CHAPTER 3

STABILITY TESTING OF ONE DIMENSIONAL RECURSIVE DIGITAL FILTERS

3.1 INTRODUCTION:

The one dimensional recursive digital filter having the transfer function \( H(z) \),

\[
H(z) = \frac{A(z)}{B(z)},
\]

(3.1)

where \( A(z) = \sum_{i=0}^{N} a_i z^{-i} \) and \( B(z) = \sum_{i=0}^{N} b_i z^{-i} \) is stable if and only if

\[
B(z) \neq 0 \quad \text{for} \quad |z| \geq 1
\]

(3.2)

where \( B(z) \) is the denominator polynomial of the transfer function. Here z-transform is defined with negative powers of \( z \). Testing of the above condition is finding the zeros of the polynomial \( B(z) \) directly or by using Jury’s table algorithm (Antoniou, 1999). For higher degree polynomials, Jury’s algorithm is efficient and accurate.
3.2 STABILITY TESTING OF 1-D DIGITAL FILTERS USING THE METHOD OF EVALUATION OF COMPLEX INTEGRALS

An alternative method for testing the stability of one dimensional recursive digital filter is discussed below:

If \( h(n) \) is the impulse response of the one dimensional filter, the Parseval’s integral gives the variance \( \sum_{n=0}^{\infty} h^2(n) \) as

\[
\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint_{|z|=1} |H(z)H(z^{-1})| \frac{dz}{z}
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} H(e^{j\omega})H(e^{-j\omega}) d\omega
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \left| H(e^{j\omega}) \right|^2 d\omega
\]

using \( z = e^{j\omega} \)

For a stable transfer function \( H(z) \), the right hand side of the equation (3.3) is always finite and positive.

In the one dimensional transfer function (Hwang, 1978) uses the Laurent series expansion and shows that if

\[
H(z) = \frac{A(z)}{B(z)},
\]

\[
\sum_{n=0}^{\infty} h^2(n) = r_o
\] (3.4)
That is

\[ H(z) = \frac{a_0 z^{-x} + a_1 z^{-(x-1)} + \cdots + a_x}{b_0 z^{-x} + b_1 z^{-(x-1)} + \cdots + b_x}, b_0 \neq 0 \]

\[ = \sum_{k=0}^{\infty} h_k z^{-k} \quad (3.5) \]

Then, \( H(z) H(z^{-1}) \) admits a Laurent Series (Hwang, 1978)

\[ H(z)H\left(z^{-1}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_i h_j z^{-i} \]

\[ = \sum_{k=-\infty}^{\infty} C_k z^k \quad (3.6) \]

where

\[ C_k = \sum_{i=0}^{\infty} h_i h_{i+k} \]

Consider, on the other hand, the decomposition

\[ H(z)H\left(z^{-1}\right) = \frac{A(z) A\left(z^{-1}\right)}{B(z) B\left(z^{-1}\right)} \]

\[ = \frac{P(z)}{B(z)} + \frac{P(z^{-1})}{B\left(z^{-1}\right)} \quad (3.7) \]

where \( P(z) = \sum_{i=0}^{\infty} p_j z^{N-i} \)

The unknown \( p \)'s are a solution of the \( N+1 \) equations and expand by long division.

\[ \frac{P(z)}{B(z)} = \frac{r_0}{2} + \sum_{k=1}^{\infty} r_k z^{-k} \quad (3.8) \]
Then, identifying (3.6) and (3.7), then

\[
\frac{1}{2\pi j} \oint H(z)H\left(z^{-1}\right) z^k \frac{dz}{z} = \sum_{i=0}^{\infty} h_i h_{i+k} = r_k
\]  

(3.9)

if \( k = 0 \)

\[
\frac{1}{2\pi j} \oint H(z)H\left(z^{-1}\right) \frac{dz}{z} = \sum_{i=0}^{\infty} h_i^2 = r_0
\]  

(3.10)

So \( r_o = 2 \frac{p_0}{b_0} \)

Since the stability of the one dimensional recursive digital filter transfer function \( H(z) \), for simplicity it can be assumed that \( H(z) \) is of the form

\[
H(z) = \frac{1}{b_0 z^N + b_1 z^{N-1} + \ldots + b_N} = \frac{1}{B(z)}
\]  

(3.11)

with \( b_0 > 0 \). The decomposition of the form given in (3.7) which results in a matrix equation can be used.

\[
\begin{bmatrix}
  b_0 & b_1 & b_2 & \cdots & b_N \\
  0 & b_0 & b_1 & \cdots & b_{N-1} \\
  0 & 0 & b_0 & \cdots & b_{N-2} \\
  \vdots & \vdots & \vdots & \ddots & \cdots \\
  0 & 0 & 0 & \cdots & b_0
\end{bmatrix}
+ \begin{bmatrix}
  b_0 & b_1 & b_2 & \cdots & b_N \\
  b_1 & \cdots & b_N & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & b_N & 0
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_N
\end{bmatrix} = \begin{bmatrix}1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

(3.12)

It has to be noted that to get (3.12), it is not necessary to assume the \( H(z) \) to be a stable transfer function. Even if \( H(z) \) is not a stable transfer function it can be expanded by long hand division like

\[
H(z) = \sum_{i=0}^{\infty} h_i z^{-i}
\]
and

\[ H(z)H\left(z^{-1}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_i h_j z^{-i} z^j \]

The constant part of the RHS will give us

\[ \sum_{i=0}^{\infty} h_i^2 = \sum_{n=0}^{\infty} h^2(n) \]

So the decomposition of the type (3.7) resulting in the matrix equation of the form (3.12) can always be used to evaluate \( \sum_{i=0}^{\infty} h_i^2 \).

So

\[ \sum_{n=0}^{\infty} h^2(n) = \frac{2p_0}{b_0} = r_o \quad (3.14) \]

There are three cases depending on the pole location of the one dimensional recursive digital filter.

**Case (i)**

All the poles of the transfer function or zeros of \( B(z) \) are inside the unit circle \( |z| = 1 \). In this case, the transfer function is a stable transfer function and the \( r_o \) value is finite and positive.
**Case (ii)**

All the poles of the transfer function are outside the unit circle |z|=1. In this case H(z) is the unstable transfer function and the value of $r_o$ will be negative.

**Case (iii)**

If some poles of the transfer function are inside the unit circle and some are outside the unit circle, $r_o$ value may be positive or negative. If $r_o$ value is negative, then it can be directly concluded that the transfer function is unstable. Suppose $r_o$ is positive, then test the following conditions.

(i) $B(1) > 0$

(ii) $(-1)^N B(-1) > 0$  \( (3.15) \)

Here if any of the conditions (i) & (ii) fails, then it can be concluded that the transfer function is unstable. The above method of testing the stability of one dimensional recursive digital filter is procedurally simpler and it may be an alternative to Jury’s table algorithm (Antoniou, 1999).

### 3.3 ILLUSTRATIVE EXAMPLES

**Example 3.1:**

Let

$$H(z) = \frac{1}{z^6 - 0.4z^5 - 0.93z^4 + 0.412z^3 + 0.1172z^2 - 0.0494z + 0.0035}$$
It has been tested that all the poles of $H(z)$ are inside the unit circle. The value of

$$r_o = 3.3368,$$
which gives

$$\sum_{n=0}^{\infty} h^2(n)$$

If $r_o$ is positive, we test for the conditions (3.15)

(i) $B(1) > 0$

(ii) $(-1)^N B(-1) > 0$

and if these conditions are satisfied it can be concluded that the given transfer function is stable. For example, it has been found that $B(1) = 0.1533$ and $(-1)^6 B(-1) = 0.2281$ and so the filter is stable.

**Example 3.2:**

Consider the transfer function

$$H(z) = \frac{1}{z^2 - z^2 - 14z + 24}$$

The $r_o$ has to been found to be equal to $r_o = -0.0026$

As the value of $r_0$ is negative, all the poles will lie outside the unit circle. Hence it is concluded that the filter transfer function is unstable.
Example 3.3:

Let us consider the following transfer function

\[ H(z) = \frac{1}{z^3 + 3.7z^2 - 1.18z + 0.08} \]

The \( r_o \) for this case to be found to be \( r_o = 0.0138 \). Now when conditions (3.15) are applied and verified for positivity

(i) \( B(1) = 3.6 > 0 \)

(ii) \( (-1)^3 B(-1) = -3.96 < 0 \)

It is found that the second condition is not satisfied. So, it can be concluded that the transfer function is an unstable one. In fact the poles of \( H(z) \) are

\[ Z_{p1} = -4, \ Z_{p2} = 0.2 \text{ and } Z_{p3} = 0.1 \]

Example 3.4:

Consider the transfer function

\[ H(z) = \frac{1}{z^{34} - 1.2z^{33} + 0.5z^{32} - 1.5z^{31} + 1.8z^{30} - 0.75z^{29} + 0.6z^2 - 0.72z + 0.2718} \]

Here \( r_o = -5581.2234 \). Since it is negative it can be said that the filter is unstable without even testing the condition (3.15)
3.4 SUMMARY

The above methods of testing the stability of one dimensional recursive filter is presented here as the following theorem.

**Theorem:** A one dimensional transfer function

\[
H(z) = \frac{A(z)}{B(z)}
\]

of order N is BIBO stable if

(i) \( \sum_{n=0}^{\infty} h^2(n) > 0 \) and finite

(ii) \( B(1) > 0 \)

(iii) \( (-1)^N B(-1) > 0 \)

This method is procedurally simpler and it can be an alternative to Jury’s table algorithm (Antoniou, 1999). Though computationally more time consuming, our method gives a double benefit as when a filter is found stable the value of \( \sum h^2(n) \) (variance) for that filter which is useful in finding out the quantization error.

In fact when a one dimensional transfer function \( H(z) \) is to be tested for stability, the knowledge of pole distribution is not known. In case (ii), above holds that the transfer function is unstable. In the case (i) and in (iii) where \( r_o \) is positive it can be concluded above stability only after checking the conditions of (3.15).