If \( n \) is even, we have \( \sin n \alpha + \omega > 0 \) and hence \( \frac{dF}{d\theta} > 0 \) in this case, corresponding to a right position of the \( \alpha \).

\[
\sin(n \alpha + \omega) = \sin \alpha = \frac{-1}{\sqrt{A + B}}.
\]

\[
\cos(n \alpha + \omega) = \cos \alpha = \frac{B}{\sqrt{A + B}}.
\]

Therefore, for a solution \( 2k \pi \) we have

\[
F_p = F_p \cos \alpha = \cos \alpha \cos \alpha = \cos^2 \alpha + \omega
\]

\[
\frac{dr_p}{dn} = -\sqrt{A + B} - \omega
\]

\[
\text{for } \frac{2\pi}{2} < \alpha < \pi \text{ or } \frac{3\pi}{2} < \alpha < 2\pi.
\]

\[
\tan \alpha = -1 > 0; 0 < \alpha < \frac{\pi}{2}.
\]

If \( n \) is even, we have \( \sin n \alpha + \omega > 0 \) and hence \( \frac{dF}{d\theta} < 0 \) and this corresponds to a left position of the \( \alpha \).

\[
\sin(n \alpha + \omega) = \sin \alpha = \frac{-1}{\sqrt{A + B}}.
\]

\[
\cos(n \alpha + \omega) = \cos \alpha = \frac{B}{\sqrt{A + B}}.
\]
\[ r_a = r_{\text{apogee}} = r + r \left( \sin \alpha \cos \omega + \sin \omega \right) \]

\[ = \left( n \tau + \omega \right) \sqrt{1 + B} \]

Hence

\[ \frac{dr_a}{dn} = r \sqrt{1 + B} > 0 \quad (2.3.4) \]

If \( n \) is odd, \( \cos \omega + \omega < 0 \) and hence \( \frac{dr_a}{dn} > 0 \) and this corresponds to perigee position. In this case

\[ \sin \omega + \omega = \sin \alpha = \frac{-\tan \theta}{\sqrt{1 + B}} \]

\[ \cos \omega + \omega = \cos \alpha = \frac{-\sin \theta}{\sqrt{1 + B}} \]

Therefore, from equation (2.2.7), we have

\[ r_b = r_{\text{perigee}} = r + r \left( \sin \alpha \cos \omega - \sin \omega \right) \]

\[ = \left( n \tau + \omega \right) \sqrt{1 + B} \]

Hence

\[ \frac{dr_b}{dn} = -r \sqrt{1 + B} < 0 \quad (2.3.5) \]

Note that whether \( n > 0 \) or \( n < 0 \), \( \omega \) is positive or negative at \( \omega = n \tau + \omega \), \( \alpha = \tan^{-1} \left( -\frac{\sin \theta}{\cos \theta} \right) \) according as \( n \) is odd or even. Further, we also note from equations

\[ \text{...} \]
Therefore, in the steady orbit at the origin, the angular position $\omega$ is fixed at $\omega + \omega_0$ and the mean angular velocity $\omega_0$ is fixed at $-\frac{1}{2}$ or $+\frac{1}{2}$ so that change is due to the magnitude $\omega_0$ of the angular velocity.

It is evident that $\omega$ is everywhere greater than $\omega_0$, since in the limit when $\omega_0 = 0$, $\omega = \frac{d\theta}{dt}$, and at $\omega = \omega_0$ the line of equation $\omega = \omega_0$ for $\phi = \omega_0$ and $\phi + d\phi < 0$, whereas for $\omega = \omega_0$ and $\phi + d\phi > 0$, we have $\omega = \omega_0$ for $\phi = 0$.

This is in accord with the theorem of total angular momentum $L$ (2.3.1) by putting $\theta = 0$ and $\phi = 0$.

We have

\[ \dot{r}_0 = \frac{2\pi}{\omega_0} \left( \frac{r_0^2}{2\pi} + \frac{2\pi}{\omega_0} \dot{r}_0 \right) \text{ or } \frac{\dot{r}_0}{\omega_0} = \frac{2\pi}{\omega_0} \left( \frac{r_0^2}{2\pi} + \frac{2\pi}{\omega_0} \dot{r}_0 \right) \]

and

\[ \dot{r}_0 = \dot{r}_0 \dot{\omega}_0 = \frac{k}{2\pi} \left( \frac{1}{\omega_0} \right) \left( 2\pi \omega \right) = \frac{k}{\omega_0} \left( 2\pi \omega \right) = \frac{2\pi k}{\omega_0} \omega, \]
\[- \frac{2}{3} \cdot \frac{\text{J} \sin e}{p} \sin e (2m + \varphi) = \frac{L_0^2}{G} (2 - \frac{14}{3} \sin e) \]
\[- \frac{\text{J} \sin e}{p} - \frac{L_0^2}{G} \cdot \frac{16 + 27 \sin^4 e - \frac{33}{36} \sin e}{7r_0^{3/2}} \]

Note that the angular position of the object remains fixed at \(-\pi/2\). The object is both solar pressure and oblateness to reduce \(r_0\) by first approximation, \(p = r_0\).

Upto \(0(10^{-4})\), the same result of \(r_0\) is not seen in the expression of \(p\) when the approximation \(r_0\) and \(p = 0\) are.

\[
\frac{dr}{dt} = \frac{3kR^3}{2} \sin e (2m + \varphi) \sin e (2 - \frac{8}{3} \sin e) \approx \frac{3kR^3}{2} \sin e (2m + \varphi) \sin e (2 - \frac{8}{3} \sin e) < 0.
\]

since \((- \frac{2}{3} \sin e) > 0\) for \(\sin e = \frac{2}{3}\), the rate of decrease of \(r_0\) with each revolution is constant. Further, the effect of oblateness is coupled with solar pressure.

The angular position of the object also remains fixed at \(-\pi/2\). The effect of oblateness is coupled with solar radiation pressure, which is the weight of the object. In first approximation, \(r_0 = r_0\) up to \(0(10^{-4})\), but it enters for \(10^{-5}\) \(0(10^{-5})\). Gain

\[
\frac{dr}{dt} = \frac{3kR^3}{2} \times 2m + \frac{L_0^2}{G} \times \frac{1}{4} (1 - \frac{9}{2} \sin e) \times 2 \varphi > 0,
\]

the time of fall of \(r_0\) is also constant. Since the rate of increase of \(p\) and rate of increase of e
of apogee is the same, the semi-major axis \( \frac{1}{2} (\text{perigee } + \text{apogee}) \) remains fixed and therefore the period of revolution is approximately constant.

(b) Suppose \( \theta = 0 \) then \( \theta = 0 \) and therefore again \( \theta = n\pi - \pi/2 \); that is, the case when the sun is fixed anywhere in the equatorial plane at \( x = -\infty \). Further, we note that

\[ \frac{dR}{dt} > 0 , \text{ when } n \text{ is even } (= 2m, \text{ say}), \text{ we get } \text{perigee}, \text{ and} \]

\[ \frac{dR}{dt} < 0 , \text{ when } n \text{ is odd } (= 2m+1, \text{ say}), \text{ we get } \text{apogee}, \text{ where} \]

\( m \) is zero or positive integer.

g et the perigee or apogee distances from equation

(2.3.1): by putting \( \theta = n\pi - \pi/2 \), \( fn = 2m \) or \( 2m+1 \) we have

\[ r_p = r_{\text{perigee}} = r_0 - \frac{K}{2\mu} \left( 3r_0^2 + 2J_3 r_0 \right) \times 2m \]

and

\[ r_a = r_{\text{apogee}} = r_0 + \frac{K}{2\mu} \left( 3r_0^2 + 2J_3 r_0 \right) (2m+1) = \]

\[ \frac{2J_3}{r_0} = \frac{3K}{r_0^2} = \frac{6K}{7r_0}. \]

We note that in this case also, the angular position of perigee remains fixed at \( -\pi/2 \). The effect of oblateness coupled with solar radiation pressure is to decrease the height of the perigee. As a first approximation \( r_p = r_0 \) (upto \( 0(10^{-4}) \)), the oblateness terms do not enter the expression of \( r_p \) but it does enter if we take \( r_p \) upto \( 0(10^{-3}) \).
where

\[ f_r = \frac{\partial \mathbf{r}}{\partial r} = -\frac{\sin^3 \theta - \sin \theta - 1}{r^2} \cdot \frac{\sin^3 \theta}{r^2} \cdot \frac{1}{7r^3} \]

\[ f_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -\frac{2\sin^3 \theta \cos \theta + \frac{\sin \theta}{7r^3}}{r^3} \]

\[ 28\sin^3 \theta \cos \theta - 6\sin 2\theta \]

and \( V \) is the potential of the Sun given by (2.3.1).

Putting \( r = r^2 \theta \) in the second equation of (2.4.1),

we have

\[ \frac{dp}{dt} = -\frac{2\sin^3 \theta \cos \theta + \frac{\sin \theta}{7r^3}}{r^3} \cdot 39\sin^3 \theta \cos \theta \]

\[ \times \cos \theta - 6\sin 2\theta - k \cdot r \cdot \sin \theta \]

Hence

\[ \frac{d^2p}{dt^2} = 2\rho \frac{dp}{dt} = 2\rho \cdot \frac{dp}{dt} \cdot \frac{dp}{dt} \]

\[ = -\frac{4\sin^3 \theta \cos \theta + \frac{2\sin \theta}{7r^3}}{r^3} \]

\[ \times (28\sin^3 \theta \cos \theta - 6\sin 2\theta) - \]

\[ - 2\rho \sin \theta \]

(2.4.3)

Integrating, we get
\[ p^2 = \frac{\mu_0^2}{r_0} \cos 2\theta + \frac{2 \omega \mu_0^4}{7 r_0^3} (7 \sin^4 \theta + 3 \cos 2\theta) + 2 \kappa r_0 \cos \theta + c_1 \quad 0 < \theta < \pi_2 \]

or

\[ p^2 = \frac{\mu_0^2}{r_0} \cos 2\theta + \frac{2 \omega \mu_0^4}{7 r_0^3} (7 \sin^4 \theta + 3 \cos 2\theta) + c_2 \quad 0 < \theta < \pi_1 \]

where \( c_1 \) and \( c_2 \) are constants of integration. Applying initial conditions \( \theta_0 = -\pi/2, \quad r_0 = \mu_0, \quad \omega = 0 \) we get

\[ c_1 = \mu_0 r_0 + \frac{2 \omega \mu_0^4}{r_0} - \frac{\omega \mu_0^4}{7 r_0^3} \]

and

\[ c_2 = \mu_0 r_0 + 2 \kappa r_0 \cos \theta + \frac{2 \omega \mu_0^4}{r_0} - \frac{\omega \mu_0^4}{7 r_0^3} \]

and finally after simplification we get

\[ p^2 = \mu_0 r_0 \left[ 1 + \frac{2 \kappa r_0}{\mu} \cos \theta + \phi(0) \right], \quad 0 < \theta < \pi_2 \quad (2.4.3) \]

\[ p^2 = \mu_0 r_0 \left[ 1 + \frac{2 \kappa r_0}{\mu} \cos \theta + \phi(0) \right], \quad 0 < \theta < \pi_1 \quad (2.4.4) \]

where

\[ \phi(0) = \frac{\omega \mu_0^4}{r_0^3} \cos \theta + \frac{2 \omega \mu_0^4}{7 r_0^3} (7 \sin^4 \theta + 3 \cos 2\theta). \]

Now, taking \( u = \frac{1}{p} \) in (2.4.1) and proceeding as in section (2.2), we get
\[ d^2 u + u = \frac{1}{p^2} - \frac{u_{0}^4}{p^2} (3\sin^4 \theta - 1) - \frac{K \cos \theta}{u^2} + \]

\[ \frac{1}{7p^2} (35\sin^4 \theta - 30\sin^2 \theta + 3) - \]

\[ \frac{1}{p^2} \left( \frac{p dp}{d\theta} \right) \]  

(2.4.5)

Putting the value of \( p^2 \) from (2.4.3) in (2.4.5), we have

\[ d^2 u + u = \left( \mu - \mu_0 \sin^2 \theta (3\sin^4 \theta - 1) - \frac{K \cos \theta}{u^2} + \right. \]

\[ \frac{1}{7p^2} (35\sin^4 \theta - 30\sin^2 \theta + 3) - \]

\[ \frac{1}{d\theta} \left( \frac{p dp}{d\theta} \right) \frac{1}{\mu_0} \left[ 1 + \frac{2K\rho_1^3}{\mu} \cos \theta + \right. \]

\[ \frac{2\mu_0^{1/2}}{r_0} \cos^2 \theta + \frac{2\mu_0^{1/4}}{7r_0^3} (7\sin^4 \theta + 3\cos 2\theta - 4) \]

and after simplification, we obtain

\[ d^2 u + u = \frac{1}{r_0^2} - \frac{3K\rho_1^3}{\mu} \cos \theta - \frac{J_{0}^4}{r_0^3} (1 + \sin^2 \theta) + \]

\[ \frac{4K\rho_1^3}{\mu_0} (1 + \cos^2 \theta) \cos \theta + \frac{6K\rho_1^3}{\mu} \cos^4 \theta + \]

\[ \frac{3\rho_1^4}{r_0^5} (21\sin^4 \theta - 24\sin^2 \theta + 8) + \]

\[ \frac{2\rho_1^4}{r_0^5} \cos^2 \theta (2 - \cos^2 \theta) \]

\[ = \frac{1}{r_0^2} - f(0); \quad 0 < r < r_0 \]  

(2.4.6)
where

\[ f(\theta) = \frac{2k}{\mu r^3} \cos \theta + \frac{2k}{r_0^3} (1 - \sin^2 \theta) - \frac{4k}{\mu r_0^3} (1 + \cos^2 \theta) \times \]

\[ \times \cos \theta - \frac{6k}{\mu^2} \cos^2 \theta - \frac{2k}{7r_0^5} (21 \sin^3 \theta \cos \theta - 24 \sin^2 \theta \cos \theta) - \frac{21k}{r_0^5} \cos^3 \theta (1 - \cos^2 \theta) \]  \hspace{1cm} (2.4.7)

and for \( 0 \leq \theta \leq \pi \), for the shadow portion, putting \( \kappa = 0 \)

in equation (2.4.5) we obtain

\[ \frac{d^2 y}{d\theta^2} + u = \frac{1}{\mu} - \frac{2k}{r_0^3} (5 \sin^2 \theta - 1) + \frac{4k}{r_0^3} \frac{1}{7} \times \]

\[ (35 \sin^4 \theta - 30 \sin^2 \theta + 3) \] \hspace{1cm} (2.4.8)

Putting the value of \( \mu' \) from (2.4.4) in (2.4.8) we get

\[ \frac{d^2 y}{d\theta^2} + u = \left( \frac{\mu - \mu \sin^2 \theta (5 \sin^2 \theta - 1) + \frac{4k}{r_0^3} \frac{1}{7} \times \right) \]

\[ (35 \sin^4 \theta - 30 \sin^2 \theta + 3) \sqrt{\frac{k}{r_0^3}} [1 + \frac{2k}{\mu} \times \]

\[ \times \cos \theta + \frac{2k}{r_0^3} \cos \theta - \frac{2k}{7r_0^5} (17 \sin^4 \theta + \]

\[ + 3 \cos 2\theta) \]  \hspace{1cm} (2.4.9)

and after simplification, we obtain

\[ \frac{d^2 y}{d\theta^2} + u = \frac{1}{r_0^3} \left[ \frac{2k}{\mu} \cos \theta_1 - \frac{2k}{r_0^3} (1 - \sin^2 \theta) \right] + \]

\[ \left( \frac{2k}{\mu} \cos \theta_1 + 2 \cos \theta_1 \cos \theta_1 + \frac{4k}{r_0^3} \frac{1}{7} \cos \theta_1 \right) \]
\[
\begin{align*}
&= \frac{1}{r_0} - f(0); \quad 0 \leq r_0 < 1 \\
&= \frac{2Kr_0^2}{\mu} \cos^2 \theta_1 + \frac{\lambda_0^2}{r_0^3} (1 + \sin \theta) - \frac{2\lambda_0^4}{r_0^3} \sin \theta \\
&\times (2 - \cos^2 \theta) \\
&\text{where} \\
f(0) &= \frac{2Kr_0^2}{\mu} \cos^2 \theta_1 + \frac{\lambda_0^2}{r_0^3} (1 + \sin \theta) - \frac{2\lambda_0^4}{r_0^3} \sin \theta \\
&\times (2 - \cos^2 \theta) \\
&\text{In order to find out the solutions of the differential equation (2.4.8) and (2.4.9), we express } f(0), \text{ from (2.4.7) and (2.4.10), by its Fourier series in the form} \\
f(0) &= \lambda_0 + \frac{1}{\mu} \cos \theta + \sum_{n = 2}^{\infty} \lambda_n \cos n \theta \\
&\text{where} \\
\lambda_0 &= \frac{1}{2\pi} \int_{0}^{2\pi} f(0) \, d\theta + \frac{1}{2\pi} \int_{0}^{2\pi} f(0) \, d\theta \\
&= I_1 + I_2, \text{ say} \\
&\text{Now}
\end{align*}
\]
\[ I_1 = \frac{1}{2} \left[ -\frac{3K E}{\mu} \sin \theta_2 + \frac{K}{2r_0^3} \left( (3 \theta_2 - \theta_1) \sin \theta_1 \cos \theta_1 \right) - \cos \theta_1 \right] + \frac{2\sin \theta_1}{\mu r_0} \left[ 7 \sin \theta_1 + \frac{1}{2} \sin 3 \theta_1 \right] - \frac{K r_0^3}{\mu^2} \left( (4 \theta_2 - \theta_1) \sin \theta_1 \cos \theta_1 \right) + \frac{2K r_0^3}{7 \times 32r_0^3} \times \right.

\[ \times \left. \{ 2 \sin 4 \theta_1 + 24 \sin 2 \theta_1 - 12 \sin 2 \theta_1 - 124 \sin - \theta_1 \} \right] + \frac{K r_0^3}{16r_0^3} \left( 8 \sin \theta_1 - \sin 4 \theta_1 - 30 \sin \theta_1 \right) \]

and

\[ I_2 = \frac{1}{2} \left[ \frac{2K E}{\mu} \theta_1 \cos \theta_1 \right] + \frac{1K}{2r_0^3} \left( (3 \theta_1 - \sin \theta_1 \cos \theta_1 \right) - \frac{K r_0^3}{\mu^2} \cos \theta_1 \left( 5 \theta_1 - \sin \theta_1 \cos \theta_1 \right) - \frac{K r_0^3}{\mu^2} \times \]

\[ \times \cos \theta_1 \left( 7 \times 32r_0^3 \right) \times \]

\[ \times \left. \left( 12 \sin 4 \theta_1 + 24 \sin 2 \theta_1 - 124 \sin - \theta_1 \right) \right] + \frac{K r_0^3}{16r_0^3} \left( 8 \sin \theta_1 - \sin 4 \theta_1 - 30 \sin \theta_1 \right) \]

Substituting the values of \( I_1 \) and \( I_2 \) in equation (2.4.12), we obtain

\[ A_0 = \frac{K r_0^3}{\mu^2} \left[ 3 \sin \theta_1 + 2 \theta_1 \cos \theta_1 \right] + \frac{2K r_0^3}{2r_0^3} \times \]

\[ + \frac{2K r_0^3}{\mu r_0} \left( 14 \sin \theta_1 \sin \frac{3}{2} \sin - \theta_1 - 5 \theta_1 \cos \theta_1 - \sin \theta_1 \cos \theta_1 \right) \]
\[ - \frac{K^2 r^3}{\mu r^3} \left[ (m-2)_1 \cos \theta_1 (3 \sin \theta_1 - 4 \theta_1 \cos \theta_1) \right] - \]

\[ - \frac{20}{56} \cdot \frac{x_{14}}{r_0} - \frac{5}{4} \cdot \frac{x_{13}}{r_0} \quad (12.4.15) \]

\[ A_1 = \frac{1}{r} \int_0^\theta f(\theta) \cos \theta \, d\theta + \frac{1}{r} \int_0^{2\pi} f(\theta) \cos \theta \, d\theta \]

\[ = \frac{K r_0}{\mu r_0} \left[ \frac{\sin 2\theta}{2} \sin 2\theta + 3 \sin 2\theta \right] - \frac{20}{56} \frac{x_{13}}{r_0} \left[ 7(n-2)_1 \right] - \]

\[ - 4 \sin 2\theta \frac{1}{2} \sin 4\theta_1 (11 \sin 6\theta_1 + \frac{1}{2} \sin 5\theta_1 \cos \theta_1) \]

\[ + \frac{K^2 r^3}{\mu r^3} (9 \sin \theta_1 + \sin 3 \theta_1 - 6 \sin^2 \theta_1 \sin \theta_1), \quad (12.4.14) \]

and

\[ m = \frac{1}{r} \int_0^\theta f(\theta) \cos m\theta \, d\theta + \frac{1}{r} \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta \]

\[ = \frac{K r_0}{\mu r_0} \left[ \frac{-(4-2m) \sin m\theta_1 \cos \theta_1}{m^2 - 1} - \frac{6 \sin(m-1)\theta_1}{m-1} \right] + \]

\[ + \frac{K^2 r^3}{\mu r^3} \left( \frac{1}{m^2 - 1} \right) \sin m\theta_1 \cos \theta_1 \]

\[ + \frac{2}{m^2 - 1} \sin(m-1)\theta_1 \]

\[ + \frac{4}{m(m-1)} \sin \theta_1 \cos \theta_1 \frac{2 \cos \theta_1}{m^2 - 4} \left[ 2 \sin(m-2) \theta_1 \right] + \]

\[ + \frac{K^2 r^3}{\mu r^3} \left( \frac{2}{m^2} \right) \sin \theta_1 \cos \theta_1 \]

\[ + \frac{2}{m^2} \left( \frac{1}{m^2 - 4} \cos \theta_1 \right) \]
\[ x \sin \theta_1 \pm \frac{1}{m-1} \left[ 2(m-2) \sin \theta_1 \cos \theta_1 \right] \]

\[ + 4 \sin(m-2) \theta_1 \]

Now taking the above expression of \( f(0) \) from equation (2.4.11), the solution of the equations (2.4.6) and (2.4.9) may be written in the form

\[ u = c_1 \cos \theta + c_2 \sin \theta + \frac{1}{r_0} - \gamma_0 - \frac{\gamma}{2} \theta \sin \theta \]

\[ + \sum_{m=2}^{\infty} \frac{\lambda_m \cos m\theta}{(m^2-1)} \]  

(2.4.15)

Applying initial conditions \( u = \frac{1}{r_0} \), \( \theta = -\pi/2 \), we get

\[ c_2 = -\gamma_0 - \frac{\gamma}{2} + 1 \]

where

\[ \gamma = \sum_{m=1}^{\infty} \frac{(-1)^m \lambda_m}{4m^2-1} \]  

(2.4.16)

and

\[ c_1 = -\left( \frac{\gamma}{2} + \gamma_0 \right) \]

where

\[ \gamma_0 = \sum_{m=1}^{\infty} \frac{(2m+1)(-1)^{m+1} \lambda_{2m+1}}{(2m+1)^2-1} \]

Putting the values of \( c_1 \) and \( c_2 \) in equation (2.4.15), we get
\[ u = \frac{1}{r_0} - r_0 - \left( \frac{1}{2} + \rho_0 \cos \theta \right) \sin \theta - \left( \frac{1}{2} - \rho_0 \sin \theta \right) \cos \theta = \frac{1}{r_0} - \left( \frac{1}{2} + \rho_0 \cos \theta \right) \sin \theta - \left( \frac{1}{2} - \rho_0 \sin \theta \right) \cos \theta \]

\[ \text{we note that } \frac{1}{m} \text{ behaves like } \frac{1}{m} \text{ and therefore } \frac{1}{m} \text{ and } \frac{1}{m^2} \text{ respectively. Also}
\]

\[ \sum_{m=2}^{\infty} \frac{\cos m \theta}{(m-1)m} \text{ behaves like } \frac{1}{m^3} \text{. All the terms of equation (2.4.17) are periodic and bounded except } \theta \sin \theta, \]

since after a no definite number of revolutions, this will be the dominant term we have

\[ \theta_{\text{apex}} = 2n\pi \pm \frac{\pi}{2} \]

Putting these values of \( \theta \) in (2.4.17), we get

\[ U_{\text{penke}} = \frac{1}{r_0} - r_0 - \left( \frac{1}{2} + \rho_0 \cos \theta \right) \sin \theta + \frac{1}{2} \cdot 2m + 1 \]

and finally, we get

\[ r_{\text{penke}} = \sqrt{2} \left[ 1 - 2n\pi \times \frac{2\pi}{2\mu} \left[ 1 - \frac{\theta}{\pi} + \frac{\sin 2\theta}{6\pi} \right] \right] + \]

\[ 2n\pi \times \frac{7\mu \sin \theta}{2\mu} \left[ 1 - \frac{\theta}{\pi} - \frac{\pi}{2} \sin 2\theta \right] - \frac{1}{28\pi} \sin 4\theta + \frac{\cos \theta}{7\pi} \sin 3\theta + \frac{k}{3} \times \]

\[ \times \sin 3\theta \] \[ = 2n\pi \times \frac{\kappa \sin \theta}{2n\pi} \left[ 9 \sin 3\theta \right] + \sin 3\theta - 8\cos \theta \sin \theta \] \[ \text{(2.4.18)} \]
and

\[ r_{\text{apogee}} = r_0 \left( 1 + 2r_0 \left( \frac{1}{A_0} - \frac{1}{A_1} \right) + \frac{1}{2} \left( \frac{A_0}{A_1} \right) ^2 \right) \]

where \( A_0 \), \( A_1 \) and \( \frac{A_0}{A_1} \) are given by equations (2.4.13), (2.4.14) and (2.4.16).

For a low to medium altitude satellite, \( A_1 \) is large and therefore \( \frac{A_0}{A_1} \) is the dominant term in equation (2.4.18). Now as \( \frac{A_0}{A_1} \) is the portion of the circumference in shadow, we note that the effect of shadow is to diminish the height of the perigee by an amount equal to the distance of the path which is in shadow. Further, for a very distant satellite \( A_1 \to 0 \) and therefore the effect of shadow is almost negligible.

2.5 Perigee and Apogee Distances and Their Locations For Elliptic Orbit

In the previous sections, the orbit of the satellite considered so far was circular. For a fixed source and the angular positions of perigee and apogee were found to remain fixed at \( \pm \pi/2 \), when the satellite move in the equatorial plane. But if the orbit of the satellite is elliptic, the position of perigee and apogee may not remain fixed.

In this section, we shall try to find special distances and their angular positions for elliptic case taking the radiation source as fixed. The technique of solution will require expansions of various expressions in terms of the
eccentricity \( e \), however the eccentricity considered is of the order of 0.1 \( \times 10^{-3} \).

As before, the potential of the Earth at an external point is given by the equation (2.4.1).

As in equation (2.4.1), the equations of motion of the satellite are given by

\[
\frac{d\mathbf{r}}{dt} = - \frac{\mathbf{r}}{r^3} + \frac{\mathbf{H} - 1.19}{r^5} \times (35 \sin^3 \theta - 30 \sin \theta \cos \theta) + \mathbf{v} \times \mathbf{v}
\]

\[
\frac{d^2 \mathbf{r}}{dt^2} = - \frac{2H \sin^3 \theta \cos \theta}{r^3} - \frac{\mathbf{H} - 1.19}{7r^3} \times (28 \sin^3 \theta \cos \theta - 6 \sin \theta \cos \theta) + \mathbf{v} \times \mathbf{v}
\]

As before, if we put \( \mathbf{F} = \mathbf{r} \times \dot{\mathbf{r}} \) in equation (2.5.2) we get

\[
\frac{d\theta}{dt} = \frac{\mathbf{H} - 1.19}{r^3} \times (28 \sin^3 \theta \cos \theta - 6 \sin \theta \cos \theta) - 6 \sin 2\theta \times \mathbf{v} \times \mathbf{v}
\]

and

\[
\frac{d(\mathbf{e})}{d\theta} = \frac{\mathbf{H} - 1.19}{r^3} \times (28 \sin^3 \theta \cos \theta - 6 \sin \theta \cos \theta)
\]

The equation of the ellipse is
\[ r = \frac{1}{1 + e \cos(\theta - \psi)} \]

where \( \psi \) is the initial angular position of the particle and \( e = \sqrt{1 - \mu} \), the eccentricity of the ellipse.

Using this value of \( e \) in Eq. (2.5.3), we have

\[ \frac{dr}{d\theta} = -2 \sin \theta \sqrt{1 - \cos^2(\theta - \psi)} \left( 2 - \frac{3}{2} \sin^2 \theta \right) - \frac{4 \mu}{3} \left( 28 \sin^2 \theta \cos \theta \right) \]

Integrating, we get

\[ r' = 2k \left( 1 - \frac{1}{2} \sin \theta \right) \left( 1 - 2 \sin^2 \theta \right) \left( 1 - \frac{3}{2} \sin^2 \theta \right) \left( 1 - \frac{7}{6} \sin^2 \theta \right) \]

\[ \times (7 \sin^2 \theta \cos \theta + 1) \]

(2.5.4)

Applying initial conditions \( \theta = 0 \), \( r' = \infty \), we obtain

\[ r_1 = \frac{1}{1 - 3 \cos \psi} \cos \psi - \frac{\sqrt{3}}{2} \sin \psi + \frac{2 \mu}{3} \]

Substituting the value of \( r_1 \) in Eq. (2.5.4), we finally get
\[ l^{-d} = \mu \left( 1 + \frac{KL^2}{\mu} \right) \left( 2\cos \theta - 3e \cos \psi (1 - \sin^2 \theta) + \frac{3}{2} e \right) \times \left( \sin \psi (20 - \sin 2\theta + \pi) + \frac{2\mu}{l^2} (1 - \sin^2 \theta) + \frac{2\mu}{7l^2} (7\sin^4 \theta + 3\cos 2\theta - 4) \right) \]  

(2.5.5)

Now, taking \( u = \frac{1}{p} \) in equation (2.5.1) and proceeding as in section (2.2), we get

\[ \frac{d^2u}{d\theta^2} + u = \frac{u}{p^2} \left( \sin^2 \theta - 1 \right) - \frac{K \cos \theta}{p^2} \times \left( \frac{2\mu}{7p^2} (35 \sin^4 \theta - 30 \sin^2 \theta + 3) - \frac{1}{p^2} \sin \theta \times \left( p \frac{du}{d\theta} \right) \right) \]  

(2.5.6)

Putting the value of \( l^{-d} \) from equation (2.5.5) in equation (2.5.6), we have

\[ \frac{d^2u}{d\theta^2} + u = \left( \frac{u - \mu A_{-d} \cdot u (\sin^2 \theta - 1) - \frac{K \cos \theta}{u} }{u} \right) \times \left( \frac{2\mu}{7p^2} (35 \sin^4 \theta - 30 \sin^2 \theta + 3) - \frac{1}{p^2} \sin \theta \times \left( p \frac{du}{d\theta} \right) \right) \times \left( 1 - \sin^2 \theta + \frac{3}{2} e \sin \psi (20 - \sin 2\theta + \pi) \right) \]

\[ + \frac{2\mu}{l^2} \left( 1 - \sin^2 \theta \right) + \frac{2\mu}{7l^2} (7\sin^4 \theta + 3\cos 2\theta - 4) \]  

and after simplification, we obtain
\[
\frac{\text{d}^2 u}{\text{d}o^2} + u = \frac{1}{l} - \frac{3k\lambda}{\mu} \left[ \cos \theta \cos \psi \cos \phi \phi + \frac{1}{2} \epsilon \sin \psi \phi \right]
\]
\[
(2 \theta - \sin 2 \theta) - \frac{4a^2}{l^3} \left( 1 + \sin^2 \theta \phi \right) + \frac{K a^4}{7 l^5} \times
\]
\[
(2 \sin \theta \psi \cos 2 \theta \phi) + \frac{4 a^2}{l^3} \left( 1 + \cos \psi \phi \right) \cdot \cos \phi (1 + \cos \psi \phi) \cdot
\]
\[
+ \frac{6 a^2 \lambda}{l^3} \cos \phi \cdot \frac{4 a^2}{l^3} \left( 1 - \sin \theta \phi \right) \cdot
\]

Solving the above differential equation, we get

\[
u = \ln \cos \theta \cdot B \sin \theta \phi + \frac{1}{l} - \frac{3k\lambda}{\mu} \left[ \frac{1}{2} \epsilon \cos \theta \sin \psi \phi - \cos \theta \cdot \frac{3 \sin \theta \psi \cos 2 \theta \phi}{2 l^3} + \frac{1}{2} \epsilon \cos \psi \cos 2 \theta \phi + \frac{1}{2} \epsilon \sin \psi \phi \cdot 2 \theta \phi \right]
\]
\[
+ \frac{1}{3} \sin 2 \theta \phi - \frac{4 a^2}{l^3} \left( 5 \cos \frac{2 \theta \phi}{3} \right) + \frac{K a^4}{280 l^5} \times
\]
\[
( - 7 \cos 4 \theta \phi + 20 \cos 2 \theta \phi + 155) + \frac{K a^4}{2 l^3} (70 \sin \theta - 1)
\]
\[
- \frac{1}{4} \cos 3 \theta \phi - \frac{3 K a^3}{l^5} \left( 1 - \cos \frac{2 \theta \phi}{3} \right)
\]
\[
- \frac{9 a^4}{30 l^5} \left( \cos 4 \theta \phi + 20 \cos 2 \theta \phi - 45 \right) (2.5.7)
\]

where \( \lambda \) and \( B \) are constants of integration.

Now we

\[
u = \frac{4 \pi e \cos (\theta - \psi)}{l}
\]

\( t \theta = -\sqrt{2} \). We want \( \psi \).
\[ u_o = \frac{1-e^\sin \psi}{a(1-e^d)} \]

and

\[
\frac{d}{do} = \frac{e \cos \psi}{a(1-e^d)}
\]

Applying the above initial values in equation (2.5.7)

\[
B = \frac{e \sin \psi}{b} - \frac{3K\lambda}{\mu} \left( \frac{a}{4} - \frac{2}{3} c \cos \psi \right) - \frac{4}{3} x
\]

\[ = - \frac{16}{5} \frac{16}{3} \frac{27}{4} \frac{1}{\mu} \]

and

\[
= \frac{e \cos \psi}{b} + \frac{3K\lambda}{\mu} \left( - \frac{1}{2} + \frac{2}{3} c \sin \psi \right)
\]

\[ + \frac{25}{8} \frac{16}{3} \frac{1}{\mu} \]

Substituting the values of \( \phi \) and \( \theta \) in equation (2.5.7), we obtain

\[
u = \frac{1-e^\cos \theta - \frac{3K\lambda}{2\mu} \cos \theta \left[ \cos \theta + \theta \sin \theta \right]}{2\mu}
\]

\[ - \frac{3K\lambda}{2\mu} \left( \frac{1}{2} \sin \psi - \cos \psi + 2 \sin \psi + \frac{1}{3} \right)
\]

\[ \times \cos (2x-\psi) - \frac{4}{3} \sin (\theta \psi) \right) = \frac{16}{61} \left( 8 \sin \theta \cos 2\theta + 9 \right)
\]

\[ + \frac{25}{260} \left( 128 \sin \theta \cos 4\theta + 20 \cos 2\theta \right)
\]

\[ + \frac{16}{240} \left( 25 \cos \theta - \cos 3\theta + 2 \sin (\theta/2 + \pi/2) \right)
\]

\[ + \frac{16}{80} \left( 4 \sin \theta - \cos 2\theta \right).
\]

(2.5.8)
All the terms in equation (2.5.8) are periodic and bounded except the terms containing \( \theta \) and after a moderate number of revolutions, these will be the dominant terms and therefore we write after some simplification

\[
U = \left( -\frac{3K}{2\mu} + \frac{7}{2} \frac{Ma^{2.5}}{\mu} \right) [\cos \theta + (\theta + \pi/2) \sin \theta]
\]

Now, we wish to examine the behaviour of perigee and apogee distances that is we require \( \frac{dU}{d\theta} = 0 \) and therefore we find the approximate solution as

\[
\theta_{\text{ap}} = 2n\pi \pm \pi/2
\]

From equation (2.5.8), we get the perigee distance as

\[
\theta_{\text{perigee}} = -\left( \frac{\nu - 3K}{2\mu} + \frac{3K}{2\mu} \left( \frac{a}{4} - \frac{2}{3} \nu \cos \psi \right) - \frac{3K}{2\mu} \left( \frac{a}{4} - \frac{2}{3} \nu \cos \psi \right) \right)
\]

\[
+ \frac{1}{2} \left[ \frac{5K}{2\mu} + \frac{3K}{2\mu} \sin \psi \cos \psi \right] - \frac{1}{2} (2n\pi - \frac{\pi}{2}) - \frac{5K}{2\mu} + \frac{3K}{2\mu} (4n\pi - \psi) \right) - \frac{1}{2} (2n\pi - \frac{\pi}{2}) + \frac{5K}{2\mu} \left( 2n\pi - \frac{\pi}{2} \right)
\]

\[
+ \frac{3K}{2\mu} \left( 2n\pi - \frac{\pi}{2} \right) + \frac{3K}{2\mu} \left( 2n\pi - \frac{\pi}{2} \right)
\]

\[
+ \frac{3K}{2\mu} \left( 2n\pi - \frac{\pi}{2} \right)
\]

Simplifying and using \( \lambda = \alpha + \omega' \), we obtain
\[ r_{\text{perigee}} = \frac{a(1 + \sin \psi)}{1 - e \sin \psi} \times 2\pi - \frac{7}{2} \frac{k_a\pi^3}{\mu} \times 2\pi \]

Similarly, we can obtain \( r_{\text{apogee}} \) by putting \( \theta = 2\pi - \frac{\pi}{2} \) in equation (2.5.8).

However, perigee is initially located at \( \theta = \psi \), but for small eccentricity after a moderate number of revolutions, it is shifted from \( \psi \) to \( -\psi + 2\pi \). And so the eccentricity is large, the orbit is such slow and the initial perigee location tends to persist.

Conclusion: when the source is taken as fixed, Levin proved that the perigee is fixed at \( -\psi / 2 \), but in our case the inclusion of obstruction results in the shifting of perigee from \( -\psi / 2 \) to \( 2\psi \theta = \tan^{-1}(-\psi / 2) \), \( \psi \) defined by equation (2.2.8). The effect of obstruction is to decrease further the perigee distance or to increase the apogee distance. The effect of shadow is to diminish the height of the perigee by an amount equal to the fraction \( 1 - \sin \theta \) the path which is in shadow and for a distant satellite, the effect of shadow is negligible. For small eccentricity though the perigee is initially located at \( \theta = \psi \), yet after a moderate number of revolutions \( \theta \) is shifted from \( \psi \) to \( -\psi / 2 + 2\pi \), and for large eccentricity the motion is much slower and the initial perigee location tends to persist.