CONTINUOUS MONOTONIC DECOMPOSITION OF
GRAPHS INTO CYCLES

In this chapter, we investigate the concept of continuous monotonic decomposition of graphs into cycles and investigate their variations.

**Definition 3.1** A decomposition of $G$ into cycles $\{C_i / i = 3, 4, \ldots, n\}$ where each $C_i$ is a cycle of length $i$ in $G$, is called a **continuous monotonic cycle decomposition (CMCD)**.

**Example 3.2** For the graph $G$ in fig 3.1, the decomposition $C_3, C_4, C_5$ is a CMCD.
Remark 3.3 Let $G$ be a $(p, q)$ graph. If $G$ admits a CMCD $\{C_i/ i = 3, 4, \ldots, n\}$, then

1. $q = \binom{n+1}{2} C_2 - 3$

2. The degree of each vertex is even.

3. $\text{Diam}(G) \geq 2$.

4. Girth of $G$ is 3 and circumference is at least $n$.

5. $G$ is Eulerian, but not conversely.

Example 3.4 The graph $G$ in fig 3.2 is Eulerian, but it does not have a CMCD.
Theorem 3.5  If $G$ admits a CMCD $(C_3, C_4, \ldots, C_{2n})$, then $\text{diam}(G) \leq n^2 - 1$.

Proof The maximum distance that a member of the decomposition can contribute to the diameter of $G$ is the diameter of the member cycle itself (see Fig 3.3).

Fig 3.3

Therefore,

$$\text{Diam}(G) \leq \text{diam}(C_3) + \text{diam}(C_4) + \ldots + \text{diam}(C_{2n-1}) + \text{diam}(C_{2n}).$$

$$= 1 + 2 + 2.3 + 2.4 + \ldots + 2(n-1) + n$$

$$= 1 + 2 \left[ 2 + 3 + 4 + \ldots (n-1) \right] + n$$

$$= 1 + 2 \left[ (n-1)n/2 - 1 \right] + n$$

$$= 1 + n^2 - n - 2 + n$$

$$= n^2 - 1$$

Hence $\text{diam}(G) \leq n^2 - 1$. \qed
**Theorem 3.6** If $G$ admits CMCD $(C_3, C_4, \ldots, C_{2n+1})$, then
\[
\text{diam}(G) \leq (n^2 - 1) + n.
\]

**Proof** Similar to theorem 3.5. \(\blacksquare\)

**Remark 3.7** If $G$ is decomposed into $C_3, C_4, \ldots, C_{2n}$, then $G$ may have a cycle of size $> 2n$. For, the graph $G$ in fig 3.4 is decomposed into cycles $C_3, C_4, C_5, C_6, C_7$ & $C_8$ and $G$ has a cycle of length 21.

![Fig 3.4](image)

**Lemma 3.8** There does not exist two integers $s$ and $n$ such that
\[
s(s+1)/2 - 3 = n(2n+1).
\]

**Theorem 3.9** $K_p$ does not admit CMCD where $p \geq 4$.

**Proof** If $p$ is even, then degree of each vertex is odd and the proof is trivial. Let $p$ be odd so that $p = 2n + 1$ for some positive
integer $n \geq 2$. Suppose $K_p$ admits CMCD. Then there exists $C_3, C_4, \ldots, C_s$ such that

$$\frac{1}{2} s (s + 1) - 3 = n (2n + 1) \tag{1}.$$  

**Case (i) $s \leq n$**

In this case, $s = n - x$ for some integer $x$ where $0 \leq x \leq n$.

From (1),

$$\frac{1}{2} (n - x) (n - x + 1) - 3 = n (2n + 1)$$

ie, $n^2 - nx + n - nx + x^2 - x - 6 = 4n^2 + 2n$

$$3n^2 + (2n + 1)x + n - x^2 + 6 = 0.$$

which is not possible since $n < x$

**Case (ii) $s \geq n$**

In this case, $s = n + x$ for some integer $x$ where $0 \leq x \leq n + 1$

From (1),

$$\frac{1}{2} (n + x) (n + x + 1) - 3 = n (2n + 1)$$

$$(n + x) (n + x + 1) - 6 = 2n (2n + 1)$$

$$n^2 + nx + n + nx + x^2 + x - 6 = 4n^2 + 2n$$

$$3n^2 - (2x - 1)n - (x^2 + x - 6) = 0$$

The discriminant of this quadratic equation is

$$= (2x - 1)^2 + 12 (x^2 + x - 6)$$

$$= 16x^2 + 8x - 71$$

$$= (4x + 1)^2 - 72.$$  

Since $n$ is a positive integer, the value of this discriminant must be a perfect square. If $a = 4x + 1$ and $m^2$ is the perfect square
for some integer m, then \(a^2 - 72 = m^2\). This implies that \(a^2 - m^2 = 72\). But this is possible only when \(a \leq 32\). Thus \(4x + 1 \leq 32\) which implies that \(x \leq 31/4\). Therefore the possible values of \(x\) are 1, 2, 3, 4, 5, 6 and 7. It is easy to see that \(x = 2\) is the only integer for which the discriminant is a perfect square. Thus \(n = 0\) or 1 which is a contradiction.

Therefore, \(K_p\) does not admit CMCD. 

**Theorem 3.10** Let \(G\) be graph with \(n(2n + 1)\) edges and \(\text{diam}(G) = n^2 - 1\). Then \(G\) admits a CMCD if and only if \(G\) has \(2n - 2\) non-isomorphic blocks each of which is a cycle \(C_i\) incident with at most two cycles at a distance \(\leq i/2\).

**Proof** Let \(G\) admit a CMCD \((C_3, C_4, \ldots, C_{2n})\) with diameter \(d_3, d_4, \ldots, d_{2n}\) respectively. Then \(d_3 + d_4 + \ldots + d_{2n} = n^2 - 1\).

**Claim 1** Each block is a cycle.

Since \(G\) admits a CMCD, \(G\) contains neither a cut edge nor a theta subgraph.

Suppose there exists a block \(B\) which is not a cycle. Then some cycle \(C_i (i \neq 3)\) has a chord, say \(e = uv\) and this must be an edge of some other cycle \(C_j\), as illustrated in fig 3.5.
Clearly \( \text{diam}(C_i) = \left\lfloor \frac{i}{2} \right\rfloor \) and \( \text{diam}(C_j) = \left\lfloor \frac{j}{2} \right\rfloor \).

Let \( x \) be a vertex of \( C_j \) and \( y \) (\( \neq x, u, v \)) be a vertex in \( C_i \). Any vertex of \( C_i \) can meet \( u \) or \( v \) at a distance \( \left\lfloor \frac{i-2}{2} \right\rfloor \) and hence

\[
d(x,y) \leq \left\lfloor \frac{i-2}{2} \right\rfloor + \left\lfloor \frac{j-2}{2} \right\rfloor < \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor ,
\]

which is not possible, since \( \text{diam}(G) = n^2 - 1 \).

Hence each block is cycle.

**Claim 2** Each cycle \( C_i \) is incident with at most two cycles.
Since $\text{diam}(G) = n^2 - 1$, $G$ has at least two end blocks. Suppose $C_i$ is a cycle which is adjacent to three cycles $C_j$, $C_k$ and $C_t$ as shown in Fig 3.6.

Let $x$ be a vertex in the component of $G - E(C_i)$ containing $C_j$, $y$ be a vertex in the component of $G - E(C_i)$ containing $C_k$ and $w$ be a vertex in the component of $G - E(C_i)$ containing $C_t$. Clearly there exists an $x$-$y$ path not passing through the component containing $C_t$.

Therefore $d(x,y) \leq \text{diam}(G - G')$ where $G'$ is the subgraph containing $C_t$ but not containing $C_i$. 

$< n^2 - 1$, which is a contradiction.
Claim 3 If $s$ and $t$ are two vertices of degree 4 in $C_i$, then $d(s,t) = \lfloor i/2 \rfloor$.

Clearly $d(s,t)$ can not be greater than $\lfloor i/2 \rfloor$.

Suppose $d(s,t) < \lfloor i/2 \rfloor$.

![Diagram of two cycles with vertices $s$ and $t$.]

Fig 3.7

Let $P$ be the shortest path (of length $m < \lfloor i/2 \rfloor$) between $s$ and $t$. Then

$$diam(G) = diam(C_3) + \ldots + diam(C_{i-1}) + m + diam(C_{i+1})$$

$$+ \ldots + diam(C_{2n}).$$

$$< n^2 - 1, \text{ since } m < \lfloor i/2 \rfloor.$$  

This is a contradiction.

Conversely, let $G$ have $2n - 2$ non-isomorphic blocks $B_1, B_2, \ldots, B_{2n-2}$ each of which is a cycle incident with at most two cycles at a distance $\lfloor i/2 \rfloor$. 

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We claim that $C_3$ is the smallest block.

Suppose not. Then $\text{diam}(G) \geq 2$ for each $i = 1, 2, \ldots, 2n - 2$.

Since there are $2n - 2$ non-isomorphic blocks (cycles),

$$\text{diam}(G) = \text{diam}(B_1) + \text{diam}(B_2) + \ldots + \text{diam}(B_{2n-2}).$$

$$\geq 2 + \text{diam}(B_2) + \ldots + \text{diam}(B_{2n-2})$$

$$> 2 + 2.2 + 2.3 + \ldots + 2(n-1) + n$$

$$> n^2 - 1,$$

which is contradiction.

Therefore $B_i = C_3$, for some $i$.

Similarly we can prove that $C_{2n}$ is the largest block in $G$.

Since all the blocks are non-isomorphic, they are $C_3, C_4, \ldots, C_{2n}$, which is a CMCD of $G$.

**Theorem 3.11** For $n \geq 2$, $K_{2n+1} - 3e$ admits CMCD, where "3e" denotes the three edges of a triangle in $K_{2n+1}$.

**Proof** We shall prove the result by induction on the number of vertices.

For $n = 2$, let $V = \{ v_1, v_2, v_3, v_4, v_5 \}$ and $C = (v_1 v_2 v_3 v_1)$ be the triangle in $K_5$. Then $K_5 - \{ v_1v_2, v_2v_3, v_3v_1 \}$ can be decomposed into $C_3 = (v_2 v_4 v_5 v_2)$ and $C_4 = (v_1 v_4 v_3 v_5 v_1)$ which is a CMCD of $K_5 - 3e$ as shown in fig 3.8.
Now assume that $K_{2n-1} - 3e$ admits CMCD where $3e$ denotes the edges of a triangle with vertices $a, b$ and $c$ in $K_{2n-1}$. Let $C_3, C_4, \ldots, C_{2n-2}$ be the CMCD of $K_{2n-1} - 3e$. Now we construct a path of length $2n - 4$, by taking one edge from each cycle as follows:

Let $e_1 = uv$ be any edge of $C_3$ and $C_i (i \geq 4)$ be a cycle which passes through $u$. Let $e_2$ be any edge of $C_i$ which also passes through $u$. Let $w$ be the other end of $e_2$. Let $C_j$ be any other cycle in the CMCD and $x (\neq a, b, c)$ be a vertex of $C_j$. Let $e_3 = wx$ be an edge of some other cycle $C_j$. Let $e_4 = xy$ be an edge of $C_j$ (see fig 3.9).
Proceeding like this, we get edges $e_5, e_6, \ldots, e_{2n-4}$ such that the induced subgraph $<e_1, e_2, \ldots, e_{2n-4}>$ is a path $P_{2n-4}$ of length $2n-4$.

Now consider $K_{2n+1} - 3e$ with vertex set $\{v_1, v_2, \ldots, v_{2n}, v_{2n+1}\}$ in which $v_1, v_2, v_{2n-1}$ form the triangle 3e. By induction, $K_{2n+1} - 3e - \{v_{2n}, v_{2n+1}\}$ admits a CMCD $C_3, C_4, \ldots, C_{2n-2}$. By construction, we can select one edge from each cycle such that the selected edges induce a path $P_{2n-4}$ of length $2n-4$. Thus the cycles $C_i$ (i $\geq 4$) are reduced to paths $P_{i-1}$ of length $i-1$. By making $v_{2n}$ adjacent to the ends of any of the $n-2$ paths and $v_{2n+1}$ adjacent to the ends of the remaining $n-2$ paths, we get $C_4, C_5, \ldots, C_{2n-1}$.

Clearly $v_1 v_{2n} v_{2n+1} v_1$ forms $C_3$ as shown in the fig 3.10.

![Fig 3.10](image)
If \( v_i \) and \( v_j \) are the origin and terminus of \( P_{2n-4} \), and \( P_4 = v_i \backslash v_{2n} \cup v_{2n+1} \backslash v_j \), then \( P_{2n-4} \cup P_4 = C_{2n} \). Thus \( K_{2n+1} - 3e \) admits CMCD \( (C_3, C_4, \ldots, C_{2n}) \).

**Remark 3.12** \( K_{2n} - 3e \) does not admit a CMCD.

**Conclusion and open problem**

In this chapter, we obtained the upper bound \( \text{diam}(G) \leq n^2 - 1 \) as a necessary condition for a graph to have a CMCD. Based on this upper bound, we characterized graphs which admit a CMCD. It is natural to think about graphs with \( \text{diam}(G) = n^2 - 2 \) and hence we have the following open problem.

1. Let \( G \) be a graph with \( n(2n+1) \) edges and \( \text{diam}(G) = n^2 - 2 \). Find a necessary and sufficient condition for \( G \) to have a CMCD.

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