CONTINUOUS MONOTONIC DECOMPOSITION OF GRAPHS

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ABSTRACT

A decomposition of a graph is a partition of its edge set. Different types of decomposition have been studied in the literature. We introduce the concept of continuous monotonic decomposition of a connected graph and investigate their variations.

1. Introduction

By a graph we mean an undirected connected graph G without loops or multiple edges. The degree of any vertex u in G is the number of edges incident with u and is denoted by d(u) and the distance between two vertices u and v of G is the length of the shortest u - v path in G and is denoted by d(u,v). A tree with exactly one vertex of degree ≥ 3 is called a spider tree and a rooted tree consisting of k-branches where the ith branch is a path of length i is called an olive tree. A wheel on p vertices is denoted by W_p. A path of length t is denoted by P_t. A graph obtained by attaching a pendant edge to each vertex of K_n is denoted by K_n. Terms not defined here are used in the sense of Harary [3].

Let G = (V, E) be a connected simple graph of order p and size q. If G_1, G_2, ..., G_n are edge disjoint subgraphs of G such that E(G) = E(G_1) ∪ E(G_2) ∪ ... ∪ E(G_n), then (G_1, G_2, ..., G_n) is said to be a decomposition of G. Different types of decomposition of G have been studied in the literature by imposing suitable conditions on the subgraphs G_i. Isomorphic decompositions are found in [6] and non-isomorphic decompositions are dealt in [2]. In [1], Alavi et. al. introduced Ascending Subgraph Decomposition (ASD) as a decomposition of G into subgraphs

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G_i (not necessarily connected) such that |E(G_i)| = i and G_i is isomorphic to a proper subgraph of G_{i+1}.

In this paper we introduce the concept of continuous monotonic decomposition of graphs and investigate their variations.

2. Definition and Examples

A decomposition \((G_1, G_2, \ldots, G_n)\) of \(G\) is said to be a continuous monotonic decomposition (CMD) if each \(G_i\) is connected and \(|E(G_i)| = i\) for each \(i = 1, 2, \ldots, n\). For the graph \(G\) in Fig 2.1, \((G_1, G_2, G_3, G_4)\) is CMD.

This concept is entirely different from the ascending subgraph decomposition. The decomposition in Fig. 2.1 is a CMD but not ASD, since \(G_3\) is not isomorphic to a proper subgraph of \(G_4\). The decomposition in Fig 2.2 is an ASD but not a CMD, since \(H_i\) is not a connected subgraph.
A CMD in which each $G_i$ is a star is said to be a continuous monotonic star decomposition (CMSD) and a CMD in which each $G_i$ is a path is said to be a continuous monotonic path decomposition (CMPD). These concepts are illustrated in Fig. 2.3.
It is clear that every CMSD/CMPD is an ASD, but not conversely. We use the following lemma to prove Theorem 3.4.

**Lemma 2.1** [4, 5]

Let \( G \) be an edge disjoint union of stars \( S_{i_1}, S_{i_2}, \ldots, S_{i_k} \) for some \( k > 0 \) such that \( q(G) = \binom{i(i+1)}{2} \). Then \( G \) can be decomposed into stars \( S_{i_1}, S_{i_2}, \ldots, S_{i_k} \).

**3. Main Results**

**Theorem 3.1** : Let \( G \) be a connected simple graph of order \( p \) and size \( q \). Then \( G \) admits a CMD \( (H_1, H_2, \ldots, H_n) \) if and only if \( q = \binom{n + 1}{2} \).

**Proof** : Let \( G \) be a connected graph with \( q = \binom{n + 1}{2} \). Let \( u, v \) be two vertices of \( G \) such that \( d(u, v) \) is maximum. Let \( N_i(u) = \{ v \in V \mid d(u, v) = i \} \). If \( d(u) \geq n \), choose \( n \) edges incident with \( u \). Let \( H_n \) be a subgraph induced by these \( n \) edges. If \( d(u) < n \), then choose \( n \) edges incident with \( u \), vertices of \( N_i(u), N_{i+1}(u), \ldots \), successively such that the subgraph \( H_n \) induced by these edges is connected. In both cases, \( G - H_n \) has a connected component \( G^1 \) with \( (n-1)n/2 \) edges. Now, consider \( G^1 \) and proceed as above to get \( H_{n-1} \) such that \( G^1 - H_{n-1} \) has a connected component \( G^2 \) of size \( (n-1)(n-2)/2 \).
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edges. Proceeding like this, we get a connected subgraph $H_2$ such that $G^{n-1}$ is a graph with one edge which is taken as $H_1$. Thus $(H_1, H_2, \ldots, H_n)$ is a CMD of $G$. The converse is obvious.

Remark 1: It is easy to see that $K_n$ admits CMSD.

**Theorem 3.2:** $K_n$ admits a CMSD for all $n \geq 1$.

**Proof:** Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_n$ with pendant edges $v_i v_i'$ for all $i = 1, 2, \ldots, n$. Let $S_n$ be the star $K_{1^n}$ rooted at $v_n$ in $K_n$. Clearly $K_n - E(S_n) \supseteq K_{n-1}$. Let $S_{n-1}$ be the star $K_{n-1}$ rooted at $v_{n-1}$ in $K_{n-1}$ and so on. Finally, $S_1$ be the star $K_{1^1}$ rooted at $v_1$ in $K_1$. Then $S_n, S_{n-1}, S_{n-2}, \ldots, S_1$ is a CMSD of $K_n$.

Now we proceed to characterise complete bipartite graphs which admits CMSD.

**Theorem 3.3:**

(i) $K_{n,2n+1}$ admits CMSD for all $n \geq 1$.

(ii) $K_{n+1,2n+1}$ admits CMSD for all $n \geq 1$.

**Proof:** Let $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_{2n+1}\}$ be the bipartition of $K_{n,2n+1}$. Let $T_i$ denote the star $K_i,2n+1$ rooted at $v_i$. Then $T_i$ can be decomposed into stars $S_i$ and $S_{2i}$. Similarly, $T_2$ can be decomposed into stars $S_2$ and $S_{2n+1}$. Continuing this process, $T_n$ can be decomposed into stars $S_n$ and $S_{2n+1}$. Thus $K_{n,2n+1}$ is decomposed into $2n$ stars of sizes $1, 2, \ldots, 2n$.

Similarly, $K_{n+1,2n+1}$ is decomposed into $2n+1$ stars of sizes $1, 2, \ldots, 2n+1$.

**Theorem 3.4:** $K_{m,r}$ (m \leq r) can be decomposed into stars $S_i, S_2, \ldots, S_{2n}$ (CMSD) if and only if $m = n - i$ and $r = 2n+1+j$ where $i, j > 0$ such that $n = i(j+1)/(j-2i)$.

**Proof:** Let $S_i, S_2, \ldots, S_{2n}$ be a CMSD of $K_{m,r}$. Then $q = n(2n + 1) = mr$ and $m < r$. Since $S_{2n} \subseteq K_{m,r}$ and $m < r = 2n+1$. Hence $m = n - i$ and $r = 2n+1+j$ where $i, j > 0$. Then $n-i)(2n+1+j) = mr = n(2n+1)$ and hence $n = i(j+1)/(j-2i)$.
Conversely, let $n = i(j+1)/(j-2i)$ where $i, j > 0$ and let $m = n-i$ and $r = 2n+1+j$.

To prove that $K_{m,r}$ can be decomposed into stars $S_1, S_2, \ldots, S_{2n}$.

Now,

$$q(K_{m,r}) = mr$$

$$= (n-i)(2n+1+j)$$

$$= n(2n+1)+n(j-2i) - i - ij$$

$$= n(2n+1) + i(j+1)(j-2i)/(j-2i) - i - ij$$

$$= n(2n+1)$$

We construct a CMSD as follows:

Let $U = \{u_1, u_2, \ldots, u_{n-i}\}$ and $V = \{v_1, v_2, \ldots, v_{2n+1}\}$ be the bipartition of $K_{m,r}$. Let $T_s$ denote the star $K_{1,2n+1}$ rooted at $u_s$ for $s = 1, 2, \ldots, (n-i)$.

Clearly,

- $T_1$ can be decomposed into two stars $S_{2}, S_{1+i}$;
- $T_2$ can be decomposed into two stars $S_{2+i}, S_{2+i}$;
- $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$
- $T_{n-i}$ can be decomposed into two stars $S_{2n-i}, S_{2n-i}$, that is, $T_{n-i}$ can be decomposed into two stars $S_{n+i}$ and $S_{n+i}$. Obviously, our decomposition requires $S_{2n}, S_{2n-1}, \ldots, S_{n+i+1}$. The remaining stars $S_1, S_2, \ldots, S_{n+i}$ must be from the stars $S_{1+i}, S_{2+i}, \ldots, S_{n+i}$.

When claim that $n+i < j+n-i$.

Since $n = i(j+1)/(j-2i)$ is a positive integer, $n+i < j+n-i$.

Now,

$$q(S_1 \cup S_2 \cup \ldots \cup S_{n+i}) = 1 + 2 + \ldots + (n+i)$$

$$= (n+i)(n+i+1)/2$$

$$= (n^2+n+2ni+i^2+i)/2$$
By lemma 2.1, $S_{n+1}, S_{n+2}, \ldots, S_{n+m}$ can be decomposed into stars $S_1, S_2, \ldots, S_{n+m}$. Thus $K_{n+m}$ can be decomposed into stars $S_1, S_2, \ldots, S_{n+m}$, which is the required CMSD.

**Theorem 3.5**: $W_p$ admits CMSD if and only if $p = 4, 6$.

**Proof**: $W_p = C_{p-1} + K_1$. Any CMSD of $W_p$ contains at most 5 edges in $C_{p-1}$ and hence $p - 1 \leq 5$ so that $p \leq 6$. Since $q(W_p) = 8$, $p \neq 5$. Hence $p = 4, 6$.

Conversely, let $p = 4, 6$. If $p = 4$, then $W_p = K_4$ which admits CMSD. If $p = 6$, then $W_6 = C_5 + K_1$. Let $V(C_5) = \{u_1, u_2, u_3, u_4, u_5\}$ and $V(K_1) = \{u_6\}$. Then $S_1 = \langle u_1, u_2 \rangle$, $S_2 = \langle u_2, u_3, u_4 \rangle$, $S_3 = \langle u_4, u_5, u_1, u_6 \rangle$, $S_4 = \langle u_1, u_6, u_2, u_6, u_3, u_6, u_4, u_6 \rangle$ is a CMSD of $W_6$.

**Notation**: For a spider tree $T$ with the unique vertex $u$ of degree $\geq 3$, let $W$ denote the set of pendant vertices of $T$ and let $x$ denote the number of vertices which are at a distance greater than one from $u$.

**Theorem 3.6**: A spider tree $T$ admits CMSD if and only if $T - W = P$, where $t \leq 3$.

**Proof**: Let $S_1, S_2, \ldots, S_n$ be the CMSD of $T$. We claim that $T - W$ is a path.

Suppose $T - W$ is not a path. Then there exists at least three pendant vertices $x_1, x_2$ and $x_3$ such that $d(u, x_i) > 2$, for all $i = 1, 2, 3$. Clearly no internal vertex of $u - x_i$ path can be centre of star of size $\geq 3$. Hence none of the pendant edges incident with $x_i$ can be fitted in any star of size $\geq 3$, which is a contradiction to the hypothesis. Hence $T - W$ is a path, say $P_t$.

We claim that $t \leq 3$. We consider two cases.
Case (i) \( u \) is the origin of \( P_t \).

Since \( T \) is a spider tree, all the internal vertices and terminus of \( P_t \) are of degree 2 in \( T \). Let \( x \) be the unique pendant vertex adjacent to the terminus of \( P_t \) and \( u_i \) be the vertex adjacent to \( u \) in \( P_t \). Then no vertex (except origin) of \( P_t \) can be a centre of a star of size \( \geq 3 \). Hence \( u_i - x \) path of \( T \) must be decomposed into non-isomorphic stars of sizes one and two so that \( d(u, x) \leq 4 \). Thus \( t \leq 3 \).

Case (ii) \( u \) is not the origin of \( P_t \).

There exists exactly two pendant vertices \( x_1 \) and \( x_2 \) which are not adjacent to \( u \) in \( T \). Let \( u_1 \) be a vertex adjacent to \( u \) in the \( x_1 - u \) section of \( P_t \) and \( u_2 \) be a vertex adjacent to \( u \) in the \( x_2 - u \) section of \( P_t \). Since \( u_1 - x_1 \) and \( u_2 - x_2 \) sections must be decomposed into \( S_1 \) and \( S_2 \), we must have \( d(u, x_1) = 2 \) and \( d(u, x_2) = 1 \). Hence \( t \leq 3 \).

Conversely, let \( T - W \) be a path of length \( \leq 3 \).

Case (i) Let \( T - W = P_3 \).

Let \( u_i \) be the vertex adjacent to \( u \) in \( P_3 \). If \( u \) is the origin of \( P_3 \), then \( u_i - x \) path can be decomposed into \( S_1 \) and \( S_2 \) and remaining part of tree is a star which can be decomposed into \( S_3, S_4, ... \), \( S_n \). If \( u \) is not the origin of \( P_3 \), let \( u_1 \) and \( u_2 \) be the vertices adjacent to \( u \) in \( P_3 \) and let \( x_1 \) and \( x_2 \) be the unique vertices adjacent to \( u_1 \) and \( u_2 \) respectively. Then \( S_1 = u_1 x_1 \) and \( S_2 = u_1 - x_1 \) path in \( T \) and the remaining part of \( T \) is a star.

Case (ii) \( T - W = P_2 \).

If \( u \) is the origin of \( P_2 \), let \( u_1 \) be adjacent to \( u \) in \( P_2 \). Then \( u - x \) path in \( T \) is decomposed into \( S_1 \) and \( S_2 \) and the remaining part is a star.

If \( u \) is not the origin of \( P_2 \), let \( u_1 \) and \( u_2 \) be two vertices adjacent to \( u \) in \( P_2 \) and let \( x_1 \) and \( x_2 \) be the unique vertices adjacent to \( u_1 \) and \( u_2 \) respectively. Then \( S_1 = u_1 x_1 \) and \( S_2 = u_1 - x_1 \) path in \( T \) and the remaining part is a star.

Case (iii) \( T - W = P_1 \).

Let \( u_i \) be the vertex adjacent to \( u \) in \( P_1 \) and \( x_i \) the unique vertex adjacent to \( u_i \). Then \( S_1 = u_i x_i \) and the remaining part is a star.

Hence the theorem.
Theorem 3.7: If a spider tree $T$ admits a CMPD, then $\Delta \leq 2x + 3$.

Proof: Let $V_1, V_2, \ldots, V_x$ be the pendant vertices not adjacent to $u$ in $T$. Let $P_1, P_2, \ldots, P_n$ be the CMPD of $T$. Suppose $\Delta \geq 2x + 4$. Then there exist pendant vertices $u_1, u_2, \ldots, u_x, u_{x+1}, u_{x+2}$ which are adjacent to $u$. Let $Q_i$ denote the $v_i - u$ path in $T$. Since each $Q_i$ can at most be extended to one $u_i$ to provide some $P_i$ and three of the $u_i$'s can be taken for $P_1$ and $P_2$, there exists an edge $uu_i$ which can not be fitted in any one of the $P_i$'s, which is a contradiction. Hence $\Delta \leq 2x + 3$.

Remark 2: For the spider tree $T$ in fig 3.3, $x = 4$ and $\Delta = 2x + 3 = 11$ and it has a CMPD.

Remark 3: The converse of the above theorem need not be true.

For the spider tree $T$ in fig 3.4, $x = 4$ and $\Delta = 2x + 3 = 11$ with $q = 15$.

But it does not admit CMPD, since $P_5$ does not exist in $T$. 
Theorem 3.8: If a spider tree $T$ with $\text{diam}(T) = n$ admits CMPD, then $\Delta \leq 2n - 1$.

**Proof:** Since $\text{diam}(T) = n$ and $T$ admits CMPD, vertex $u$ is the centre of $T$. Let $P_1, P_2, \ldots, P_n$ be the CMPD of $T$. Since $P_1$ can contribute at most one degree to $u$ and each of the remaining $P_i$'s can contribute at most two degrees to $u$, $\Delta \leq 2n - 1$.

**Remark 4:** If a spider tree $T$ admits CMPD, then $\text{diam}(T) \geq n$. But the converse need not be true. For the spider tree $T$ in fig 3.5, $\text{diam}(T) = 7$ and $q = 28$. But it does not admit CMPD, since $P_6$ and $P_7$ do not exist simultaneously.

![Fig. 3.5](image)

**Theorem 3.9:** Let $T$ be a spider tree with $\text{diam}(T) = n$ (even). If $d(u,z) = n/2$ for all $z$ in $W$, then $T$ admits a CMPD.

**Proof:** Let $u$ be the origin of $(n+1)$ paths of length $n/2$. Let $Q_1, Q_2, \ldots, Q_{n+1}$ be the path of length $n/2$. Clearly

$$Q_i \cup Q_j = P_n$$

for all $i \neq j, i, j = 1, 2, \ldots, n$.

Hence each pair can be decomposed into $P_i$ and $P_{n-i}$ for all $i = 0, 1, \ldots, (n-2)/2$ and $Q_{i+1}$ is taken for $P_{n+2}$, which is the required CMPD.

**Theorem 3.10:** Let $T$ be a spider tree with $\text{diam}(T) = n$ (odd). If $d(u,z) = (n+1)/2$ for exactly one vertex $z$ in $W$ and $d(u,z) = (n-1)/2$ for all other vertices in $W$, then $T$ admits a CMPD.
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Proof: Let $Q_1, Q_2, \ldots, Q_{n+1}$ be the paths of length $(n-1)/2$ and $Q_{n+2}$ be the path of length $(n+1)/2$. Clearly $Q_i \cup Q_j = P_n$ for all $i \neq j, i, j = 1, 2, \ldots, n$.

Hence each pair can be decomposed into $P_1$ and $P_{n+1}$ for all $i = 0, 1, \ldots, (n-3)/2$ and $Q_{n+1}, Q_{n+2}$ are taken for $P_{(n-1)/2}$ and $P_{(n+1)/2}$ respectively, which is the required CMPD.

Remark 5: Converse of Theorem 3.9 need not be true. For the spider tree in Fig. 3.6, $\text{diam}(T) = 6$ and $P_6, P_5, P_4, \ldots, P_1$ exist but $d(u, z) \neq n/2$.

Fig. 3.6

Remark 6: Every olive tree admits a CMPD.

Theorem 3.11: Any olive tree $T$ with $k$ branches admits CMSD if $k \leq 3$.

Proof: Suppose an olive tree $T$ with $k$ branches admits CMSD. Since every olive tree is a spider, by Theorem 3.6, $T=WP_t$ where $t \leq 3$. This is possible only when $k \leq 3$. Converse is obvious.

REFERENCES


