CHAPTER 6

STABILITY TESTING OF NSHP FILTERS AND GENERAL FILTERS

6.1 INTRODUCTION

In non-symmetric half plane filters, the input and output filter coefficients have supports in any of the two adjacent quadrants with vertex (0,0) and with the sector angle less than \( \pi \). Totally, there are eight classes of non-symmetric half plane recursive digital filters (NSHP). Whose regions of support are denoted as \( R_{\Theta^+}, R_{\Theta^-}, R_{+\Theta}, R_{-\Theta}, R_{\Theta^+}, R_{\Theta^-}, R_{+\Theta}, R_{-\Theta} \). The NSHP filters are more frequently used because these filters have positive definite magnitude characteristics. In general filters the filter coefficients have supports with vertex (0, 0) and with the sector angle less than \( \pi \).

In this stability testing method, the NSHP filter or general filter is transformed into first quadrant quarter plane filter. The transformation is suggested by O’conner and Huang (O’conner and Huang, 1978).
6.2 TRANSFORMATION OF NSHP FILTER OR GENERAL FILTER TO FIRST QUADRANT QUARTER PLANE FILTER

Consider two sectors $S_1 = \{(M_1, N_1), (M_2, N_2)\}$ and $S_2 = \{(P_1, Q_1), (P_2, Q_2)\}$ with $D = M_1N_2 - M_2N_1 \neq 0$ and $E = P_1Q_2 - P_2Q_1 \neq 0$.

There exists many linear mapping of the form

$$m = k_1 m^1 + k_2 n^1$$
$$N = k_3 m^1 + k_4 n^1$$  \quad (k_i \in \mathbb{Z}) \quad (6.1)$$

that map $S_1$ one to one into $S_2$. One mapping is

$$k_1 = (N_2 P_1 - N_1 P_2) \text{Sgn}(D)$$
$$k_2 = (M_1 P_2 - M_2 P_1) \text{Sgn}(D)$$
$$k_3 = (N_2 Q_1 - N_1 Q_2) \text{Sgn}(D)$$
$$k_4 = (M_1 Q_2 - M_2 Q_1) \text{Sgn}(D)$$  \quad (6.2)$$

with this mapping $S_2$ is the smallest sector containing the mapped points. If $S_2$ is the first quadrant in the lattice plane, then (6.1) can be used to derive the following mapping of $S_1$ in to $S_2$ by setting $P_1 = 1$, $Q_1 = 0$, $P_2 = 0$, and $Q_2 = 1$

$$k_1 = \text{Sgn}(D) N_2$$
$$k_2 = -\text{Sgn}(D) M_2$$
$$k_3 = -\text{Sgn}(D) N_1$$
$$k_4 = \text{Sgn}(D) M_1$$
Consider the $R_{+}$ to be the class of NSHP filter of second order and transform it to the first quadrant. Let the transfer function of the second order NSHP filter is

$$
H_{+}(z_1, z_2) = \sum_{i, j} \sum_{R_{+}} b_{ij} z_1^i z_2^j = \frac{1}{B(z_1, z_2)} \tag{6.3}
$$

where,

$$
B(z_1, z_2) = (b_{22} z_1^2 z_2^2 + b_{21} z_1^2 z_2^{-1} + b_{12} z_1^{-1} z_2^2 + b_{11} z_1^{-1} z_2^{-1})
+ b_{10} z_1 z_2 + b_{02} z_1^2 + b_{01} z_2 + b_{00}
$$

By using the transformation method suggested above we obtain the first quadrant polynomial $B(z_1, z_2)$

$$
B(z_1, z_2) = (b_{22} z_1^2 + b_{21} z_2^2 + b_{20} z_2 + b_{11} z_1^2 + b_{10} z_1 z_2 + b_{02} z_1^2 + b_{01} z_2 + b_{00})
$$

The corresponding first quadrant filter transfer function is

$$
H(z_1, z_2) = \frac{1}{B(z_1, z_2)} \tag{6.5}
$$
6.3 SIMPLE METHOD TO TEST THE STABILITY OF 2-D NSHP AND GENERAL FILTERS

Here \( a, d, g \) and \( a^{-1}, d^{-1}, g^{-1} \) are identified from equation (6.4) as

\[
\begin{align*}
  a &= b_{22} z_2^6 + b_{21} z_2^5 + b_{02} z_2^4 + b_{24} z_2^3 + b_{2-2} z_2^2 \\
  d &= b_{12} z_2^4 + b_{11} z_2^3 + b_{10} z_2^2 + b_{1-1} z_2 + b_{1-2} \\
  g &= b_{02} z_2^2 + b_{01} z_2 + b_{00}
\end{align*}
\]

and the corresponding \( a^{-1}, d^{-1}, g^{-1} \) are the ones obtained by replacing \( z_2 \) by \( z_2^{-1} \) in \( a, d \) and \( g \) respectively. It may be noted that this transformed first quadrant polynomial \( B(z_1, z_2) \) is of second degree in \( z_2 \) and using the Hwang’s decomposition

\[
\frac{q_0(z_2^{-1})}{b_0(z_2^{-1})} \quad \text{after identifying} \quad b_0(z_2^{-1}) = a^{-1}
\]

\[
\frac{q_0(z_2^{-1})}{b_0(z_2^{-1})} = \frac{A_1(z_2)}{B_1(z_2)}
\]

(6.6)

\( A_1(z_2) \) and \( B_1(z_2) \) being self-inversive polynomials. It is suggested that to test the stability of \( H(z_1, z_2) \), simply find out the root distribution of \( B_1(z_2) \). If \( H(z_1, z_2) \) is unstable some zeros of the self-inversive polynomial \( B_1(z_2) \) will be on the unit circle or else \( B_1(z_2) \) will have negative sign prefixed. If the 2-D transfer function \( H(z_1, z_2) \) is stable, \( B_1(z_2) \) will be decomposable as the product of \( B_2(z_2) \) \( B_2(z_2^{-1}) \) with \( B_2(z_2) \) having all its zeros inside the unit circle and \( B_2(z_2^{-1}) \) having its zeros all outside the unit circle. This transformed first quadrant filter is stable then the corresponding NSHP or general filter is stable otherwise the filter is unstable.
6.4 ILLUSTRATIVE EXAMPLES

Example 6.1:

Let the NSHP filter transfer function of the second order and of class $R$ be

\[
H(z_1, z_2) = \frac{1}{(z_1^2 z_2^4 + 0.8 z_2^3 + 0.7 z_1 z_2^3 + 0.5 z_1 z_2^2 + 0.6 z_1^2 z_2^2 \\
+ 0.9 z_1^2 z_2 + 0.3 z_1^2 + 0.9 z_1 z_2^2 + 1.5 z_1 z_2 + 0.9 z_1 \\
+ 0.3 z_2^2 + 0.9 z_2 + 0.6)}
\]

The transformed first-quadrant transfer function is

\[
H(z_1, z_2) = \frac{1}{a z_1^2 + d z_1 + g} = \frac{A(z_1, z_2)}{B(z_1, z_2)}
\]

Where

\[
a = 0.6 z_2^6 + 0.9 z_2^5 + 0.3 z_2^4 + 0.8 z_2^3 + z_2^2
\]
\[
d = 0.9 z_2^4 + 1.5 z_2^3 + 0.9 z_2^2 + 0.5 z_2 + 0.7
\]
\[
g = 0.3 z_2^2 + 0.9 z_2 + 0.6
\]

Here $B(z_1, z_2)$ is a second degree polynomial in $z_1$ and identifying $b_0(z_2^{-1}) = a^{-1}$, equation (6.6) becomes.
\[
q_0(z_2^{-1}) = \frac{+1.02 z_2^2 + 1.38 z_2^3 + 0.6 z_2^4}{b_0(z_2^{-1})} = \frac{+1.02 z_2^2 + 1.38 z_2^3 + 0.6 z_2^4}{(0.0702 z_2^3 + 0.4095 z_2^7 + 0.8172 z_2^6 + 1.0926 z_2^5 + 1.9680 z_2^4 + 2.8755 z_2^3 + 2.8416 z_2^2 + 3.3624 z_2 + 2.8416 z_2^2 + 2.26784 z_2^3 + 2.8755 z_2^4 + 1.9680 z_2^5 + 1.0926 z_2^6 + 0.8172 z_2^7 + 0.4095 z_2^8 + 0.0702 z_2^9)}
\]

Then the root distribution of \(B_1(z_2)\) are found out to check the stability. It is found from the result, that \(B_1(z_2)\) which is a self inverisive polynomial has 3 pairs of complex conjugate zeros on the unit circle. They are

\[
0.5600 \pm j 1.2054
\]

\[
0.7851 \pm j 0.6194
\]

and \(-0.9144 \pm j 0.4048\)

So the polynomial \(B_1(z_2)\) is not decomposable into the product of two polynomials one having all zeros inside the unit circle and the other having zeros all outside the unit circle. So the transfer function \(H(z_1, z_2)\) of the 2\(^{nd}\) order NSHP polynomial is unstable. The same conclusion was drawn in (Huang, 1981) also.
Example 6.2:

Consider the following general recursive filter transfer function (Huang, 1981 p.146)

$$H_g(z_1, z_2) = \frac{1}{0.5z_1^{-1}z_2 + 1 + 0.89z_1 + 0.1z_1z_2 + 0.5z_1^2z_2^{-1}}$$

where the z-transform is defined with positive powers of $z_1$ and $z_2$.

The transformed first-quadrant transfer function is

$$\overline{H}(z_1, z_2) = \frac{1}{1 + 0.5z_1 + 0.5z_2 + 0.89z_1z_2 + 0.1z_1^2z_2^3}$$

Since the stability testing is dealing with 2-D transfer functions where z-transform is defined with negative powers, replacing $z_1$ by $(1/z_1)$ and $z_2$ by $(1/z_2)$ in $\overline{H}(z_1, z_2)$ and following first quadrant recursive digital transfer function is derived.

$$H(z_1, z_2) = \frac{z_2^2z_3}{(z_1^2z_2^3 + 0.5z_1z_2^3 + 0.5z_1^2z_2^2 + 0.89z_1z_2^2 + 0.1)}$$

$$= \frac{z_2^2z_3}{(z_2^3 + 0.5z_2^2)z_1^1 + (0.5z_2^3 + 0.89z_2^2)z_1^1 + 0.1}$$

$$= \frac{z_2^2z_3}{az_1^2 + d z_1 + g}$$

$$= \frac{A(z_1, z_2)}{B(z_1, z_2)}$$
where
\[
\begin{align*}
a &= z_2^3 + 0.5z_2^2 \\
d &= 0.5z_2^3 + 0.89z_2^2 \\
g &= 0.1
\end{align*}
\]

Here also the 2-D polynomial \( B(z_1, z_2) \) is of second degree in \( z_1 \).

Using the equation (6.6) and identifying
\[
a^{-1} = b_0(z_2^{-1})
\]

we get
\[
q_0(z_2^{-1}) = \frac{(0.5z_2^{-1} + 1.24 + 0.5z_2)}{(0.0125z_2^{-4} + 0.0695z_2^{-3} + 0.1561z_2^{-2} + 0.2375z_2^{-1} + 0.2796 + 0.2375z_2 + 0.1561z_2^2 + 0.0695z_2^3 + 0.0125z_2^4)}
\]
\[
= \frac{A_1(z_2)}{B_1(z_2)}
\]

The self inversive polynomial \( B_1(z_2) \) has got two pairs of complex conjugate zeros on the unit circle as given below

\[
0.0801 \pm j0.9968 \\
-0.3033 \pm j0.9529
\]

Therefore, the filter transfer function is unstable as concluded in the literature.

**Example 6.3 :**

If in the transfer function taken in example 6, the coefficient value 0.89 is changed to 0.85 it has been found that the corresponding \( B_1(z_2) \) is given by
\[ B_1(z) = 0.0125z_2^4 + 0.0675z_2^3 + 0.1586z_2^2 + 0.2905z_2^{-1} + 0.3872 + 0.2905z_2 + 0.1586z_2^2 + 0.0675z_2^3 + 0.0125z_2^4 \]

It has no zeros on the unit circle. So the transfer function is stable, as concluded in (Huang, 1981).

6.5 SUMMARY

In this method of stability testing, the NSHP filter or general filter is first transformed into first quadrant quarter plane filter. The transformation is suggested by O’conner and Huang (O’conner and Huang, 1978).

After transforming the NSHP or general filter into first quadrant quarter plane filter, stability is checked by using the stability testing of 2-D digital filters using the method of evaluation of complex Integrals or the simplified method. If the transformed first quadrant filter is stable then the corresponding NSHP or general filter is stable otherwise, the filter is unstable.