CHAPTER 5

SIMPLIFIED METHOD OF TESTING THE STABILITY OF THE FIRST QUADRANT QUARTER PLANE FILTERS

5.1 ELEGANT METHOD TO TEST THE STABILITY OF 2-D RECURSIVE FILTERS

A very simple and computationally less time consuming method for testing the stability of two dimensional recursive digital filters is proposed in this chapter. As it is not necessary to find the variance $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h^2(m,n)$, 2-D transfer function is considered to be of the form.

$$H(z_1, z_2) = \frac{1}{\sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} z_1^{-i} z_2^{-j}} = \frac{z_1^N z_2^N}{B(z_1, z_2)}$$

That is

$$H(z_1, z_2) = \frac{z_1^N z_2^N}{\sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} z_1^i z_2^j} \tag{5.1}$$
Obviously, 2-D transfer functions which have non-essential singularities of the second kind are not considered. Though the method proposed are valid for any value of N. However, the stability is tested for second order transfer function of the filter (N=2). Thus, the transfer function tested for stability is of the form

\[
H(z_1, z_2) = \frac{z_1^2 z_2^2}{(b_{22} z_1^2 + b_{21} z_1 + b_{12} z_2^2 + b_{11} z_2 + b_{10}) (z_1^2 + b_{21} z_1 + b_{11} z_2 + b_{10})}
\]

That is

\[
H(z_1, z_2) = \frac{z_1^2 z_2^2}{z_2^2 + (b_{22} z_1^2 + b_{12} z_1 + b_{10}) z_2 + (b_{21} z_1 + b_{11} z_2 + b_{10})}
\]

If

\[
\begin{align*}
a & \Delta b_{22} z_1^2 + b_{12} z_1 + b_{10} \\
d & \Delta b_{21} z_1^2 + b_{11} z_1 + b_{01} \\
g & \Delta b_{20} z_1^2 + b_{10} z_1 + b_{00} \\
-1 \ a & \Delta b_{22} z_1^{-2} + b_{12} z_1^{-1} + b_{02} \\
-1 \ d & \Delta b_{21} z_1^{-2} + b_{11} z_1^{-1} + b_{01} \\
-1 \ g & \Delta b_{20} z_1^{-2} + b_{10} z_1^{-1} + b_{00}
\end{align*}
\]
Then this is decomposed as \( H(z_1, z_2) H(z_1^{-1}, z_2^{-1}) \) as

\[
H(z_1, z_2) H(z_1^{-1}, z_2^{-1}) = \frac{p_1(z_1) z_2 + p_2(z_1)}{a z_2^2 + d z_2 + g} + \frac{q_0(z_2^{-2}) z_2^{-2} + q_1(z_1^{-1}) z_2^{-1} + q_2(z_1^{-1})}{a^{-1} z_2^{-2} + d^{-1} z_2^{-1} + g^{-1}} \quad (5.4)
\]

like in equation (4.3). Equating the line coefficients on both sides of (5.4) the following matrix equation is obtained

\[
\begin{bmatrix}
  a & d & g & d^{-1} & g^{-1} \\
  0 & a & d & g^{-1} & 0 \\
  0 & 0 & a & 0 & 0 \\
  d & g & 0 & a^{-1} & d^{-1} \\
  g & 0 & 0 & 0 & a^{-1}
\end{bmatrix}
\begin{bmatrix}
  q_o(z_1^{-1}) \\
  q_1(z_1^{-1}) \\
  q_2(z_1^{-1}) \\
  p_1(z_1) \\
  p_2(z_2)
\end{bmatrix}
= \begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} \quad (5.5)
\]

Solving the matrix equation (5.5) get \( q_o(z_1^{-1}) \). Then \( \frac{q_o(z_1^{-1})}{b_o(z_1^{-1})} \) is formed as

\[
\frac{q_o(z_1^{-1})}{b_o(z_1^{-1})} = \frac{(aa^{-1} - gg^{-1})}{aa^{-1}(aa^{-1} - gg^{-1}) - d(-dg^{-1}a^{-1} + ad^{-1}a^{-1}) + g(-g^{-1}dd^{-1} + ad^{-2} - aa^{-1}g^{-1} + gg^{-2})}
= \frac{A_1(z_1)}{B_1(z_1)} \quad (5.6)
\]
where \( A_1(z_1) \) and \( B_1(z_1) \) are self inversive polynomials. \( B_1(z_1) \) will be of degree 8 in positive powers of \( z_1 \). In general, it will be of degree \( N^2 \) in both negative and positive powers of \( z_1 \) for any \( N \). If the 2-D transfer function 
\[
H(z_1, z_2)
\]
is stable, \( B_1(z_1) \) will be decomposable as the product of 
\[
B_2(z_1)B_2(z_1^{-1})
\]
with \( B_2(z_1) \) having all its zeros inside the unit circle and 
\[
B_2(z_1^{-1})
\]
having its zeros all out side the unit circle. In this case the integral

\[
\frac{1}{(2\pi j)} \int_{|z_1|=1} \frac{q_0(z_1^{-1})}{b_0(z_1^{-1})} \frac{dz_1}{z_1}
\]
can be evaluated and this gives positive and finite value for \( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} h^2(m, n) \).

On the other hand if the transfer function \( H(z_1, z_2) \) is not stable it will not be able to decompose \( B_1(z_1) \) as \( B_2(z_1)B_2(z_1^{-1}) \). Though \( B_1(z_1) \) is a self inversive polynomial, will have some zeros on the unit circle and the decomposition will not yield stable \( B_2(z_1) \). This makes it impossible to use the 1-D method of Hwang (Hwang 1978) and obtain the variance of the 2-D filter. It has been shown that (Hwang 1981) if we evaluate the integral

\[
\frac{1}{(2\pi j)} \int_{|z_1|=1} \frac{q_0(z_1^{-1})}{b_0(z_1^{-1})} \frac{dz_1}{z_1}
\]
by the residue method, is obtained negative valued variance; this indicating that the given transfer function \( H(z_1, z_2) \) is unstable. Thus, evaluating the integral by the residue method is very time consuming.
To improve, it is suggested that after getting $B_i(z_i)$ of (5.6) to test the stability of $H(z_1, z_2)$, simply find out the root distribution of $B_i(z_i)$. If $H(z_1, z_2)$ is unstable some zeros of the self-inversive polynomial $B_i(z_i)$ will be on the unit circle or else $B_i(z_i)$ will have negative sign prefixed. Finding the root distribution by any method is very simple least time consuming. Thus, this method is more accurate and less time consuming in comparison with all the existing methods of testing $H(z_1, z_2)$ for stability.

5.2 ILLUSTRATIVE EXAMPLES

To illustrate the proposed stability testing procedure, a few examples are carried out.

Example 5.1:

Consider problem solved in the literature (Hwang, 1981) is taken for justification purpose.

\[
H(z_1, z_2) = \frac{z_1 z_2}{0.5 z_1 z_2 + 0.2 z_2 + 0.5 z_1 + 1}
\]

\[
= \frac{z_1 z_2}{(0.5 z_1 + 0.2)z_2 + (0.5 z_1 + 1)}
\]
The matrix equation to be solved for $q_0(z^{-1})$ is

\[
\begin{bmatrix}
(0.5z + 0.2) & (0.5z + 1) & (0.5z^{-1} + 1) \\
0 & (0.5z + 0.2) & 0 \\
(0.5z + 1) & 0 & (0.5z^{-1} + 0.2)
\end{bmatrix}
\begin{bmatrix}
q_0(z^{-1}) \\
q_1(z^{-1}) \\
p_1(z_1)
\end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Solving for $q_0(z^{-1})$, we get

\[
q_0(z^{-1}) = \frac{-2.5z_1}{(z_1 + 1.8633)(z_1 + 0.5367)}
\]

This is the case, where $B_1(z)$, though does not have zeros on the unit circle, it is prefixed with negative sign. So $H(z_1, z_2)$ is unstable.

**Example 5.2**

Consider problem solved in the literature (Huang, 1981) is taken for justification purpose.

\[
H(z_1, z_2) = \frac{z_1z_2}{(z_1^2z_2^2 - 1.2z_1^2z_2 + 0.5z_1^3 - 1.5z_1z_2^3 + 1.8z_1z_2 + 0.75z_1 + 0.6z_2^2 - 0.72z_2 + 0.2718)}
\]
It is found that the value of $a$, $d$, $g$ are

$$ a = z_1^2 - 1.5 z_1 + 0.6 $$
$$ d = -1.2 z_1^2 + 1.8 z_1 - 0.72 $$
$$ g = 0.5 z_1^2 - 0.75 z_1 + 0.2718 $$

and $a^{-1}$, $d^{-1}$, $g^{-1}$ are obtained by replacing $z_1$, by $(z_1^{-1})$ in $a$, $d$ and $g$, respectively. Using a Matlab program for (5.6)

$$ \frac{q_0(z_1^{-1})}{b_0(z_1^{-1})} = \frac{(0.464 z_1^2 - 1.821 z_1 + 2.7237 - 1.821 z_1^{-1} + 0.464 z_1^{-2})}{(0.0737 z_1^4 - 0.5805 z_1^3 + 2.048 z_1^2 - 4.0903 z_1 + 5.106 - 4.0903 z_1^{-1} + 2.048 z_1^{-2} - 0.5805 z_1^{-3} + 0.0737 z_1^{-4})} $$

So

$$ B_1(z_1) = 0.0737 z_1^8 - 0.5805 z_1^7 + 2.048 z_1^6 - 4.0903 z_1^5 + 5.106 z_1^4 - 4.0903 z_1^3 + 2.048 z_1^2 - 0.5805 z_1 + 0.0737 $$

Self inversive polynomial $B_1(z_1)$ has got two pairs of complex conjugate zeros on the unit circle as given below:

$$ 0.66250259550876 \pm j 0.74905961775027 $$

and

$$ 0.92245865178430 \pm j 0.3860958893180 $$

Hence, the transfer function $H(z_1, z_2)$ is unstable.
Example 5.4:

Consider problem solved in the literature (Huang, 1981) is taken for justification purpose.

\[
H(z_1, z_2) = \frac{z_1 z_2}{(z_1^2 - 1.2 z_1 z_2 + 0.5 z_1^2 + 1.8 z_1 z_2 + 0.75 z_1 + 0.6 z_2^2 - 0.72 z_2 + 0.29)}
\]

The \( B_1(z_1) \) for is found to have no zeros on the unit circle. So the filter is stable.

Example 5.4

Consider (Huang, 1981, p.124)

\[
H(z_1, z_2) = \frac{z_1^2 z_2^2}{(z_1^2 z_2^2 - 0.75 z_1^2 z_2 + 0.9 z_1^2 - 1.5 z_1 z_2^2 - 1.2 z_1 z_2 + 1.3 z_1 + 1.2 z_2^2 + 0.9 z_2 + 0.5)}
\]

It has been found by using the Matlab program developed by as that

\[
\begin{align*}
q_0(z_1^{-1}) &= A_1(z_1) \\
 b_0(z_1^{-1}) &= B_1(z_1) \\
A_1(z_1) &= \frac{A_1(z_1)}{(2.7428 z^6 + 8.5549 z^7 + 9.8173 z^8 + 1.5159 z^5 - 5.4042 z^4 + 1.5159 z^3 + 9.8173 z^2 + 8.5549 z + 2.7428)}
\end{align*}
\]

The root distribution of \( B_1(z_1) \) shows that it has two zeros on the unit circle. They are
0.84496738998511 ± j 0.53481782 866855

Hence, the filter transfer function is unstable.

5.3 SUMMARY

The method proposed for testing the stability of second order two dimensional recursive digital filter is a very simple method and procedurally simple though not computationally. This method actually boils down to test the zero distributions of an eighth degree polynomial. This method is very accurate like any other method since it gave exactly the same result as other methods when applied to some barely stable or barely unstable filter transfer functions.