp}{(1 - 2p \cos t + p^2)^{1+\alpha/2}} \right\}
\]

\[
= O \left\{ \int_0^\pi \Omega_p^a(t) \, dt \right\} = O \left\{ \int_{\pi/2}^{\pi} \frac{\Omega_p^a(t)}{t^{1+\alpha}} \, dt \right\}
\]

so that
\[
\int_0^{N_2} dp = O \left\{ \int_0^\pi \frac{\Omega_p^a(t) \, dt}{t^{1+\alpha}} \right\}.
\]
Similarly
\[ \int_{0}^{\pi} N_1 \, d\rho = O \left\{ \int_{-\pi}^{\pi} \frac{\Omega^\alpha p(t) \, dt}{|t|^{1+\alpha}} \right\}. \]

Therefore
\[ \int_{0}^{\pi} M_2 \, d\rho = O \left\{ \int_{-\pi}^{\pi} \frac{\Omega^\alpha p(t) \, dt}{|t|^{1+\alpha}} \right\} \]

\[ = O(1), \]

by virtue of the hypothesis.

Also
\[ \int_{0}^{\pi} M_1 \, d\rho = \bigg\{ \int_{0}^{1+\rho} \frac{1}{(1+\rho)^{2}} \, d\rho \bigg\} \]

\[ = O(1). \]

This completes the proof of the theorem.