Chapter 2

CONTINUOUS MONOTONIC DECOMPOSITION OF GRAPHS

In this chapter, we introduce the concept of continuous monotonic decomposition of graphs and investigate their variations. The contents of this chapter have been published in [12].

Definition 2.1 Let $G = (V, E)$ be a connected simple graph of order $p$ and size $q$. If $G_1, G_2, \ldots, G_n$ are edge disjoint subgraphs of $G$ such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then $(G_1, G_2, \ldots, G_n)$ is said to be a decomposition of $G$.

Different types of decomposition of $G$ have been studied in the literature [3, 5, 6, 8, 11, 13, 15, 17, 18] by imposing suitable conditions on the subgraphs $G_i$. Isomorphic decompositions are found in [2, 7, 16, 20] and non-isomorphic decompositions are dealt in [1, 10, 12, 19].

Definition 2.2 A decomposition with isomorphic subgraphs is called an isomorphic decomposition. For the graph $G$ given in fig 2.1, $(G_1, G_2, G_3)$ is an isomorphic decomposition of $G$. 
Definition 2.3 A decomposition with non-isomorphic subgraphs is called an *non-isomorphic decomposition*. For the graph G given in fig 2.2, \((G_1, G_2, G_3)\) is a non-isomorphic decomposition of G.
In [1], Alavi et al. introduced *Ascending Subgraph Decomposition* (ASD) of a graph.

**Definition 2.4** A decomposition of $G$ into subgraphs $G_i$ (not necessarily connected) such that $|E(G_i)| = i$ and $G_i$ is isomorphic to a proper subgraph of $G_{i+1}$, is called an *Ascending Subgraph Decomposition*.

**Example 2.5** For the graph $G$ in Fig 2.3, the decomposition $(G_1, G_2, G_3, G_4)$ of $G$ are such that $|E(G_i)| = i$ for all $i = 1, 2, 3, 4$ and $G_i$ is isomorphic to a proper subgraph of $G_{i+1}$. Therefore the decomposition $(G_1, G_2, G_3, G_4)$ is an ascending subgraph decomposition of $G$. 

Fig 2.2
We define continuous monotonic decomposition as follows.

**Definition 2.6** A decomposition \( (G_1, G_2, \ldots, G_n) \) of \( G \) is said to be a **continuous monotonic decomposition (CMD)** if each \( G_i \) is connected and \( |E(G_i)| = i \) for each \( i = 1, 2, \ldots, n \).

**Example 2.7** For the graph \( G \) in fig 2.4, \((G_1, G_2, G_3, G_4)\) is a CMD.
This concept is entirely different from the ascending subgraph decomposition. The decomposition in fig 2.4 is a CMD but not ASD, since $G_3$ is not isomorphic to a proper subgraph of $G_4$. The decomposition in fig 2.3 is an ASD but not a CMD, since $H_4$ is not a connected subgraph.

**Definition 2.8** A CMD in which each $G_i$ is a star is said to be a *continuous monotonic star decomposition (CMSD)*.

**Example 2.9** For the graph $G$ in fig 2.5, $(S_1, S_2, S_3, S_4)$ is a CMSD of the graph $G$.

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**Fig 2.5**
**Definition 2.10** A CMD in which each $G_i$ is a path is said to be a *continuous monotonic path decomposition* (CMPD).

**Example 2.11** For the graph $G$ in fig 2.6, $(P_1, P_2, P_3, P_4)$ is a CMSD of $G$.

![Graph G with paths $P_1$, $P_2$, $P_3$, $P_4$]

**Remark 2.12** It is clear that every CMSD / CMPD is an ASD, but not conversely.

Theorem 2.13 gives a necessary and sufficient condition for a graph to have a CMD.
Theorem 2.13 Let $G$ be a connected simple graph of order $p$ and size $q$. Then $G$ admits a CMD $(H_1, H_2, \ldots, H_n)$ if and only if $q = \binom{n+1}{2}$.

Proof Let $G$ be a connected graph with $q = \binom{n+1}{2}$. Let $u, v$ be two vertices of $G$ such that $d(u, v)$ is maximum. Let $N_r(u) = \{v \in V/ d(u, v) = r \}$. If $d(u) \geq n$, choose $n$ edges incident with $u$. Let $H_n$ be a subgraph induced by these $n$ edges. If $d(u) < n$, then choose $n$ edges incident with $u$, vertices of $N_1(u), N_2(u), \ldots$, successively such that the subgraph $H_n$ induced by these edges is connected. In both cases, $G - H_n$ has a connected component $G^1$ with $(n-1)n/2$ edges. Now, consider $G^1$ and proceed as above to get $H_{n-1}$ such that $G^1 - H_{n-1}$ has a connected component $G^2$ of size $(n-1)(n-2)/2$ edges. Proceeding like this, we get a connected subgraph $H_2$ such that $G^{n-1}$ is a graph with one edge which is taken as $H_1$. Thus $(H_1, H_2, \ldots, H_n)$ is a CMD of $G$. The converse is obvious. 

Now we proceed to investigate some special class of graphs which admit a CMSD.

Remark 2.14 It is easy to see that $K_n$ admits CMSD.
Theorem 2.15 $K^+_n$ admits a CMSD for all $n \geq 1$.

**Proof** Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_n$ with pendant edges $v_i v'_i$ for all $i = 1, 2, \ldots, n$. Let $S_n$ be the star $K_{1,n}$ centered at $v_n$ in $K^+_n$. Clearly $K^+_n - E(S_n) \supseteq K^+_n - 1$.

Let $S_{n-1}$ be the star $K_{1,n-1}$ centered at $v_{n-1}$ in $K^+_n - 1$ and so on.

Finally, $S_1$ be the star $K_{1,1}$ centered at $v_1$ in $K^+_1$. Then $S_1, S_2, \ldots, S_n$ is a CMSD of $K^+_n$.

Theorem 2.16 (i) $K_{n+2n+1}$ admits CMSD for all $n \geq 1$.

(ii) $K_{n+1,2n+1}$ admits CMSD for all $n \geq 1$.

**Proof** Let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_{2n+1}\}$ be the bipartition of $K_{n,2n+1}$. Let $T_i$ denote the star $K_{1,2n+1}$ centered at $v_i$. Then $T_1$ can be decomposed into stars $S_1$ and $S_{2n}$; $T_2$ can be decomposed into two stars $S_2$ and $S_{2n-1}$. Continuing this process, $T_n$ can be decomposed into stars $S_n$ and $S_{n+1}$. Thus $K_{n,2n+1}$ is decomposed into $2n$ stars of sizes $1, 2, \ldots, 2n$.

Similarly $K_{n+1,2n+1}$ is decomposed into $2n+1$ stars of sizes $1, 2, \ldots, 2n+1$.

Example 2.17 The decomposition of $K_{3,7}$ into continuous monotonic stars is illustrated in fig 2.7
We use the following suitcase lemma to prove theorem 2.19.

**Lemma 2.18 [18, 19]**

Let $G$ be an edge disjoint union of stars $S_{i+1}, S_{i+2}, ..., S_{i+k}$ for some $k > 0$ such that $q = i (i + 1) / 2$. Then $G$ can be decomposed into stars $S_1, S_2, ..., S_i$.

**Theorem 2.19** $K_{m,r}$ ( $m \leq r$ ) can be decomposed into stars $S_1, S_2, ..., S_{2n}$ (CMSD) if and only if $m = n - i$ and $r = 2n + 1 + j$ where $i, j > 0$ such that $n = i (j + 1)/(j - 2i)$. 
Proof Let $S_1, S_2, \ldots, S_{2n}$ be a CMSD of $K_{m',r}$. Then $q = n (2n + 1)$

= $m \cdot r$ and $m < r$. Since $S_{2n} \subseteq K_{m',r}$ and $m < r$, $r > 2n + 1$. Hence

$m = n - i$ and $r = 2n + 1 + j$ where $i, j > 0$. Then $(n - i) (2n + 1 + j) =

m \cdot r = n (2n + 1)$ and hence $n = i (j + 1)/(j - 2i)$.

Conversely, let $n = i (j + 1)/(j - 2i)$ where $i, j > 0$ and let

$m = n - i$ and $r = 2n + 1 + j$. We have to prove that $K_{m',r}$ can be decomposed into stars $S_1, S_2, \ldots, S_{2n}$.

Now, $q(K_{m',r}) = m \cdot r$

= $(n - i)(2n + 1 + j)$

= $n (2n + 1) + n j - 2ni - i - ij$

= $n (2n + 1) + n (j - 2i) - i - ij$

= $n (2n + 1) + i (j + 1)(j - 2i)/(j - 2i) - i - ij$

= $n (2n + 1)$

We construct a CMSD as follows:

Let $U = \{u_1, u_2, \ldots, u_{n-i}\}$ and $V = \{v_1, v_2, \ldots, v_{2n+1+j}\}$ be the bipartition of $K_{m',r}$. Let $T_s$ denote the star $K_{1,2n+1+j}$ centered at $u_s$ for $s = 1, 2, \ldots, (n - i)$.

Clearly,

$T_1$ can be decomposed into two stars $S_{2n}$ and $S_{j+1}$;
T₂ can be decomposed into two stars S₂n - 1 and S_j + 2;

Tₙ - i can be decomposed into two stars S₂n - (n - i - 1) and Sₖ₊ₙ - i.

That is, Tₙ - i can be decomposed into two stars Sₙ₊ᵢ₊₁ and Sₖ₊ₙ - i.

Obviously, our decomposition requires S₂ₙ, S₂ₙ₋₁, ..., Sₙ₊ᵢ₊₁. The remaining stars S₁, S₂, ..., Sₙ₊ᵢ must be from the stars Sₖ₊₁, Sₖ₊₂, ..., Sₖ₊ₙ - i.

Since \( n = \frac{j + 1}{j - 2i} \) is a positive integer, \( n + i < j + n - i \).

Now, \( q(S₁US₂U...Sₙ₊ᵢ) = 1 + 2 + ... + (n + i) \)

\[ = \frac{(n + i)(n + i + 1)}{2} \]

\[ = \frac{n^2 + 2ni + i^2 + i}{2} \]

Also,

\[ q(Sₖ₊₁U Sₖ₊₂U ... U Sₖ₊ₙ - i) = q(Kₘ,r) - q(S₂ₙUS₂ₙ₋₁U...US₂ₙU (n - i - 1)) \]

\[ = n(2n + 1) - \{2n(n - i) - [1 + 2 + ... + (n - i - 1)]\} \]

\[ = n(2n + 1) - \{2n(n - i) - (n - i - 1)(n - i)/2\} \]

\[ = \frac{n^2 + n + 2ni + i^2 + i}{2} \]

By lemma 2.18, Sₖ₊₁, Sₖ₊₂, ..., Sₖ₊ₙ - i can be decomposed into stars S₁, S₂, ..., Sₙ₊ᵢ. Thus Kₘ,r can be decomposed into stars S₁, S₂, ..., S₂ₙ which is the required CMSD. \[ \]

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Theorem 2.20 \( W_p \) admits CMSD if and only if \( p = 4, 6 \).

Proof \( W_p = C_{p-1} + K_1 \). Any CMSD of \( W_p \) contains at most 5 edges in \( C_{p-1} \) and hence \( p - 1 \leq 5 \) so that \( p \leq 6 \). Since \( q(W_5) = 8 \), \( p \neq 5 \). Hence \( p = 4, 6 \).

Conversely, let \( p = 4, 6 \). If \( p = 4 \), then \( W_p = K_4 \) which admits CMSD. If \( p = 6 \), then \( W_6 = C_5 + K_1 \). Let \( V(C_5) = \{u_1, u_2, u_3, u_4, u_5\} \) and \( V(K_1) = \{u_6\} \). Then \( S_1 = \langle u_1 u_2 \rangle \),
\[
S_2 = \langle u_2 u_3, u_3 u_4 \rangle ,
\]
\[
S_3 = \langle u_4 u_5, u_5 u_1, u_5 u_6 \rangle ,
\]
\[
S_4 = \langle u_1 u_6, u_2 u_6, u_3 u_6, u_4 u_6 \rangle \text{ is a CMSD of } W_6. \]

Example 2.21 A decomposition of \( W_6 \) into continuous monotonic stars is illustrated in fig 2.8.

![Diagram of W_6](image-url)
Notation: For a spider tree $T$ with the unique vertex $u$ of degree $\geq 3$, let $W$ denote the set of pendant vertices of $T$ and let $x$ denote the number of vertices which are at a distance greater than one from $u$.

**Theorem 2.22** A spider tree $T$ admits CMSD if and only if $T - W = P_t$ where $t \leq 3$.

**Proof** Let $S_1, S_2, \ldots, S_n$ be the CMSD of $T$.

We claim that $T - W$ is a path.
Suppose $T - W$ is not a path. Then there exists at least three pendant vertices $x_1, x_2$ and $x_3$ such that $d(u, x_i) > 2$, for all $i = 1, 2, 3$ (see fig 2.9). Clearly no internal vertex of $u - x_i$ path can be a center of star of size $\geq 3$. Hence none of the pendant edges incident with $x_i$ can be fitted in any star of size $\geq 3$, which is a contradiction to the hypothesis. Hence $T - W$ is a path, say $P_t$.

![Fig 2.9](image)

We claim that $t \leq 3$.

We consider two cases.

**Case (i) $u$ is the origin of $P_t$.**

Since $T$ is a spider tree, all the internal vertices and terminus of $P_t$ are of degree 2 in $T$. Let $x$ be the unique pendant vertex adjacent to the terminus of $P_t$ and $u_1$ be the vertex adjacent to $u$ in $P_t$ (see fig 2.10).
Then no vertex (except origin) of $P_i$ can be a center of a star of size $\geq 3$. Hence $u_1 - x$ path of $T$ must be decomposed into non-isomorphic stars of sizes one and two so that $d (u , x) \leq 4$. Thus $t \leq 3.$

Case (ii) $u$ is not the origin of $P_i$.

There exists exactly two pendant vertices $x_1$ and $x_2$ which are not adjacent to $u$ in $T$. Let $u_1$ be a vertex adjacent to $u$ in the $x_1 - u$ section of $P_i$ and $u_2$ be a vertex adjacent to $u$ in the $x_2 - u$ section of $P_i$ (see fig 2.8). Since $u_1 - x_1$ and $u_2 - x_2$ sections must be decomposed into $S_1$ and $S_2$, we must have $d (u , x_1 ) \leq 3$ and $d (u , x_2 ) \leq 2$. Hence $t \leq 3.$
Conversely, let $T - W$ be a path of length $\leq 3$. We consider three cases.

**Case (i)** Let $T - W = P_3$.

Let $u_1$ be the vertex adjacent to $u$ in $P_3$. If $u$ is the origin of $P_3$, then the $u_1 - x$ path can be decomposed into $S_1$ and $S_2$ and remaining part of tree is a star which can be decomposed into $S_3$, $S_4$, $\ldots$, $S_n$ (see fig 2.12).

![Fig 2.12](image)

If $u$ is not the origin of $P_3$, let $u_1$ and $u_2$ be the vertices adjacent to $u$ in $P_3$ and $d(u_1, x_1) = 2$ and $d(u_2, x_2) = 1$. Then $S_1 = u_2x_2$ and $S_2 = u_1 - x_1$ path in $T$ and the remaining part of $T$ is a star (see fig 2.13).

![Fig 2.13](image)
Case (ii) \( T - W = P_2 \).

If \( u \) is the origin of \( P_2 \), let \( u_1 \) be adjacent to \( u \) in \( P_2 \). Then the \( u - x \) path in \( T \) is decomposed into \( S_1 \) and \( S_2 \) and the remaining part is a star (see fig 2.14).

![Fig 2.14](image)

If \( u \) is not the origin of \( P_2 \), let \( u_1 \) and \( u_2 \) be two vertices adjacent to \( u \) in \( P_2 \) and let \( x_1 \) and \( x_2 \) be the unique vertices adjacent to \( u_1 \) and \( u_2 \) respectively. Then \( S_1 = u_1 x_1 \) and \( S_2 = u - x_2 \) path in \( T \) and the remaining part is a star (see fig 2.15).

![Fig 2.15](image)

Case (iii) \( T - W = P_1 \).
Let \( u_1 \) be the vertex adjacent to \( u \) in \( P_1 \) and \( x_1 \) be the unique vertex adjacent to \( u_1 \). Then \( S_1 = u_1 x_1 \) and the remaining part is a star (see fig 2.16).

![Diagram of a spider tree](image)

**Fig 2.16**

Hence the theorem. \( \square \)

Now we consider spider trees which admit CMPD.

**Theorem 2.23** If a spider tree \( T \) admits a CMPD, then \( \Delta \leq 2x + 3 \).

**Proof** Let \( v_1, v_2, \ldots, v_x \) be the pendant vertices not adjacent to \( u \) in \( T \). Let \( P_1, P_2, \ldots, P_n \) be the CMPD of \( T \). Suppose \( \Delta \geq 2x + 4 \). Then there exists pendant vertices \( u_1, u_2, \ldots, u_x, u_{x+1}, u_{x+2}, u_{x+3}, u_{x+4} \) which are adjacent to \( u \). Let \( Q_i \) denote the \( v_i - u \) path in \( T \). Since each \( Q_i \) can at most be extended to one \( u_i \) to provide some \( P_i \) and three of the \( uu_i \)'s can be taken for \( P_1 \) and \( P_2 \), there exists an edge \( uu_i \) which
can not be fitted in any one of the $P_i$'s, which is a contradiction.

Hence $\Delta \leq 2x + 3$.  

**Remark 2.24** For the spider tree $T$ in fig 2.17, $x = 4$ and $\Delta = 2x + 3 = 11$ and it has a CMPD.

![Fig 2.17](image)

**Remark 2.25** The converse of the above theorem need not be true.

For the spider tree $T$ in fig 2.18, $x = 4$ and $\Delta = 2x + 3 = 11$ with $q = 15$. But it does not admit CMPD, since $P_5$ does not exist in $T$.

![Fig 2.18](image)
Theorem 2.26 If a spider tree $T$ with $\text{diam}(T) = n$ admits CMPD, then $\Delta \leq 2n - 1$.

Proof Since $\text{diam}(T) = n$ and $T$ admits CMPD, vertex $u$ is the centre of $T$. Let $P_1, P_2, \ldots, P_n$ be the CMPD of $T$. Since $P_1$ can contribute at most one degree to $u$ and each of the remaining $P_i$'s can contribute at most two degrees to $u$, $\Delta \leq 2n - 1$.

Remark 2.27 If a spider tree $T$ admits CMPD, then $\text{diam}(T) \geq n$. But the converse need not be true. For the spider tree $T$ in fig 2.19, $\text{diam}(T) = 7$ and $q = 28$. But it does not admit CMPD, since $P_6$ and $P_7$ do not exist simultaneously.

Fig 2.19

Theorem 2.28 Let $T$ be a spider tree with $\text{diam}(T) = n$ (even). If $d(u, z) = n/2$ for all $z$ in $W$, then $T$ admits a CMPD.
Proof Let $u$ be the origin of $(n + 1)$ paths of length $n/2$. Let $Q_1, Q_2, \ldots, Q_{n+1}$ be the paths of length $n/2$. Clearly $Q_i \cup Q_j = P_n$ for all $i \neq j$ and $i, j = 1, 2, \ldots, n$. Hence each pair can be decomposed into $P_i$ and $P_{n-i}$ for all $i = 0, 1, \ldots, (n - 2)/2$ and $Q_{n+1}$ is taken for $P_{n/2}$, which is the required CMPD.

Theorem 2.29 Let $T$ be a spider tree with $\text{diam}(T) = n$ (odd). If $d(u, z) = (n + 1)/2$ for exactly one vertex $z$ in $W$ and $d(u, z) = (n - 1)/2$ for all other vertices in $W$, then $T$ admits a CMPD.

Proof Let $Q_1, Q_2, \ldots, Q_{n+1}$ be the paths of length $(n - 1)/2$ and $Q_{n+2}$ be a path of length $(n + 1)/2$. Clearly $Q_i \cup Q_j = P_n$ for all $i \neq j$ and $i, j = 1, 2, \ldots, n$. Hence each pair can be decomposed into $P_i$ and $P_{n-i}$ for all $i = 0, 1, \ldots, (n - 3)/2$ and $Q_{n+1}, Q_{n+2}$ are taken for $P_{(n-1)/2}$ and $P_{(n+1)/2}$ respectively, which is the required CMPD.

Remark 2.30 Converse of theorem 2.28 need not be true. For the spider tree in fig 2.20, $\text{diam}(T) = 6$ and $P_6, P_5, P_4, \ldots, P_1$ exist but $d(u, z) \neq n/2$. 

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Remark 2.31 Every olive tree admits a CMPD.

Theorem 2.32 Any olive tree $T$ with $k$ branches admits CMSD if and only if $k \leq 3$.

Proof Suppose an olive tree $T$ with $k$ branches admits CMSD. Since every olive tree is a spider, by theorem 2.22, $T - W = P_t$ where $t \leq 3$. This is possible only when $k \leq 3$. Converse is obvious. $lacksquare$

Conclusion and open problem.

In this chapter, we introduced the concept of continuous monotonic decomposition in graphs. We obtained a necessary and sufficient condition for a graph to have a CMD. We characterized
complete bipartite graphs, spiders, wheels and olive trees which admit CMSD. Then we considered spider trees which admit CMPD. Some necessary conditions and some sufficient conditions were obtained. But no necessary and sufficient condition was obtained. Thus we have the following open problem.

1. Find a necessary and sufficient condition for a spider tree to have a CMPD.

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