CHAPTER 2
LITERATURE REVIEW FOR STABILITY TESTING OF TWO DIMENSIONAL RECURSIVE DIGITAL FILTERS

The methods for testing the stability of two dimensional recursive digital filters fall into two categories. First type is algebraic methods and the second type is mapping methods.

2.1 ALGEBRAIC METHODS

The algebraic methods can exactly determine the stability in finite number of steps. These testing methods rely on certain algebraic properties of one-dimensional complex polynomials. It is found in literatures that algebraic methods are exclusively developed for first quadrant quarter plane filters. Most of the algebraic methods can be reduced to two steps. First, one dimensional algebraic polynomial root distribution test is applied to the two dimensional polynomial which is viewed as a one dimensional polynomial with polynomial coefficients. Secondly this usually yield real coefficient one dimensional polynomials which must be checked for sign definiteness over a certain real interval. If this step is validated, then the filter is stable. In some cases, the second step can be reduced to or replaced by checking whether a real polynomial has real roots in a given interval or whether a self-inverse polynomial satisfies a specific zero distribution condition.
The first algebraic test to appear in the literature was by Huang (Huang, 1972). In this paper, the two-dimensional bilinear transform was utilized to transform the system function so that multidimensional continuous theory would be used. Instead of testing $B(z_1, z_2) \neq 0$ on $|z_1| \leq 1, |z_2| \leq 1$, the change of variables $s_1 = (1-z_1)/(1+z_1), \quad s_2 = (1- z_2)/(1+ z_2)$ is performed on $H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}$ to obtain $G(s_1, s_2) = \frac{E(s_1, s_2)}{F(s_1, s_2)}$ and $F(s_1, s_2)$ is tested for zeros in $\zeta = \{ (s_1, s_2) \mid \Re \{s_1\} \geq 0, \Re\{s_2\} \geq 0 \}$. If $A(z_1, z_2)$ and $B(z_1, z_2)$ were mutually prime polynomials then $B(z_1, z_2)$ would be free of zeros on $|z_1| \leq 1, \quad |z_2| \leq 1$ if and only if $F(s_1, s_2)$ is free of zeros on $\zeta$. But this method has been shown to be incorrect in the literature (Goodman, 1977). Several additional one-dimensional tests must be performed to validate the conditions. However, once these one-dimensional procedures have been performed the Hermite algebraic procedure employed by Huang can be made valid.

Algebraic method reduces the stability problem as to determine whether a $M \times M$ symmetric polynomial matrix $D(z_1)$, whose entries are real polynomials of order $2M$, is positive definite for all values of $z_1$, Huang suggested that it could be verified by showing that the successive principal minors of $D(z_1)$ are positive for all real value of $z_1$. Sturms method could be used to test positivity of each minor. However, this procedure can also be simplified considerably by using the result of Siljak (Siljak, 1974). It can be proved that it is not necessary to test all leading minors for positivity. The following conditions should be satisfied:

(i) $D(0)$ be positive definite
(ii) $\text{Det} [D] > 0$ for all real $z_1$
A major difficulty arises in the determination of the polynomial \( \text{det}[D(z_1)] \). On a digital computer, it is very difficult to compute the coefficients of the determinant of a polynomial matrix in an efficient and programmable manner. Even with the above simplifications, these tests are not as efficient as some other algebraic methods especially for lower order filters. This is mainly due to the calculation of the bilinear transformation.

Maria and Fahmy (Maria and Fahmy, 1973) tested the condition that \( B(z_1, z_2) \neq 0 \) on \( |z_1| = 1 \) and \( |z_2| \leq 1 \) by again considering \( B(z_1, z_2) = \sum_{n=0}^{M} a_n(z_1)z_2^n \) where \( a_n(z_1) = \sum_{i=0}^{M} b(i, n)z_1^i \) and then employing the Jury Marden table test. This table test performs basically the same function for determining a complex polynomials root distribution with respect to the unit circle. In fact, a \( M^{th} \) order complex polynomial has all its zeros outside the unit circle if and only if the first column of the table is positive. In two dimensional filters, the table entries are polynomials in \( z_1 \) and \( z_1^{-1} \). Moreover, the first column entries that are to be checked for positivity of self-inversive polynomials, that is \( Z_{10} \) is a root so is \( Z_{10}^{-1} \). Two schemes can be employed for checking positivity of such a polynomial. They are Sturms and Cohns (Anderson and Jury, 1973) methods. Both methods require large number of complex procedures and complex multiplications. In order to carryout Maria and Fahmy’s test completely, the positivity of \( M \) polynomials of order \( M \) to \( 2^{M-1}M \) must be checked. This means 16 x 16 quarter plane filter stability test requires over a trillion calculations to guarantee stability.

Anderson and Jury (Anderson and Jury, 1973) tested the condition \( B(z_1, z_2) \neq 0 ; \ |z_1| = 1, |z_2| \leq 1 \) by again considering \( B(z_1, z_2) \) as a one dimensional polynomial with polynomial coefficients \( a_n(z_1) \) and then employing the Schur-Cohn test. This is as in Huang’s procedure concluded in
testing a $M \times M$ Hermitian polynomial matrix $C(z_1)$ with polynomial entries of order $2M$ for positive definiteness on $|z_1| = 1$. Siljak (Siljak, 1975) later showed that this condition can be significantly simplified to check that $C(1)$ be positive definite and $\det[C(z_1)] > 0$ on $|z_1| = 1$. It is difficult to express the coefficients of $\det[C(z_1)]$ in terms of the coefficients of the polynomial entries of $C(z_1)$. However, there do exist several efficient, though complicated, methods for computing the exact determinant of matrices with polynomial coefficients. It is found in literatures, several methods were compared and it was concluded that a congruence method was the most efficient method when Schur-Cohn test is appropriately implemented; it becomes more efficient than the previous table test. However, for lower order filters, the Jury’s table test may be preferred because of its ease of implementation. In addition, Anderson, Jury and Siljak’s procedure cannot be terminated early as in the case of Maria and Fahmy’s test and hence it must always be computed continuously all the way to determine whether a filter is stable or not. If the congruence method is used in the implementation of Huang’s Hermite procedure, approximately the same number of calculations are required.

Schussler presented a new stability theorem for one-dimensional polynomials with real coefficients. Bose (Bose, 1977) extended this technique to implement a new algebraic stability test for two-dimensional filters. The most complicated step of this test is in determining the resultant $R(x)$ of two $2M$ order polynomials $D_1(x, z)$ and $D_2(x, z)$ derived from $B(z_1, z_2)$ and examining it for real roots in the interval $-1 \leq x \leq 1$. Unfortunately, it is very difficult to compute $R(x)$. If Sylvester’s matrix is employed, $R(x)$ can be found by taking the determinant of $4M$ by $4M$ polynomial matrix. There do exist methods where the resultant can be computed using lower order matrices. However, the computational burden is never less than the Schur-Cohn algebraic test.
In (Karan and Srivastava, 1986), authors presented a new stability test for two dimensional recursive digital filters, where there is no need for obtaining the determinant of the Schur – Cohn matrix (Silzak, 1975) and the resultant of $D_1(x, z_2)$ and $D_2(x, z_2)$ as in (Bose, 1979) and applying Strum’s test. The method is based on the development of table which requires large number of computations and complex procedures.

In (Nie and Unbehauen, 1989), authors presented the testing of two dimensional recursive digital filters based on the theorem of continued fractional expansion of complex discrete reactance functions. This novel algorithm is derived for the stability test of digital filters directly in $z$-domain. The algorithm is formulated in a table form which is similar to the Jury Marden table test. In (Gu and Lee, 1990), authors proposed the numerical algorithm for stability testing of 2-D recursive digital filters. It consists of a Cholesky factorization and a discrete cosine transform program as well as a table method for testing zero locations of one-dimensional polynomials.

In (Jury and Hu, 1994), authors developed the modified polynomial array, which eliminates the redundant information introduced in the basic table technique and leads to a greater reduction in the order of the polynomial entries of the array from $n_2 2^{n-1}$ to $n_1n_2$ where $n_1$ and $n_2$ are the orders of $B(z_1, z_2)$ in $z_1$ and $z_2$ respectively. This method consists of two steps;

First one is for a one dimensional auxiliary polynomial $B(z_2)$ formed from $B(z_1, z_2)$ at $z_1 = 1$. This step provides a simple way to check the appropriate necessary conditions.
Second, the modified polynomial array is constructed for $B(z_1, z_2)$ on the boundary $|z_1| = 1$. This positivity test for the last entry of the array is then carried out.

In (Bistritz, 1999), author presented a algebraic method to test the stability of two dimensional recursive filter in finite number of arithmetic operations. The test consists of a sequence of centro-symmetric matrices, referred as a 2-D table, that is constructed by a three-term recursion of 2-D polynomials and of a few accompanying conditions on 1-D polynomial that may be examined by unit circle zero location tests. This method is based on location of zeros of a one-dimensional polynomial with respect to the unit circle. This is considered to be the so called immittance counterparts of the Jury-Marden’s tables and Schur-Cohn algorithms for stability testing.

In (Bistritz, 2001), author presented a new algebraic procedure that solves the problem in a very low count of arithmetic operations. This stability test avoids the construction of the table and shows that it is possible instead of telescopes, (bring forth) the last 1-D polynomial of the 2-D table by interpolation. This approach, hence called telepolation, replaces the construction of the 2-D table by testing the stability of finite number of 1-D polynomials of degree $n_2$ (or $n_1$) using a certain associated 1-D stability testing algorithm. In (Bistritz,2002) the same author gives an alternative version for the procedure. It is obtained by applying telepolation to a different immittance type tabular 2-D stability test.

In (Bistritz, 2004), author tested the condition $B(z_1, z_2) \neq 0, |z_1| = 1, |z_2| \leq 1$ by evaluating the stability of a large number of one dimensional polynomials with real coefficients. This is implemented by using a modified form of the 1-D stability test. This test consists of telepolation in a tabular test, which is obtained by the combination of Jury’s tabular test along with that of
doubling the degree technique. Normally, doubling the degree technique increases the computational time. The Jury Marden algorithm requires more number of one-dimensional polynomials with complex coefficients that are to be tested for the stability. On comparing this new procedure (Bistritz, 2004) with Jury Marden algorithm, it has more computational complexity for obtaining the one dimensional polynomials with real coefficients. So the computation time is also as high as Jury Marden algorithm.

Although algebraic methods can, in theory, determine the stability of any recursive filter, there exist a number of practical difficulties in their implementations, even a low-order filter requires a large number of calculations, a digital computer must be utilized to implement these complicated algorithms. Therefore, the computer’s finite word length effects, that is both the round off error and coefficient quantization effects must be considered. In practice, this implies that the algebraic methods are no longer accurate. Hence, these methods have difficulties in determining the stability of critically stable or unstable filters. These are not suitable for testing higher order filters.

2.2 MAPPING METHODS

The mapping methods need an infinite number of steps to determine stability of the digital filter exactly. The other name of mapping methods is approximate methods. In mapping methods, checking the condition \( B(z_1, z_2) \neq 0 \) for \( |z_1| = 1, |z_2| \leq 1 \), is to effectively map the circle \( |z_1| = 1 \) into \( z_2 \) plane based on \( B(z_1, z_2) = 0 \) and to determine if the image lies outside the disc \( |z_2| \leq 1 \). Here the condition is satisfied iff none of these curves intersect the unit disc. If any of these curves do intersect the unit disc, implies that there exists \( B(z_{10}, z_{20}) = 0 \). Therefore, in theory, the stability of a filter can be determined exactly by checking the zero distribution of the one dimensional
polynomial in $z_2$, $B(z_{10}, z_2)$ for each fixed $z_{10}$ with $|z_{10}| = 1$. Since $z_1$ has unity magnitude, we will alternatively specify these one-dimensional polynomials with a parameter $u$ where $z_1 = e^{ju}$ for $0 \leq u < 2\pi$ as

$$B\left(e^{-ju}, z_2\right) = \sum_{n=0}^{N} \left[ \sum_{m=0}^{N} b(m, n) e^{-jum} \right] z^n \quad (2.1)$$

The methods which attempt to obtain stability of two dimensional recursive filters by examining $B\left(e^{-ju}, z_2\right)$ for $0 \leq u \leq 2\pi$ will be called mapping methods. For each fixed $u_1$, $B\left(e^{-j_u}, z_2\right)$ is a one dimensional polynomial with complex coefficients. If $B\left(e^{-j_u}, z_2\right)$ has no zeros inside or on the unit circle for all $u_1$, that is, $B\left(e^{-j_u}, z_2\right)$ must be stable one dimensional polynomials. Therefore, for each value of $u$, any root distribution method which can handle polynomials with complex coefficients may be used to check the stability of $B\left(e^{-ju}, z_2\right)$

In practice, the one dimensional stability of $B\left(e^{-ju}, z_2\right)$ cannot be checked for all the possible values of $u$. Only for a finite number of $u_j$, $B\left(e^{-j_u}, z_2\right)$ can be examined for stability. Once the interval $(0, 2\pi)$ is sampled on a countable grid the accuracy of the method suffers. For particular $u_j$, $B\left(e^{-j_u}, z_2\right)$ has zeros inside or on the unit circle, then the filter is definitely unstable. However, if $B\left(e^{-j_u}, z_2\right)$, $j = 0, 1, \ldots, L-1$ are all stable, this does not imply that $B(z_1, z_2) \neq 0$ on $|z_1| = 1$, $|z_2| \leq 1$. Even if we check $B\left(e^{-ju}, z_2\right)$ on a countably very large set, stability cannot be guaranteed.
Although there exist examples, where the mapping methods will fail, for the vast majority of filters a fine enough sampling of \((0, 2\pi)\) will yield good results. Considering \(U_0\) and \(V_0\) as a set of all points in the interval \((0,2\pi)\). From the fact that the root maps of \(B(z_1, z_2) = 0\) form continuous curves and hence if there exists a \((u_0, z_0)\) with \(|z_0|<1\) such that \(B(e^{-ju}, z_1) = 0\) then there exists an interval around \(u_0\) such that if \(u \in V_o\), \(B(e^{-ju}, z_2)\) will have at least one root inside the unit circle. Therefore, if for a \(L\), the number of samples on \((0, 2\pi)\) is large enough, then a \(u_j\) will fall in the interval \(U_0\) and the test will determine that the filter is unstable. Hence, the number of samples, \(L\), determines the accuracy of the test. If larger \(L\) is chosen, more polynomials \(B(e^{-ju}, z_1)\) \(j = 0,1,2 \ldots L-1\), are checked for stability and hence the test is more time consuming.

The root mapping method is based on \(B(z_1, z_2) \neq 0\) on \(|z_1| =1, |z_2| \leq 1\) iff the root maps of \(B(e^{-ju}, z_2)\) as a function of \(z_2\) do not intersect the unit disc \(|z_2| \leq 1\). Huang and Shanks (Huang, 1972), (Shanks, 1972) implemented this test by calculating the minimum magnitude root of \(B(e^{-ju}, z_2)\) for a finite number of \(u_j\). These roots are then plotted in the \(z_2\) plane to form a curve. The filter is unstable if this curve intersects the unit disc. This method uses FFT algorithm which requires \(L\log_2 L\) operations. This implementation becomes increasingly more efficient as the value of \(L\) is increased.

In (Huang, 1981), author presented a integrated logarithmic derivative method checking the condition \(B(z_1, z_2) \neq 0\) for \(|z_1|=1, |z_2| \leq 1\) is to determine for each \(z_1\) with \(|z_1| = 1\) if \(B(z_1, z_2) = 0\) for \(|z_2| \leq 1\). Cauchy principal value formula states that
\[ N(Z_i) = \frac{1}{2\pi j} \int_{|z|=1}^{z} \frac{\partial B(z_1, z_2)}{\partial z_2} \left[ B(z_1, z_2) \right]^{-1} dz_2 \] (2.2)

where \( N(z_1) \) for a fixed \( z_1 \) is the number of zeros in \( z_2 \) of \( B(z_1, z_2) \) that lie in the region \(|z_2| \leq 1\). Therefore, \( B(z_1, z_2) \neq 0 \) on \(|z_1| = 1, |z_2| \leq 1\) iff \( N(z_1) \) is considered as a function of \( z_1 = e^{-ju} \), always equals zero. \( N(z_1) \) can also be interpreted as the winding number of the polynomial \( B(z_1, z_2) \) for a fixed \( z_1 \). Interpreted in this way, equation (2.2) implies that no linear phase component of \( B(z_1, z_2) \) can exist for \( B(z_1, z_2) \neq 0 \) on \(|z_1| = 1, |z_2| \leq 1\) or that \( B(z_1, z_2) \) has no zeros inside or on \(|z_2| \leq 1\) for each \( z_1 \) with \(|z_1| = 1\). Therefore in theory (2.2), is equivalent to the previously discussed procedures. However equation (2.2) can be employed to yield another computational procedure for testing the condition \( B(z_1, z_2) \neq 0 \) for \(|z_1| = 1, |z_2| \leq 1\). Since \( z_1 = e^{-ju} \) and since the integration is over \( z_1 = e^{-ju} 0 \leq v \leq 2\pi \), then

\[ N(e^{-ju}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial B(e^{ju}, e^{jv})}{\partial V} \left[ B(e^{ju}, e^{jv}) \right]^{-1} dv \] (2.3)

For each fixed \( u \), this method is equivalent to, approximately determining the number of zeros outside the unit circle. The direct evaluation of equation (2.3) is not so easy. So adaptive phase unwrapping is an accurate method for evaluating equation (2.3).

In Nyquist test (Huang, 1981), \( B(e^{ju}, e^{jv}) \) will reveal that this is just the two dimensional Fourier transform of \( b(m, n) \). For convenience, it will be written as \( B(u, v) \). In this notation, the Nyquist test can be interpreted as a procedure which determines if slices of \( B(u, v) \) in the \( v \) direction are functions which encircle or pass through the origin; that is, \( B(z_1, z_2) \) is free of zeros in
$|z_1|=1,$ $|z_2|=1$ if $\text{Ind}[B(u,v)] = 0$ for all $0 \leq u \leq 2\pi$, where $\text{Ind}$ indicates the number of encirclements of the origin.

The above discussion suggests that all of the stability information about $b(m,n)$ is present in its two dimensional Fourier transform $B(u,v)$. Furthermore, the one dimensional Nyquist criterion can be interpreted using the unwrapped phase function. Therefore, $B(z_1,z_2)$ is free of zeros on $|z_1|=1$, $|z_2|\leq1$ if the unwrapped phase along the $v$ direction can be defined and has no linear phase component for each $u$; with the help of this interpretation and (Tribolet, 1977) an efficient algorithm for testing condition $B(z_1,z_2) \neq 0$ for $|z_1|=1$, $|z_2|\leq1$ can be developed. First use FFT procedure to acquire $D(k,n)$. That is Fast Fourier Transform of each row of $b(m,n)$ to obtain $M$ complex sequences $C_n(k)$ of length $L$. Form two dimensional array with $C_n(k)$, $k=0,1,\ldots,L-1$ as the rows, $D(k,n)$ is an $L$ by $M+1$ complex array.

For each $k$, unwrap the phase of the sequence $D_k(n)$ using the adaptive phase unwrapping algorithm (Tribolet, 1977). Here $D_k(n)$ is considered to be a time sequence, that is phase unwrapping gives information about the roots of $\sum_{n=0}^{M} D_k(n)z^n$. If for any $k$, the unwrapped phase along the $u$ direction contains a linear phase term, then the filter is definitely unstable. However, if none of the sequences contain a linear phase component, and $L$ is also large, then $B(z_1,z_2)$ has no zeros on $|z_1|=1$, $|z_2|\leq1$. The accuracy of this test is enhanced by increasing $L$, the FFT size.

Decarlo implemented his Nyquist test by literally mapping $B(e^{ju},e^{jv})$ for each fixed $u$ and looking for the number of encirclements of the origin. The inefficiency arises in this method because the FFT algorithm is not used. The accuracy problem arises because the mapping method can lead to ambiguities. This does not occur in the adaptive unwrapping algorithm.
because it has the potential of adapting. It appears that the adaptive phase unwrapping algorithm as described in (Tribolet, 1977) is the most accurate as well as the most efficient for a given accuracy unit circle Nyquist testing procedure. However, since the sequences \( D_k(n) \) have length \((M+1)\), for most recursive filter applications the value is less than twenty or so, it follows that this algorithm is not the most efficient means of extracting zero distribution information. Jury’s table is probably one of the most efficient root distribution procedures for low order sequences.

One consequence of applying the phase unwrapping procedure to a stable filter is that it almost computes the sampled two-dimensional unwrapped phase. The only reason that the above method may not yield the correct sample unwrapped phase is that there may be a phase ambiguity along the \( u \) axis. This may allow the phase values along a particular column (\( v \) direction, \( u \) fixed) to, all be offset by the same integral multiple of \( 2\pi \). This ambiguity does not affect the above test because only the presence of a linear phase term need be detected. In order to obtain the correct unwrapped phase, the phase along the \( u \) axis must be computed first. This can be easily found by unwrapping the phase of the one-dimensional sequence \( c(m) = \sum_{n=0}^{M} b(m, n) \). By unwrapping the phase \( c(m) \), the sampled version of the true continuous phase \( \theta(u, v) \) along the \( u \) axis, \( \theta(k, 0) \) is obtained. While unwrapping the phase of \( D_k(n) \) will give \( \theta(k, l) + 2\pi a_k \), where \( a_k \) is an integer which is a function of \( k \). Combining this information with \( \theta(k, 0) \) will yield the true sampled unwrapped phase \( \theta(k, l) \).

An interesting consequence of unwrapping the phase of \( c(m) \) is that it gives root distribution information about \( B(z, 1) = \sum_{m=0}^{M} c(m)z_1^m \). Therefore the unwrapped phase contains all the stability information. If it is continuous,
odd, and periodic, then no linear phase components exist so the filter is stable and vice versa.

Another interesting result is that if any of the sequences $D_k(n)$ has an unwrapped phase with linear phase term while $B(z_1, 1)$ and $(1, z_2)$ have no such term then the unwrapped phase of $b(m, n)$ is not well defined. This statement is also true if $b(m, n)$ is complex and not a recursive filter array. The phase is ill defined for two reasons. First, Since $B(z_1, 1)$ and $B(1, z_2)$ have no linear phase term, the existence of a linear phase term must mean that there exists a $(u_o, v_o) \leq u_o, v_o < 2\pi$, such that $B(u_o, v_o) = 0$. Therefore, the phase cannot be defined at $(u_o, v_o)$. The phase function cannot be patched up as the one dimensional case where a zero on the unit circle can be interpreted to mean that the phase jumps by $\pi$ at that point because it is not known which way the phase jumps. Secondly, if the above algorithm is applied to the columns of $b(m, n)$ so that $D(m, l) = D_i(m)$, phases are unwrapped.

Instead of employing root calculations, phase unwrapping to determine if $B(u, z)$ has any roots in $|z| \leq 1$ for $0 \leq u \leq 2\pi$, one can use the Jury table (Huang, 1981). An efficient implementation of this method is to apply the table procedure to each sequence $D_k(n)$. If any of these sequences is not stable, then the original filter is unstable. In practice, virtually all two-dimensional recursive filters will have orders less than twenty or so. It appears (Huang, 1981) that the above implementation of the Jury test is the most efficient mapping method. Furthermore, it is very easy to program it.
Figure 2.1 Jury’s row algorithm and column algorithm

Jury’s row algorithm and column algorithm as given in figure 2.1 is computationally less complex, but the algorithm uses the FFT and the size of the FFT determines the accuracy of the method. Suppose L is the size of the FFT used \((L/2 + 1)\) number of second degree one dimensional polynomials of complex coefficients are to be tested for stability using Jury Marden algorithm.
In Jensen’s method (Huang, 1981),

If \( B(z_1, z_2) \neq 0 \) on \(|z_1|=1, |z_2|\leq 1\) then for each fixed \( u \)

\[
I(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|B(u,v)| \, dv
\]

\[
= \log \left| \sum_{m=0}^{M} b(m,0) e^{-jum} \right|
\]

\[
= \log |E(u)| \quad (2.4)
\]

where \( E(u) = \sum_{m=0}^{M} b(m,0) \). The right hand side of (2.4) is the Fourier transform of the first row of \( b(m, n) \), namely \( b(m, 0) \), \( m=0, \ldots, M \). This follows from Jensen’s formula which states that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|x(e^{jut})| \, dt = \log|x(0)| - \sum_{i} \log|a_i + \log|c_i| \quad (2.5)
\]

where \( a_i \) and \( c_i \) are zeros and poles of \( x(z) \) inside the unit circle. If equation (2.4) is not valid for a given \( u_o \), then by (2.5) there exists a \( (u_o, z_0) \) with \(|z_0| < 1\) such that \( B(e^{-ju_o}, z_0) = 0 \). Therefore, (2.4) is both necessary and sufficient for \( B(z_1, z_2) \neq 0 \) on \(|z_1| = 1, |z_2| < 1\). In theory, this is not as general as the previous methods because zeros cannot be located on \( T^2 \). However in practice, since only a finite number of samples can be checked. An algorithm for implementing this method is given below:
1. FFT first row of $b(m, n)$ (length $L$) to obtain

$$E(k) = \sum_{m=0}^{M} b(m,0) e^{-j\pi mk/L}$$

2. For each $k$ FFT $D_k(n)$ (length $k$). This results in a column of 2D-FFT of $b(m, n)$, namely $B(k, l)$, $l = 0, \ldots, k$

3. For each $k$ compute

$$\log |B(k, l)|$$

and compare to

$$\log |E(k)|$$

4. If these quantities differ for any $k$, then the filter is unstable.

One difficulty with this algorithm is choosing a value for $k$ such that the value of sum in step(3) approximates the integral in equation (2.4). This is the same problem that occurs in the integrated logarithmic derivative approach. This problem is enhanced when this algorithm is used to determine if one root of magnitude slightly less than unit is inside $|z_2| \leq 1$. By equation (2.4) such a root contributes a very small bias to the integral. Therefore, it is difficult to determine if this small difference is due to this bias or just integration error.

If $E(u)$ of (2.4) satisfies

$$I_E = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |E(u)| du = \log |b(0, 0)|$$

(2.6)

Then $B(z_1, 0)$ has no zeros in $|z_1| < 1$. Furthermore if $B(z_1, 0)$ has no zeros in $|z_1| < 1$, then $I_E = \log |b(0, 0)|$. By equation (2.5) if $B(z_1, 0)$ has zeros in $|z_1| < 1$, then $I_E > \log |b(0, 0)|$. If equation (2.4) and (2.6) hold, then
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} I(u) du = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log |B(u, v)| dv \, du
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |E(u)| \, du = \log |b(0, 0)| \tag{2.7}
\]

Moreover, if equation (2.4) does not hold then

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} I(u) du \neq \log |E(u)|
\]

because there exists an interval U of finite measure such that if \( u \in U \), then \( I(u) > \log |E(u)| \) while for the other values \( I(u) \geq \log |E(u)| \). Therefore

\[
\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log |B(u, v)| dv \, du = \log |b(0, 0)| \tag{2.8}
\]

iff \( B(z_1, z_2) \neq 0 \) on \( |z_1| = 1, \ |z_2| < 1 \) and \( B(z_1, 0) \neq 0 \) on \( |z_1| < 1 \). Equation (2.8) also implies that \( B(z_1, z_2) \neq 0 \) on \( |z_1| < 1, \ |z_2| = 1 \) and \( B(0, z_2) \neq 0 \) on \( |z_2| < 1 \) because the above development can be repeated for this case. Therefore, equation (2.8) holds if \( B(z_1, z_2) \neq 0 \) on \( U^2 - T^2 \). Hence the equation (2.8) can be used as an approximate stability testing procedure.

The two dimensional FFT can be utilized to approximate the equation (2.8). This is not as efficient and as accurate as other algorithms. First the previous algorithm may terminate early thus obviating the entire 2D-FFT calculations which are in (2.8). Secondly, equation (2.8) may average out or obscure instabilities which may have been easily detected using the first
method. However, the implementation of equation (2.8) is slightly simpler because only one comparison is required.

Figure 2.2 Block diagram of mapping algorithms
In essence, these procedures attempt to determine if $B(z_1, z_2)$ is free of zeros on $U^2$, where $U^2 = \{ (z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1 \}$. In particular, these algorithms try to show that $B(z_1, z_2) \neq 0$ on $|z_1| = 1, |z_2| \leq 1$ and $B(z_1, 0) \neq 0$ on $|z_1| \leq 1$, because these are necessary and sufficient conditions for $B(z_1, z_2) \neq 0$ on $U^2$. All these methods as mentioned in Figure 2.2 rely on the parameterization of $z_1 = e^{-ju}$ and try to check for each $u$ if $B(u, z_2) = 0$ on $|z_2| \leq 1$. In practice, two main problems occur in mapping methods.

(i) Only a finite number of $u$ can be chosen.

(ii) Algorithms which determine if $B(u, z_2)$ is stable may not be exact.

This second difficulty is quite apparent when Jensen’s formula or integrated logarithmic derivative method is employed. This is virtually no problem when root calculation, Jury test and Nyquist test are used. However, even with these methods, errors may occur when implemented on a finite precision computer. However, the accuracy can be increased to almost any desired level by increasing the sampling rate. Efficient implementations of the above algorithms must employ the FFT algorithm to compute these samples.

Although algebraic methods are not affected by limitation, numerical errors due to finite precision arithmetic and coefficient quantization may be severe. Furthermore, for filters of order greater than four, these techniques are extremely difficult to program. These methods also require an excessive amount of calculations for even moderate order filters. On the other hand, the mapping algorithms are easy to program and are able to handle virtually any order filters without any modifications. In particular, the Jury table scheme appears to be the most efficient and accurate approach for lower orders.
order filters while adaptive phase unwrapping should be more accurate and some times more efficient for higher order filters.

In the case where it is absolutely necessary to determine if a given filter is stable, it may be argued that approximate methods should not be used. However, a alternative can be employed to overcome part of this objection. Let \( a_1 \) and \( a_2 \) be two real numbers with magnitudes slightly greater than one. If an approximate test with the appropriate accuracy is applied to \( B(a_1 z_1, a_2 z_2) \) and it is found to be stable, then it is virtually certain that \( B(z_1, z_2) \) is stable. This is equivalent to checking the stability of the filter \( C(m, n) = a_1^m a_2^n b(m,n) \).

Other than algebraic and mapping methods for testing stability of two dimensional recursive digital filter, some complex methods are available. One is cepstral stability method. In this method (Ekstrom and Woods, 1976), the basic premise is that a general recursive digital filter is stable if and only if its two-dimensional complex cepstrum exists. Further more, the complex cepstrum must be supported in the same minimum angle sector \( \beta \). If the filter is stable, it satisfies the conditions for the existence of complex cepstrum. Therefore, if the FFT size is large enough to minimize aliasing, the result will correspond closely to the true cepstrum with support on \( \beta \). However, if the filter is unstable then the calculated complex cepstrum will not correspond to the true cepstrum and it will usually have support out side of \( \beta \). Therefore, if the complex cepstrum does not have the support on \( \beta \), then the filter is definitely unstable. However, because of aliasing it is some times difficult to say with certainty, whether the computed cepstrum is identically zero outside of \( \beta \) (Dudgeon, 1977).

The original cepstrum stability test (Ekstrom and Woods, 1976) of which the above method is an extension is based on the following premises: If a filter \( b(m, n) \) is stable then its complex cepstrum exists and is restricted to \( \beta \);
since $\beta$ has angle less than $\pi$ then $b(m, n)$ is completely specified by its even part $\hat{b}(m, n)$

$$\hat{b}(m, n) = \frac{1}{2} \left[ \hat{b}(m, n) + \hat{b}(-m, n) \right]$$

$$\hat{b}(m, n) = \text{FT}^{-1} [\log |B(u, v)|]$$

$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log |B(u, v)| e^{jum} e^{jum} du dv$$

(2.9)

The following algorithm given by Ekstrom and Woods listed bellow, follows the above arguments.

1. Given $b(m, n)$ a recursive filter array
2. Calculate $|B(u, v)|^2$ using DFT
3. Compute $\log |B(u, v)|^2$
4. Compute IDFT $[\log |B(u, v)|^2]$
5. Window $\beta$ call $\hat{b}$
6. Compute DFT $(\hat{b})$
7. Calculate $\exp[\text{DFT}(\hat{b})] = B^1(u, v)$
8. Compute IDFT $[B^1(u, v)] = b^1(m, n)$

$b^1(m,n)$ is compared with $b(m, n)$, if equal then $b(m, n)$ is stable.

Another method of testing the stability of the filter is row and column concatenation algorithm. It is based on the theorem that $b(m, n)$ is a stable recursive filter array if and only if,

(a) $B(z_1, z_2) \neq 0$ on $T^2$ 

(2.10)
(b) $B(\lambda, 1)$ and $B(1, \lambda)$ have no linear phase terms, i.e.

$$\text{Ind} (B(\lambda, 1)) = 0 = \text{Ind} (B(1, \lambda)),$$

where Ind indicates the number of encirclements of the origin.

Since only the existence of a linear phase term or equivalently the appropriate zero distribution with respect to the unit circle of this one-dimensional sequence need be determined. Either an analytic method or a phase unwrapping algorithm can be employed to extract this information. Generally, this one-dimensional sequence will have several hundred or more elements. A modified version of Tribolet’s phase unwrapping algorithm (Tribolet, 1977) works faster and more accurately then any other procedure.
Figure 2.3 Row and column concatenation algorithm

The one dimensional sequence associated with column concatenation has a representation of $B(\lambda^N, \lambda)$ while row concatenation has one of $B(\lambda, \lambda^M)$. These sequences can have both positive and negative indices. $b(m, n)$ is stable then for any integers $M$ and $N$

$$\text{Ind } [B(\lambda^N, \lambda)] = \text{Ind}[B(\lambda, \lambda^M)] = 0$$  \hspace{1cm} (2.11)

However if $b(m, n)$ is not stable and if
\[ \text{Ind}[B(\lambda, 1)] = \text{Ind}[B(1, \lambda)] = 0 \]

Then in most cases, if \( N \) or \( M \) is large enough in magnitude, then

\[ \text{Ind}[B(\lambda^N, \lambda)] \neq 0 \quad (2.12) \]

and/or

\[ \text{Ind}[B(\lambda, \lambda^M)] \neq 0 \]

Therefore if for any integer value of \( M \) or \( N \) equation (2.11) does not hold then \( b(m, n) \) is unstable as described in Figure 2.3. Moreover, if equation (2.12) holds for very large values of \( M \) or \( N \) then we can be “almost sure” that \( b(m, n) \) is stable.

Notice that these row and column concatenation stability tests like the row and column algorithms presented earlier cannot guarantee the stability of a filter. However, the accuracy of these tests can be increased to almost any desired level by proper selection of \( N, M \) values.