CONTINUOUS MONOTONIC DECOMPOSITION OF LOBSTER INTO STARS

In this chapter, we investigate Lobsters which can be decomposed into continuous monotonic stars.

Definition 5.1 Let L be Lobster. Then the vertices with degree at least three is called a junction of L. An edge e = uv such that u is adjacent to a junction and v is adjacent to another junction is said to be a junction-neighbor.

Definition 5.2 A set of \((n - 2)\) vertices whose degrees are 3, 4, ..., \(i - 1, i+1, i + 1, \ldots, n\) is said to be a \((3, i, n)\) - set. A set of \((n-1)\) vertices whose degrees are 3, 3, 4, ..., \(i, i + 1, \ldots, n\) is said to be a \((3, 3, n)\) - set.

Theorem 5.3 If the distance between any two nearest junctions is greater than 5, then L does not admit a CMSD.

Proof Suppose \(u_i\) and \(u_j\) are two nearest junctions such that \(d(u_i, u_j) \geq 6\), then the \(u_i - u_j\) path \(P\) contains a \(u_{i-1} - u_{j-1}\) subpath \(P'\) of length at
least 4. Since P' cannot be decomposed into stars, L does not admit CMSD, which is a contradiction.

Theorem 5.4 If L admits a CMSD \((S_1, S_2, \ldots, S_n)\), then \(\text{diam}(L) \leq 2n - 1\).

Proof Let P be a longest path in L. Since \((S_1, S_2, \ldots, S_n)\) can take at most 1, 2, \ldots, 2 edges in P,

\[
\text{diam}(L) = 1 + (2 + 2 + \ldots \text{n-1 times})
= 1 + 2(n - 1)
= 2n - 1
\]

Therefore \(\text{diam}(L) \leq 2n - 1\).

Theorem 5.5 Let L be a Lobster with \(\text{diam}(L) = 2n - 1\) and \(q = n(n + 1)/2\). Then L admits a CMSD \((S_1, S_2, \ldots, S_n)\) if and only if L is a caterpillar with \((n - 2)\) non-adjacent junction supports whose degrees are the distinct integers 3, 4, 5, \ldots, n and there is at most one junction-neighbor.

Proof Let L admit a CMSD \((S_1, S_2, \ldots, S_n)\). Since \(\text{diam}(L) = 2n - 1\), the centers of all the stars must be in a longest path and hence L is a caterpillar. Further, all the centers are distinct and the centers of \(S_3, S_4, \ldots, S_n\) are supports and hence L has at least \((n - 2)\) supports say \(u_3, u_4, \ldots, u_n\). Since \(S_1\) and \(S_2\) lie in the
longest path and the origin and terminus of $S_1$ and $S_2$ are not supports, $L$ has exactly $(n-2)$ non-adjacent supports $u_3, u_4, \ldots, u_n$.

Since the center $u_i$ of $S_i$ is incident with two edges of the longest path, it must be incident with $(i-2)$ pendant edges and hence $d(u_i) = i$ for all $i = 3, 4, \ldots, n$.

**Claim** $L$ has at most one junction - neighbor.

Suppose $e_1$ and $e_2$ are two distinct junctions, then there exist pairs of junction supports $u_i, u_j$ and $u_r, u_s$ such that $d(u_i, u_j) = d(u_r, u_s) = 3$. Clearly they are not in any of the stars $S_3, S_4, \ldots, S_n$ (see fig 5.1)

Then $<E(L) - E(S_3 \cup S_4 \cup \ldots \cup S_n) > = 3K_2$ which cannot be decomposed into $S_1$ and $S_2$ which is a contradiction. Hence there is at most one junction - neighbor.

**Conversely**, let $L$ be a caterpillar with $(n-2)$ non-adjacent supports $u_3, u_4, \ldots, u_n$ whose degrees are the distinct integers $3, 4, \ldots, n$ and there is at most one junction - neighbor. Let $E_i$ be the set
of i edges incident with \( u_i \). Take \( S_i = <N[u_i]> \) for all \( i = 3, 4, \ldots, n \).

Since there is at most one junction neighbor, \( <E(L) - E(S_3 \cup S_4 \cup \ldots \cup S_n) > = P_1 \cup P_2 \) or \( P_3 \) which can be decomposed into \( S_1 \) and \( S_2 \).

Hence \( L \) admits a CMSD. \( \blacksquare \)

**Theorem 5.6** Let \( L \) be Lobster with \( \text{diam}(L) = 2n - 2 \) and \( q = n(n +1)/2 \). Then \( L \) admits a CMSD \( (S_1, S_2, S_3, \ldots, S_n) \) if and only if

(a) \( L \) has \( (n - 2) \) non adjacent junction supports which lie on a path \( P \) and with distinct degrees \( 3, 4, 5, \ldots, n \) with no junction neighbor (or)

(b) \( L \) has \( (n - 2) \) junction supports which are non-adjacent (except possibly 2) and form a \( (3, i, n) \) - set, \( (i \geq 4) \) with at most one junction-neighbor. (or)

(c) \( L \) has \( (n - 2) \) non adjacent junction supports with degrees \( 3, 4, \ldots, n -1, n +1 \) with no junction-neighbor. (or)

(d) \( L \) has \( (n - 1) \) junction supports which are non-adjacent (except possibly one junction support of degree 3) form a \( (3,3,n) \) set with at most two junction-neighbors.
Proof Let \( L \) admit a CMSD \((S_1, S_2, S_3, \ldots, S_e)\). Since \( \text{diam}(L) = 2n - 2 \), the centers of \( S_3, S_4, \ldots, S_n \) form junction supports. Further by Theorem 5.5, the contribution of the stars to the longest path \( P \) is reduced by one.

**Case (1) \( S_1 \) does not contribute an edge to \( \text{diam}(L) \).**

In this case, let \( S_1 = e = uv \). Then either \( u \) or \( v \) is a pendant vertex. Without loss of generality, let \( v \) be the pendant vertex. Then \( u \) belongs to \( P \) or \( N_2 = \{v\} \). In both cases \( L - \{v\} \) is a caterpillar which can be decomposed into \( S_2, S_3, \ldots, S_n \).

**Subcase (a) \( N_2 = \{v\} \).**

By Theorem 5.5, \( L - \{v\} \) must have \((n - 2)\) non-adjacent junction supports whose degrees are 3, 4, 5, \ldots, \( n \).

**Claim** There is no junction - neighbor.

Suppose an edge \( e_1 \) is a junction - neighbor, then there exists two junctions \( u_i \& u_j \) such that \( d(u_i, u_j) = 3 \). Since the ends of \( e_1 \) are not supports, it is not in any of the stars \( S_3, \ldots, S_n \) (see fig 5.2).

![Fig 5.2](image-url)
Then $<E(L) - E(S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n)> = 2K_2$, which does not form $S_2$, which is a contradiction. Hence there is no junction-neighbor.

**Subcase (b) $u \in P$**

In this case, $u$ may be a center of some $S_i \ (3 \leq i \leq n)$ or the common vertex for two other stars. In both the cases, $L$ has $(n - 2)$ or $(n - 1)$ junction supports.

Assume that $L$ has $(n - 2)$ junction supports. Then from subcase (a), $L$ has no junction-neighbor. If $u$ is the center of $S_3$, then the degrees of the $(n - 2)$ junction supports are $4, 4, 5, 6, \ldots, n$. If $u$ is a center of some $S_i \ (3 < i < n)$, then the $(n - 2)$ junction supports form a $(3, i, n)$-set. If $u$ is a center of $S_n$, then the degrees of $(n - 2)$ junction supports are $3, 4, 5, \ldots, n - 1, n + 1$.

Now assume that $L$ has $(n - 1)$ junction supports. Then $u$ is the center of $S_2$ or the origin/terminus of $S_2$ or any common vertex for two stars. In the first and last case, $L$ has no junction-neighbor. In the middle case, the edge of $S_2$ not incident with $u$ is a junction-neighbor in $L$. But in all the cases, the degrees of the $(n - 1)$ junction supports are $3, 3, 4, 5, \ldots, n$ (see fig 5.3).
Case (2) $S_1$ contributes an edge to diam($L$).

Then exactly one of the stars $S_2, S_3, \ldots, S_n$ contribute exactly one edge to diam($L$). If $S_2 = u_1u_2w$ contributes only one edge say $u_1u_2$ for diam($L$), then $w \in N_1$ and $u_2$ is a junction support with $d(u_2) = 3$. Therefore $L$ has $(n - 1)$ junction supports with degrees $3, 3, 4, 5, \ldots, n$. Now let $S_1 = u_3u_4$ in $P$. If $S_1 \cap S_2 = \emptyset$, then $u_3u_4$ and $u_1u_2$ are two junction-neighbors in $L$. If $S_1$ is incident with $u_1$ such that $u_4 = u_1$, then $u_3u_1$ is a junction-neighbor in $L$. If $S_1$ is incident with $u_2$ such that $u_2 = u_3$, then $L$ has no junction-neighbor (see fig 5.4).
If $S_1 = uv$ is incident with any of the center of $S_i$, $i > 2$ then it can be taken under subcase (b) of case (1).

If $S_i$, $(i > 2)$ contribute only one edge to diam(L), then the center of $S_i$ must be adjacent to the center of some other $S_j$, $(j > 2)$ and $S_1 \& S_2$ wholly lie in the longest path of L. Then there are $(n - 2)$ junction supports. If $S_1 \cap S_2 \neq \varnothing$, then L has no junction - neighbor. Otherwise $S_1 = u_1u_2$ is a junction - neighbor (see fig 5.5).
Conversely, let \( L \) be a Lobster with \( \text{diam}(L) = 2n - 2 \) satisfying conditions (a), (b), (c) or (d).

**Case (a)** \( L \) has \( n - 2 \) non-adjacent junction supports which lie on a path with distinct degrees 3, 4, ..., \( n \) with no junction-neighbor.

Let \( u_3, u_4, \ldots, u_n \) be the non-adjacent supports with distinct degrees 3, 4, 5, ..., \( n \) respectively. Take \( S_i = \langle N[u_i] \rangle \) for all \( i = 3, 4, \ldots, n \). Since \( L \) has \( n(n +1)/2 \) edges, we have to decompose the remaining three edges into \( S_3 \) and \( S_2 \). Since \( L \) has no junction-neighbor, \( H = \langle E(L) - E(S_3US_4U \ldots US_n) \rangle \neq 3K_2 \). Since \( \text{diam}(L) = 2n - 2 \) one of these three edges does not lie in the longest path of \( L \) and since the degrees of junction supports are distinct, it is not incident with \( u_i \) for all \( i \). Hence \( H \) contains an edge \( e_1 \) which has
one end in $N_2$ and two adjacent edges $e_2$ and $e_3$ in the longest path. Take $S_1 = e_1$ and $S_2 = <e_2, e_3>$ (see fig 5.6).

Fig 5.6

Case (b) $L$ has $(n - 2)$ junction supports which are non-adjacent (except possibly 2) and form a $(3, i, n)$-set $(i \geq 4)$ with at most one junction-neighbor.

Since the $(n - 2)$ junction supports form a $(3, i, n)$-set, let $d(u_3) = 3$, $d(u_4) = 4$, ..., $d(u_{i-1}) = i - 1$, $d(u_i) = i + 1$, $d(u_{i+1})$, ..., $d(u_n) = n$ for some $i \neq 4$

**Subcase (1)** All the junction supports are non-adjacent

In this case, take $S_k = <N[u_k]>$ for all $k = 3, 4, ..., n$ and $k \neq i$. Also since $<N[u_i]> = S_{i+1}$, it can be taken as $S_1$ and $S_i$. Consider the two-edge subgraph $H = <E(L) - E(S_1US_3U ... US_n)>$. Since diam$(L) = 2n - 2$, the two edges of $H$ must lie in the longest
path of L. Also since L has at most one junction neighbor, H ≠ 2K₂ and hence it must be S₂.

**Subcase (2)** Let uᵢ and uᵢ₊₁ be two adjacent junction supports. In this case, take Sᵢ = <N[uᵢ]> for all k = 3, 4, ..., n and k ≠ i and Sᵢ = <N[uᵢ] - uᵢuᵢ₊₁> (see fig 5.7).

![Fig 5.7](image)

In this case, all the three edges of H = <E(L) - E(S₃US₄U ..... USₙ)> must lie in P. Since L has one junction-neighbor, H ≠ 3K₂. Therefore H = P₃ or P₁UP₂ which can be decomposed into S₁ and S₂.

**Case (c)** L has (n - 2) non adjacent junction support with degrees 3, 4, ..., n - 1, n + 1 with no junction - neighbor.

Let u₃, u₄, ..., uₙ₋₁, uₙ be the (n - 2) junction supports with degrees d(u₃) = 3, d(u₄) = 4, ..., d(uₙ₋₁) = n - 1, d(uₙ) = n + 1. Take Sᵢ = <N[uᵢ]> for all i = 3 ≤ i ≤ n - 1, Sₙ₊₁ = <N[uₙ]> . Take Sₙ₊₁ = SₙUS₁.

The induced subgraph H = <E(L) - E(S₁US₃US₄U ..... USₙ)> = P₂, (since L has no junction-neighbor) which can be taken as S₂.
Case (d) L has \((n - 1)\) junction supports which are non-adjacent (except possibly one junction support of degree 3) which form a \((3,3,n)\) - set with at most two junction - neighbors.

Let \(u_2, u_3, u_4, ..., u_n\) be the \((n - 1)\) non-adjacent junction supports with degrees \(d(u_3) = 3, d(u_4') = 3, d(u_4) = 4, ..., d(u_n) = n\).

Take \(S_i = <N[u_i]>\) for all \(i = 3, 4, ..., n\).

**Subcase (1)** All the junctions are non - adjacent

In this case, the three edges of \(H = <E(L) - E(S_3US_4U....US_n)>\) is nothing but \(<N[u'_3]> = K_{1,3}\) (see fig 5.8) which gives \(S_1\) and \(S_2\).

![Fig 5.8](image)

**Subcase (2)** Let \(u'_3\) be adjacent to some \(u_j\)

In this case, the three edges of \(H = <E(L) - E(S_3US_4U....US_n)\) \(\neq 3K_2\), since \(L\) has at most two junction- neighbors. Therefore \(H = P_3\) or \(P_1UP_2\) which can decomposed into \(S_1\) and \(S_2\) as illustrated in fig 5.9.
Fig 5.9

Hence in all the cases, L admits a CMSD.

**Definition 5.7** Let $C = \{c_1, c_2, \ldots, c_k\}$ be the set of adjacent junctions in the longest path $P$ of the Lobster $L$. If the induced subgraph $<c_1, c_2, \ldots, c_k>$ = $H$, then $L$ is called a $H$-Lobster.

Example of $K_2$-Lobster, $P_2$-Lobster, $2K_2$-Lobster are given in fig 5.10.

![Diagram of K2-Lobster and P2-Lobster]
**Theorem 5.8** Let $L$ be a Lobster with $\text{diam}(L) = 2n - 3$ and $N_2 \neq \emptyset$. If $L$ admits a CMSD, then all the vertices of $N_2$ are adjacent to exactly one vertex of $N_1$.

**Proof** Let $L$ admit a CMSD and $N_2 \neq \emptyset$.

**Claim** All the vertices of $N_2$ are adjacent to a common vertex of $N_1$.

If $|N_2| = 1$, then the claim is obvious.

Now let $|N_2| \geq 2$. Suppose the claim is not true.

Then there exists at least two vertices $v_1$ and $v_2$ in $N_2$ which are adjacent to two distinct vertices $w_1$ and $w_2$ in $N_1$. Then $w_1$ and $w_2$ must be the center of at least two stars $S_i$ and $S_j$ with $i$ and / or $j \geq 2$. Therefore $S_i$ and $S_j$ do not contribute any edge to $\text{diam}(L)$. Hence $\text{diam}(L) \leq 2n - 4$, which is a contradiction.$\blacksquare$
**Remark 5.9** For the same reason stated in the Theorem 5.8, the edges incident with the vertex of $N_1$ can not form two stars of the decomposition.

**Theorem 5.10** Let $L$ be a Lobster with $\text{diam}(L) = 2n - 3$ and $|N_2| = 1$. If $L$ has no adjacent junction, then $L$ admits a CMSD if and only if $L$ has exactly one junction-neighbor with

(a) $(n - 2)$ junction supports which form a $(3,i,n)$ - set, $i \neq 4$

(b) $(n - 2)$ junction supports with distant degrees $4, 4, 5, 6, 7, \ldots, n$.

(c) $(n - 2)$ junction supports with distinct degrees $3, 4, 5, 6, \ldots, n-1, n+1$.

**Proof** Let $L$ admit a CMSD $(S_1, S_2, \ldots, S_n)$. Clearly the centers of $S_3, S_4, \ldots, S_n$ form $(n - 2)$ junction supports. Further the edge contribution of the stars to the longest path $P$ is reduced by 2. Since $|N_2| = 1$, let $w$ be the unique vertex of degree 2 in $N_1$ and let $N_2 = \{v\}$. Let $u$ be the other vertex (junction) adjacent to $w$. Now either $S_1$ or $S_2$ lie on $P$ and in both cases, we have exactly one junction-neighbor (see fig 5.11).
If \( d(u) \neq 4, (n + 1) \), then the degrees of \((n - 2)\) junction supports form a \((3, i, n)\) set. If \( d(u) = 4 \), the degrees of \((n - 2)\) junction supports are 4, 4, ..., \( n \). If \( d(u) = n + 1 \), then the degrees of \((n - 2)\) junction supports are 3, 4, 5, ..., \( n - 1, n + 1 \).

Conversely, we take the unique junction neighbor as \( S_1 \) and the unique path of length two with origin and terminus \( v \) as \( S_2 \). Now let \( u_3, u_4, ..., u_n \) be the \((n - 2)\) junction supports which form a \((3, i, n)\) - set. Let \( d(u_3) = 3, d(u_4) = 4, ..., d(u_{i-1}) = i - 1, d(u_i) = i + 1, d(u_{i+1}) = i + 1, ..., d(u_n) = n \) for \( i \neq 4, n \). In this case, \( S_k = < N[u_k] > \) for all \( k = 3, 4, 5, ..., n \) and \( k \neq i \). Since \( < N[u_i] > = S_{i+1} \) for \( u = u_i \) and \( w \) is adjacent to \( u \), we take \( S_i = < N[u_i] - w > \).

If \( u_3, u_4, ..., u_n \) are junction supports with degrees 4, 4, 5, 6, ..., \( n \), we take \( S_k = < N[u_k] > \), where \( k = 4, 5, 6, ..., n \) and
$S_3 = <N[u_3] - w>$. If $u_3, u_4, \ldots, u_n$ are the junction supports with degrees 3, 4, 5, $\ldots$, $n - 1, n+1$. In this case, we take $S_k = <N[u_k]>$ where $k = 3, 4, 5, \ldots, n - 1$ and $S_n = <N[u_n] - w>$.

Hence the theorem. ■

**Notation 5.11** Let $L_i$ be a Lobster and $u$ be a vertex in the longest path of $L_i$. If $u$ is adjacent to $i$ junctions, then the Lobster obtained by attaching a pendant vertex of a star $S_j (j \geq 2)$ to $u$ is denoted by $L_i * S_j$. Examples of $L_2 * S_j, L_1 * S_j$ and $L_0 * S_j$ are given in fig 5.12.

**Fig. 5.12**
Notation 5.12 Let $L'$ be a caterpillar. Then the Lobster obtained by attaching a pendant vertex of a star $S_j (j \geq 3)$ with some junction is denoted by $L' * S_j$.

Theorem 5.13 Let $L$ be a Lobster with $\text{diam}(L) = 2n - 3$ and $|N_2| = n^2 - 2$. If $L$ has no adjacent junction, then $L$ admits a CMSD if and only if $L = L'_0 * S_i$ or $L' * S_i$ where $L'$ has at most one junction-neighbor and $(n - 3)$ junction supports with degrees $3, 4, \ldots, i - 1, i + 1, \ldots, n. (i = n_2 + 1)$

Proof Let $L$ admit a CMSD $(S_1, S_2, \ldots, S_n)$ with no adjacent junction. Since $n_2 \geq 2$ and we can attach the pendant vertex of only one star $S_i (i > 2)$ in the longest path $P$ of $L$, $S_2$ lies on the longest path $P$ of $L$. Suppose $L$ has two junction-neighbors $e_1$ and $e_2$. Then $<E(L') - E(S_3 \cup \ldots \cup S_i \cup S_{i+1} \cup \ldots \cup S_n)> = 3K_2$, which implies that there is no $S_2$, which is a contradiction. Therefore, $L$ has at most one junction-neighbor.

Since $L$ has no adjacent junctions, the only possibility is to attach a pendant vertex of a star $S_i (i = n_2 + 1)$ at the center of $S_2$ or $S_j (j \geq 3)$.

If $S_i$ is attached at the center of $S_2$, let $L'_0 = L - S_i$. Since $L$ admits a CMSD, $L'_0$ is decomposed into $S_1, S_2, S_3, \ldots, S_{i-1}, S_{i+1}$,
\ldots, S_n, which gives (n - 3) junction supports with distinct degrees 3, 4, \ldots, i - 1, i + 1, \ldots, n.

If $S_i$ is attached to a center $S_j (j \geq 2, j \neq i)$, let $L' = L - S_i$. Since $L$ admits a CMSD $L'$ is decomposed into $S_1, S_2, S_3, \ldots, S_{i - 1}, S_{i + 1}, \ldots, S_n$, which gives (n - 3) junction supports with distinct degrees 3, 4, 5, \ldots, i - 1, i + 1, \ldots, n (See fig 5.13).

Conversely, let $L = L'_0 * S_i$ or $L = L' * S_i$ where $L'$ has at most one junction-neighbor and (n - 3) junction supports with distinct degrees 3, 4, 5, \ldots, i - 1, i + 1, \ldots, n.
If \( L = L_0' \ast S_i \) then \( L_0' = L - S_i \) has \((n - 3)\) junction supports \( u_j \), where \( j = 3, 4, \ldots, i - 1, i + 1, \ldots, n \) with distinct degrees \( 3, 4, 5, \ldots, i - 1, i + 1 \). Take \( S_k = <N[u_k]> \) for all \( k = 3, 4, \ldots, n \) and \( k \neq i \).

Then \( <E(L_0') - E(S_3U\ldotsUS_iUS_{i+1}U\ldotsUS_n)> = P_3 \) or \( P_2UP_1 \) which can be taken as \( S_1 \) and \( S_2 \). Hence \( L \) can be decomposed into \( S_1, S_2, \ldots, S_n \). Similarly we can dispose of the case \( L = L_0' \ast S_1 \).

**Lemma 5.14** Let \( L \) be a Lobster with at least two adjacent junctions, \( \text{diam}(L) = 2n - 3 \) and \( N_2 \neq \varnothing \). If \( L \) admits a CMSD, then \( <C> = K_2 \) or \( P_2 \).

**Proof** It is enough to prove that \( |C| = 2 \) or \( 3 \). Suppose \( |C| \geq 4 \) let \( C = \{u_i, u_j, u_r, u_s\} \) be the set of adjacent junctions such that \( <C> \) has no isolates. If \( <C> = P_3 = u_i u_j u_r u_s \) then the stars centered at \( u_i, u_j, u_r, \) and \( u_s \) together does not contribute three edges to the longest path \( P \) of \( L \) (see fig 5.14).

![Fig 5.14](image_url)
If \( <C> = 2K_2 \), then the stars centered at each of the adjacent junctions do not contribute two edges and the star centered at \( w \) (since \( N_2 \neq \emptyset \)) does not contribute at least one edge to \( P \). In both the cases \( \text{diam}(L) \leq 2n - 4 \), which is a contradiction.

**Theorem 5.15** Let \( L \) be \( P_2 \)-Lobster with \( \text{diam}(L) = 2n - 3 \) and \( N_2 \neq \emptyset \). Then \( L \) admits a CMSD if and only if \( L \) has at most one junction-neighbor and \( L = L'_2 \ast S_2 \) (or) \( L''_2 \ast S_i (i > 2) \), where \( L' \) is a Lobster with \( (n - 2) \) junction supports with degrees 3, 4, 5, ..., \( n \) and \( L'' \) is a Lobster with \( (n - 3) \) junction supports with degrees 3, 4, 5, ..., \( i - 1, i + 1, \ldots, n \).

**Proof** Let \( L \) admit a CMSD \( (S_1, S_2, \ldots, S_n) \). Suppose \( L \) has two junction-neighbors \( e_1 \) and \( e_2 \). Since these edges cannot be in \( S_i \) for all \( i \geq 3 \), \( <E(L) - E(S_3 \cup \ldots \cup S_n)> = 3K_2 \). Thus there is no \( S_2 \) which is a contradiction (see fig 5.15).

![Fig 5.15](image-url)

Therefore \( L \) has at most one junction-neighbor.
Since all the vertices of $N_2$ are adjacent to exactly one vertex $w$, it must be the center of exactly one star of size $i = n_2 + 1$.

Let $L'' = L - < N_2 \cup \{w\} >$. Since $L$ admits a CMSD, $L''$ is decomposed into $S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$, which gives $(n-3)$ junction supports with distinct degrees $3, 4, 5, \ldots, i-1, i+1, \ldots, n$ (see fig 5.16).

![Diagram of L'']

If $n_2 = 1$, then $L' = L - < N_2 \cup w \rangle = L - S_2$ is decomposed into $S_1, S_3, \ldots, S_n$ which corresponds to $(n-2)$ junction supports with distinct degrees $3, 4, 5, \ldots, n$

Conversely, let $L = L'_2 \ast S_2$ or $L''_2 \ast S_i$.

If $L = L'_2 \ast S_2$, then $L' = L - S_2$ has $(n-2)$ non-adjacent junction supports $u_3, u_4, \ldots, u_n$ with degrees $3, 4, 5, \ldots, n$. Let $S_i = < N[u_i] >$ for all $i = 3, 4, 5, \ldots, n$. Therefore $< E(L') - E(S_3 \cup \ldots \cup S_n) > = K_2$, which can be taken as $S_1$. 

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If \( L = L'_2 * S_i \) and \( L'_2 \) is a Lobster with \((n - 3)\) non-adjacent junction supports \( u_3, u_4, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n \), let \( S_i = \langle N[u_k] \rangle \) for all \( k = 3, 4, 5, \ldots, i - 1, i + 1, \ldots, n \). Then \( <E(L''_2) - E(S_3U_4U\ldotsUS_{i-1}U \\ldotsUS_n) >= P_3 \) or \( P_2 U P_1 \) which can be decomposed into \( S_1 \) and \( S_2 \).

Hence \( L \) admits a CMSD \((S_1, S_2, S_3, \ldots, S_n)\).

**Theorem 5.16** Let \( L \) be a Lobster with \( \text{diam}(L) = 2n - 3 \), \( N_2 = \varnothing \) and \( L \) has no adjacent junction. Then \( L \) admits a CMSD if

(a) \( L \) has exactly one junction neighbor and \((n - 2)\) junction supports with degrees \( 3, 4, \ldots, i - 1, i + 1, i + 2, i + 2, \ldots, n \)

(or) \( 4, 5, 5, 6, 7, \ldots, n \)

(or) \( 3, 4, 5, \ldots, n - 1, n + 2 \)

(or) \( b \) \( L \) has at most one junction neighbor and \((n - 3)\) junction supports with degree \( 3, 4, 5, \ldots, i - 1, i + 1, \ldots, j - 1, j + 1, \ldots, i + j + 1, \ldots, n \), if \( i + j < n \)

(or) \( 3, 4, \ldots, i - 1, i + 1, \ldots, j - 1, j + 1, \ldots, n, i + j, \) if \( i + j > n \).

**Proof** Let \( L \) admit a CMSD \((S_1, S_2, \ldots, S_n)\). Since \( N_2 = \varnothing \), \( \text{diam}(L) = 2n - 3 \), and \( L \) has no adjacent junctions, exactly two stars, say \( S_i \) and \( S_j \) must have the same center in the longest path \( P \) of \( L \).
Clearly \( i, j \neq 1 \). If \( j = 2 \), then \( S_1 \) lies in \( P \) and it is the only junction neighbor. Further \( L \) has \((n - 2)\) junction supports. If \( i = 3 \), then the degrees are \( 4, 5, 6, 7, \ldots, n \) and if \( i = n \), then the degrees are \( 3, 4, 5, 6, \ldots, n - 1, n + 2 \). Otherwise, then the degrees are \( 3, 4, 5, \ldots, i - 1, i + 1, i + 2, i + 2, \ldots, n \). If \( j > 2 \) then \( S_1 \) and \( S_2 \) must lie in \( P \). If \( <S_1US_2> \) is connected then \( L \) has no junction neighbor. Otherwise, \( S_1 \) is the only junction neighbor. Since \( S_i \) and \( S_j \) have the same center, \( L \) has \((n - 3)\) junction supports. If \( i + j \leq n \), then the degree are \( 3, 4, 5, \ldots, i - 1, i + 1, \ldots, j - 1, j + 1, \ldots, i + j - 1, i + j, i + j + 1, \ldots, n \). Otherwise, the degree are \( 3, 4, \ldots, i - 1, i + 1, \ldots, j - 1, j + 1, \ldots, n, i + j \).

**Conversely**, let \( L \) be the given Lobster satisfying (a) or (b)

**Case (a)** Let \( u_3, u_4, \ldots, u_n \) the \((n - 2)\) junction supports.

Take the unique junction - neighbor as \( S_1 \). If the degrees are \( 3, 4, 5, \ldots, i - 1, i + 1, i + 2, i + 2, \ldots, n \), let \( d(u_k) = k, 3 \leq k \leq n, k \neq i \) & \( d(u_i) = i + 2 \). Take \( S_k = <N[u_k]> \) for all \( k(\neq i) \). Further \( <N[u_i]> = S_{i+2} \) which gives \( S_2 \) & \( S_i \). If the degree are \( 4, 5, 5, 6, \ldots, n \), let \( d(u_3) = 5, d(u_4) = 4, d(u_5) = 5, d(u_6) = 6, \ldots, d(u_n) = n \). Take \( <N[u_3]> = S_4, <N[u_i]> = S_i (5 \leq i \leq n) \). Further \( <N[u_4]> = S_5 \) which
gives \( S_2 \) & \( S_3 \). If the degrees are 3, 4, 5, \ldots, \( n - 1 \), \( n + 2 \), let \( d(u_k) = k, \ 3 \leq k \leq n - 1 \) and \( d(u_n) = n + 2 \). Take \( <N[u_i]> = S_i \) for all \( 3 \leq i \leq n - 1 \). Further \( <N[u_n]> = S_{n+2} \) which gives \( S_2 \) & \( S_n \) (see fig 5.17).

![Fig 5.17](image)

**Case (b)** Let \( u_4, u_5, \ldots, u_n \) be the \((n - 3)\) junction supports.

If \( i + j \leq n \), then the degree are 3, 4, 5, \ldots, \( i - 1 \), \( i + 1 \), \ldots, \( j - 1 \), \( j + 1 \), \ldots, \( i + j - 1 \), \( i + j \), \( i + j \) + 1 \ldots, \( n \). Take \( d(u_4) = 3, d(u_5) = 4, \ldots, d(u_i) = i - 1, d(u_{i+1}) = i + 1, \ldots, d(u_{j-1}) = j - 1, d(u_j) = j + 1, \ldots, d(u_{i+j-2}) = i + j - 1, d(u_{i+j-1}) = i + j, d(u_{i+j}) = i + j, \ldots, d(u_n) = n \). Take \( <N[u_4]> = S_3, \ldots, <N[u_i]> = S_{i-1}, <N[u_{i+1}]> = S_{i+1}, \ldots, <N[u_{j-1}]> = S_{j-1}, <N[u_j]> = S_j, \ldots, <N[u_{i+j-2}]> = S_{i+j-1}, <N[u_{i+j}]> = S_{i+j}, \ldots, <N[u_n]> = S_n \). Further \( <N[u_{i+j-1}]> = S_i + j \) which can be decomposed into \( S_i \) & \( S_j \). Also \( H = <E(L) - E(S_3 U S_4 U \ldots U S_n)> = P_3 \) or \( P_1 U P_2 \) which gives \( S_1 \) & \( S_2 \) (see fig 5.18).
Fig 5.18

Hence the theorem.

**Theorem 5.17** Let $L$ be a Lobster with $N_{2} = \varnothing$ and $\text{diam}(L) = 2n - 3$. If $L$ admits a CMSD, then $<C> = K_{2}$ or $2K_{2}$ or $P_{2}$ or $P_{3}$ or $P_{2} \cup K_{2}$.

**Proof** First we claim that $|C| \leq 5$. Suppose $|C| \geq 6$. Then there exists at least six vertices $u_{1}, u_{2}, \ldots, u_{6}$ such that $H = <u_{1}, u_{2}, \ldots, u_{6}>$ is isolate-free. Hence $H = 3K_{2}$ or $P_{2} \cup P_{2}$ or $P_{3} \cup K_{2}$ or $P_{5}$. In all these cases, the contribution of edges to $\text{diam}(L)$ by the stars centered at these vertices is reduced by at least 3. Therefore $\text{diam}(L) \leq 2n - 4$, which is a contradiction.

If $|C| = 2$ or 3, then clearly $<C> = K_{2}$ or $P_{2}$ respectively. If $|C| = 4$, then $<C> = 2K_{2}$ or $P_{3}$. If $|C| = 5$, $<C> \neq P_{4}$, for otherwise, the contribution of edges to $\text{diam}(L)$ by the stars centered at these vertices is reduced by at least 3 which is a contradiction.
Therefore \( <C> = P_2 \cup K_2 \). Hence \( <C> = P_2 \) or \( 2K_2 \) or \( P_3 \) or \( P_2 \cup K_2 \) or \( K_2 \). \( \square \)

**Theorem 5.18** Let \( L \) be \( P_2 \) - Lobster with \( \text{diam}(L) = 2n - 3 \) and \( N_2 = \emptyset \). Then \( L \) admits a CMSD if and only if

(a) \( L \) has exactly one junction - neighbor and \((n-1)\) junction supports with degrees 3, 4, 4, \ldots, \( n \)

(or) 3, 3, 4, 5, \ldots, i, \ldots, j - 1, j + 1,

\( j + 1, j + 2, \ldots, n \).

(or) (b) \( L \) has at most one junction neighbor and \((n-2)\) junction supports with degrees 3, 4, \ldots, i, \ldots, \( j - 1, j + 1, j + 2, j + 2, \ldots, k, \ldots, n \).

**Proof** Let \( L \) be a \( P_2 \) - lobster which admits CMSD \((S_1, S_2, \ldots, S_n)\).

Since \( N_2 = \emptyset \), \( \text{diam}(L) = 2n - 3 \), there is exactly one junction \( u_i \) which is adjacent to two junctions \( u_i \) and \( u_k \) in the longest path \( P \) of \( L \).

We consider two cases.

**Case (1)** \( d(u_i) \) and \( d(u_k) \geq 4 \).

Since \( L \) is a \( P_2 \) - Lobster, \( d(u_j) \neq 3 \). If neither \( u_iu_j \) nor \( u_ju_k \) is an edge of a star centered at \( u_j \), then there must be another junction support with degree \( j \). If \( d(u_j) = 4 \), then \( u_j \) is a center of \( S_2 \) and
hence L has \((n - 1)\) junction supports with degrees 3, 4, 4, 5, \ldots, \(n\).

In this case, \(S_1\) must lie in the longest path \(P\) of \(L\) but not incident with any junction supports and hence it must be junction-neighbor.

If \(d(u_j) > 4\), then both \(S_1\) and \(S_2\) lie in the longest path with \(<S_1 \cup S_2> = P_3\) or \(P_1 \cup P_2\) and hence \(L\) has \((n - 2)\) junction supports with degrees 3, 4, \ldots, \(i\), \ldots, \(j - 1\), \(j + 1\), \(j + 2\), \(j + 2\), \ldots, \(k\), \ldots, \(n\) with at most one junction-neighbor.

Case (2) Let \(d(u_k) = 3\).

If \(u_k\) is the center of \(S_3\) then as in case (1) \(L\) has \((n - 2)\) junction supports with degree 3, 4, \ldots, \(i\), \ldots, \(j - 1\), \(j + 1\), \(j + 2\), \(j + 2\), \ldots, \(n\) and at most one junction-neighbor. If \(u_k\) is the center of \(S_2\), then \(<S_1 \cup S_2>\) is either connected or disconnected. In both cases, \(L\) has \((n - 1)\) junction supports with degrees 3, 3, 4, \ldots, \(i\), \ldots, \(j - 1\), \(j +1\), \(j +1\), \(j + 2\), \ldots, \(n\) and exactly one junction-neighbor.

Conversely, let \(L\) be a \(P_2\) - Lobster satisfying the conditions (a) or (b).

Case (a) Let the \((n - 1)\) junction supports be \(u_3, u_4, u_4', u_5, \ldots, u_n\).

Let \(d(u_i) = i\), \(3 \leq i \leq n\) and \(d(u_4') = 4\). Take \(S_1 = <N[u_i]>\), \(3 \leq i \leq n\) and the unique junction neighbor as \(S_1\). Let \(w_1\) and \(w_2\) be
the pendant vertices adjacent to $u'$. Take $S_2 = \langle w_1, u_4', w_2 \rangle$ (see fig 5.19).

Fig 5.19

Let $u_3, u_3', u_4, \ldots, u_n$ be the $(n - 1)$ junction supports. Let $d(u_3') = 3, d(u_3) = 3, d(u_4) = 4, \ldots, d(u_i) = i, \ldots, d(u_{j-1}) = j - 1, d(u_j) = j + 1, d(u_{j+1}) = j + 1, d(u_{j+2}) = j + 2, \ldots, d(u_n) = n$. Take $S_i = \langle N[u_i] \rangle, 3 \leq i \leq n, i \neq j$ and $S_j = \langle N[u_j] - u_i u_j \rangle$. Then subgraph $<E(L) - E(S_3 U S_4 U \ldots U S_n) >= P_3$ or $P_2 U P_1$ which can be taken as $S_1$ and $S_2$ (see fig 5.20).

Fig 5.20

Case (b) Let $u_3, u_4, \ldots, u_n$ be the $(n - 2)$ junction supports with degrees $3, 4, \ldots, i, \ldots, j - 1, j + 1, j + 2, j + 2, \ldots, n$. 

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Let \( d(u_3) = 3, d(u_4) = 4, \ldots, d(u_i) = i, \ldots, d(u_{j-1}) = j - 1, d(u_j) = j + 2, d(u_{j+1}) = j + 1, d(u_{j+2}) = j + 2, \ldots, d(u_k) = k, \ldots, d(u_n) = n. \) Take \( S_i = <N[u_i]>\), \( 3 \leq i \leq n, i \neq j \) and \( S_j = <N[u_j] - u_iu_j, u_ju_k>. \) The subgraph \(<E(L) - E(S_3U S_4U \ldots U S_n)> = P_3 \) or \( P_2 U P_1 \) which can be decomposed into \( S_1 \) and \( S_2 \) (see fig 5.21).

Fig 5.21

Hence the theorem. \( \blacksquare \)

**Theorem 5.22** Let \( L \) be a \( P_2 U K_2 - \) Lobster with \( \text{diam}(L) = 2n - 3 \) and \( N_2 = \phi. \) Then \( L \) admits a CMSD if and only if \( L \) has \( n \) junction supports with degrees \( 3, 3, 3, 4, 5, \ldots, n. \)

**Proof** Let \( L \) be a \( P_2 U K_2 - \) Lobster which admits a CMSD \( (S_1, S_2, \ldots, S_n). \) Since \( N_2 = \phi \) and \( \text{diam}(L) = 2n - 3, \) there is exactly one junction \( u_j \) which is adjacent to two junctions \( u_i \) and \( u_k \) and exactly two adjacent junctions \( u_r \) and \( u_s \) such that \( <u_i, u_j, u_k, u_r, u_s> = P_2 U K_2. \) If \( d(u_j) > 3 \) or \( d(u_r) > 3, \) then the contribution of edges to \( \text{diam}(L) \) by the stars centered at \( u_i, u_j, u_k \) and \( u_s \) are reduced by at
least 3, which is a contradiction. Hence \( d(u_j) = d(u_r) = 3 \). Further \( u_j \) and \( u_r \) can not be the center of \( S_i, i \geq 3 \). Therefore \( L \) has \( n \) junction supports with degrees 3,3,3,4,5,......,n

Conversely, let \( u_3, u'_3, u''_3, u_4, u_5, \ldots, u_n \) be the \( n \) junction supports with \( d(u_i) = i \) where \( 3 \leq i \leq n \) and \( d(u'_3) = d(u''_3) = 3 \). Take \( S_i = <N[u_i]>, 3 \leq i \leq n \). Let \( w_1 \) be a pendant vertex adjacent to \( u'_3 \) take \( S_1 = u'_3w_1 \). Let \( w_2 \) be the pendant vertex adjacent to \( u''_3 \) and \( u_0 \) be a vertex in the longest path adjacent to \( u''_3 \). Take \( S_2 = <u_0u''_3w_2> \). Thus \( L \) admits a CMSD. \( \blacksquare \)

**Theorem 5.20** Let \( L \) be a 2\( K_2 \) - Lobster with \( \text{diam}(L) = 2n - 3 \) and \( N_2 = \varnothing \). Then \( L \) admits a CMSD if and only if 

(a) \( L \) has at most two junction neighbors and \((n - 1)\) junction supports with degrees 3, 3, 4, 5,......, \( i - 1, i +1, i +1, i + 2, \)

......, n. (or)

(b) \( L \) has at most one junction neighbor and \((n - 2)\) junction supports with degrees 3, 4, 5, ......, \( i - 1, i +1, i +1, i + 2, \)

......, k - 1, k + 1, \ k +1, k +2, ......, n.

**Proof** Let \( L \) be a 2\( K_2 \) - Lobster. Since \( N_2 = \varnothing \) and \( \text{diam}(L) = 2n - 3 \), there are exactly four junctions \( u_i, u_j, u_r \) and \( u_s \) such that \( u_i \) is adjacent to \( u_j \) and \( u_r \) is adjacent to \( u_s \). Clearly \( u_j \) is not adjacent to
Now let $L$ admit a CMSD $(S_1, S_2, \ldots, S_n)$. If one of these four vertices say $u$ is of degree three and it is the center of $S_2$, then $L$ has $(n - 1)$ junction supports with degrees $3, 3, 4, 5, \ldots, i - 1, i + 1, i + 1, i + 2, \ldots, n$. If $<S_1 US_2>$ is connected, then $L$ has no junction neighbor; otherwise $S_1$ and one of the edges of $S_2$ in the longest path $P$ are the two junction neighbors. If $u$ is the center of $S_3$ or all the four vertices are of degree $\geq 4$, then $L$ has $n - 2$ junction supports with degrees $3, 4, 5, \ldots, i - 1, i + 1, i + 1, i + 2, \ldots, k - 1, k + 1, k + 1, k + 2, \ldots, n$. If $<S_1 US_2>$ is connected then $L$ has no junction neighbor; otherwise $S_1$ is the only junction neighbor in $L$.

Conversely, let $L$ be $2K_2$-Lobster satisfying (a) or (b). If $u_3, u_4, \ldots, u_n$ are the $(n - 1)$ junction supports with $d(u_r) = r, 3 \leq r \leq n, r \neq i, d(u_i) = i + 1$ and $d(u'_3) = 3$. Here $u'_3 = u_k$. Take $S_r = <N[u_r]>, 3 \leq r \leq n, r \neq i$ and $S_i = <N[u_i] - u_iu_j>$ and $S_2 = <N[u'_3] - u_ku_i>$. The remaining edge can be taken as $S_1$.

If $u_3, u_4, \ldots, u_n$ be the $(n - 2)$ junction supports with $d(u_r) = r, 3 \leq r \leq n, r \neq i, k$ and $d(u_i) = i + 1$ and $d(u_k) = k + 1$, take $S_r = <N[u_r]>$ where $r \neq i, k$ and $S_i = <N[u_i] - u_iu_j>$ and $S_k = <N[u_k] - u_ku_i>$. Since $L$ has at most one junction neighbor, the
induced subgraph \( < E(L) - E(S_3 \cup S_4 \cup \ldots \cup S_n) > = P_3 \) or \( P_2 \cup P_1 \) which gives \( S_1 \) and \( S_2 \).

**Theorem 5.21** Let \( L \) be \( K_2 \) Lobster with \( \text{diam} (L) = 2n - 3 \) and \( N_2 = \emptyset \). Then \( L \) admits a CMSD if and only if \( L \) has no junction neighbor with

(a) \((n-2)\) junction supports with degrees

\[ 3, 4, 5, \ldots, i - 1, i + 1, i + 2, \ldots, j - 1, j + 1, j + 2, \ldots, n \]

(or) \[ 3, 4, \ldots, i - 1, i + 1, i + 1, i + 2, \ldots, j, \ldots, r - 1, r + 1, r + 1, r + 2, \ldots, n \]

(or) \[ 3, 4, 5, \ldots, i - 1, i + 1, i + 1, i + 1, \ldots, j, \ldots, n - 1, n + 1 \]

(b) \((n-1)\) junction supports with degrees

\[ 3, 3, 4, 5, \ldots, i - 1, i + 1, i + 1, i + 2, \ldots, j, \ldots, n \]

(or) \[ 3, 4, 4, 5, \ldots, i - 1, i + 1, i + 1, i + 2, \ldots, n \]

(or) \[ 3, 3, 4, 5, \ldots, k - 1, k + 1, k + 1, k + 2, \ldots, n \]

(or) \[ 3, 4, 4, 5, \ldots, n \]

(or) \[ 3, 3, 4, 5, \ldots, n - 1, n + 1 \]

**Proof** Let \( u_i \) and \( u_j \) be the adjacent junction supports. First we assume that \( d(u_i) \) and \( d(u_j) \geq 4 \). Since \( N_2 = \emptyset \) and \( \text{diam}(L) = 2n - 3 \),
S_1 or one of the edges of S_2 should not lie in the longest path P.

If S_1 is incident with u_i (i ≠ 3, n), then the centers of S_3, S_4, ..., S_n are the (n - 2) junction supports with degrees 3, 4, 5, ..., i - 1, i + 1, i + 1, i + 2, ..., j - 1, j + 1, j + 1, j + 2, ..., n.

If S_1 is incident with some other junction supports u_r (r ≠ 2, i, j), then L has (n - 2) junction supports with degrees 3, 4, ..., i - 1, i + 1, i + 1, i + 2, ..., j, ..., r - 1, r + 1, r + 1, r + 2, ..., n.

If S_1 is incident with u_n, then L has (n - 2) junction supports with degrees 3, 4, 5, ..., i - 1, i + 1, i + 1, i + 2, ..., j, ..., n - 1, n + 1. In all these cases, since S_2 wholly lies in P, L has no junction-neighbor.

If S_1 is incident with the center of S_2 (S_2 wholly lies in P), then L has (n - 1) junction supports with degrees 3, 3, 4, 5, ..., i - 1, i + 1, i + 1, i + 2, ..., j, ..., n. Since no junction supports can be the center for two stars of size ≥ 2, we consider the case where u_i is the center for S_1 and S_2. In this case L has (n - 1) junction supports with degrees 3, 4, 4, 5, ..., i - 1, i + 1, i + 1, i + 2, ..., n.

Now we assume that at least one of u_i and u_j is of degree 3. Let d(u_i) = 3. If u_i is the center of S_2, then S_1 is incident with some other junction supports u_k (k ≠ 3, n). Therefore L has (n - 1) junction
supports with degrees 3, 3, 4, ...., k - 1, k + 1, k + 1, k + 2, ...., n. If \( u_i \) is the center of \( S_2 \) and \( S_1 \) is incident with the junction support \( u_3 \), then \( L \) has \((n - 1)\) junction supports with degrees 3, 4, 4, 5, ...., n. If \( u_i \) is the center of \( S_2 \) and \( S_1 \) is incident with \( u_n \) then \( L \) has \((n - 1)\) junction supports with degrees 3, 3, 4, ...., n - 1, n + 1.

Conversely, let \( L \) be a \( K_2 \)-Lobster with \( \text{diam}(L) = 2n - 3 \) and \( N_2 = \emptyset \) satisfying conditions (a) or (b). Let \( u_3, u_4, ...., u_n \) be the given \((n - 2)\) junctions with \( d(u_k) = k \) for all \( 3 \leq k \leq n \), \( k \neq i \) and \( j \). \( d(u_i) = i + 2 \) and \( d(u_j) = j + 1 \). Take \( S_k = \langle N[u_k] \rangle \) for all \( 3 \leq k \leq n \), \( k \neq i, j \) and \( S_i = \langle N[u_i] - u_iu_j \rangle \). Let \( w \) be any pendant vertex incident with \( u_i \). Take \( S_1 = u_i w \). Then the induced subgraph \( \langle E(L) - E(S_1US_3U ....U S_n) \rangle = P_2 \) which can be taken as \( S_2 \) (see fig 5.22).

![Fig 5.22](image)

If the degrees are 3, 4, 5 ...., \( i - 1 \), \( i + 1 \), \( i + 2 \), ...., \( j \), ...., \( r - 1 \), \( r + 1 \), \( r + 2 \), ...., \( n \), then \( d(u_k) = k \) for all \( 3 \leq k \leq n \) and \( k \neq i, r \). \( d(u_i) = i + 1 \), \( d(u_r) = r + 1 \). Take \( S_k = \langle N[u_k] \rangle \), \( 3 \leq k \leq n \) and \( k \neq i, r \).
Thm 4.8:
Let $\mathcal{L}$ be a hoistet with underlying path $P_0$ and $m = \frac{1}{2} \left( t + \sqrt{1+4t} \right)$ with $t = \frac{1}{2}m^2(m+1)$

and $n_a = 8m-4$. Then $\mathcal{L}$ admits a CNP $\mathcal{P}$.

Can $P_0$ can be partitioned into paths $P_0', P_1'$ 

$\ldots , P_m$ such that the origin

and terminus of
$S_{r+1} = <N[u_r]>$ which can be decomposed into $S_r$ and $S_1$ and $S_i = <N[u_i] - u_j>$. Since $L$ has no junction neighbor, $<E(L) - E(S_1 \cup S_3 \cup \ldots \cup S_n)> = P_2$ which can be taken as $S_2$ (see fig. 5.23).

Fig 5.23

Now let $u_2, u_3, \ldots, u_n$ be a $(n-2)$ junctions with respective degrees $4, 4, 5, \ldots, i-1, i+1, \ldots, n$. Then take $S_k = <N[u_k]>$ for all $3 \leq k \leq n, k \neq i$ and $S_i = <N[u_i] - u_j>$. The induced subgraph $<E(L) - E(S_3 \cup S_4 \cup \ldots \cup S_n)> = P_2U_P_1$ where $P_2$ is on $L$ and $P_1$ is not on $L$ which can be taken as $S_1$ and $S_2$.

If the degrees of the $(n-2)$ junctions are $3, 4, 5, \ldots, i-1, i+1, \ldots, n-1, n+1$, take $S_k = <N[u_k]>$ for all $3 \leq k \leq n-1, k \neq i$ and $S_i = <N[u_i] - u_j>$. Then the induced subgraph $<E(L) - E(S_1 \cup S_3 \cup \ldots \cup S_n)> = P_2$.

Now let $u_3, u'_3, u_4, \ldots, u_n$ be a $(n-1)$ junction supports with respective degrees $3, 3, 4, \ldots, i-1, i+1, i+1, i+2, \ldots, j, \ldots, n$. Then we can take $S_k = <N[u_k]>$ where $3 \leq k \leq n$ and $k \neq i$. $S_i =$
\( <N[u_i] - u_j> \). Further \( <N[u'_3]> = K_{1,3} \) which can be decomposed into \( S_1 \) and \( S_2 \) (see fig 5.24).

![Fig 5.24](image)

If \( 3, 3, 4, 5, \ldots, k - 1, k + 1, k + 1, k + 2, \ldots, n \) are the degrees of the \((n - 1)\) junction supports, take \( S_r = <N[u_i]> \) for \( 3 \leq r \leq n \) and \( r \neq k \), and \( <N[u_k]> = S_{k+1} \) which can be decomposed into \( S_k \) and \( S_1 \). Then \( S_2 = <N[u'_3] - u_j> \) (see fig 5.25).

![Fig 5.25](image)

If \( 3, 4, 4, 5, \ldots, n \) are the degrees of the \((n - 1)\) junction supports, let \( d(u_k) = k \) for \( 4 \leq k \leq n \), \( d(u_3) = 4 \), \( d(u'_3) = 3 \). Take \( S_k = <N[u_k]> \) for \( 4 \leq k \leq n \). Further \( <N[u_3]> = S_4 \) which can be decomposed into \( S_3 \) and \( S_1 \) and \( S_2 = <N[u'_3] - u_j> \) (see fig 5.26).
If $3, 3, 4, 5, \ldots, n - 1, n + 1$ are the degrees of the $(n - 1)$
junction supports, take $S_k = \langle N[u_k] \rangle$, $3 \leq k \leq n - 1$ and $\langle N[u_n] \rangle = S_{n + 1}$ which can be decomposed into $S_n$ and $S_1$ and $S_2 = \langle N[u'_3] - u_j \rangle$ (see fig 5.27).

Thus in all cases, we get a CMSD. □

**Theorem 5.22** Let $L$ be $P_3$.Lobster with $\text{diam}(L) = 2n - 3$ and $N_2 = \emptyset$. Then $L$ admits a CMSD if and only if $L$ has $(n - 1)$ junction supports with degrees $3, 3, 4, 5, \ldots, i - 1, i + 1, i + 1, i + 2, \ldots, j, \ldots, n$ and no junction - neighbor.
**Proof** Let $L$ be a $P_3$ Lobster with $C = \{u_i, u_j, u_k, u_l\}$. Since $\text{diam}(L) = 2n - 3$, all the vertices of $C$ can not be of degree $\geq 4$ and there cannot be a vertex (of degree $\geq 4$) which is adjacent to two vertex of degree $\geq 4$. One of the internal vertices (say $u_k$) of $P_3$ is of degree 3. Therefore $S_1$ must be incident with $u_k$. Hence $L$ has $(n - 1)$ junction supports with degrees $3, 3, 4, 5, \ldots, i - 1, i + 1, i + 2, \ldots, j, \ldots, n$. Since $S_1$ is not on $P$, $L$ has no junction-neighbor.

Conversely, let $L$ have $(n - 1)$ junction supports $u_3, u'_3, u_4, \ldots, u_n$ with respective degrees $3, 3, 4, 5, \ldots, i - 1, i + 1, \ldots, j, \ldots, n$ and $L$ has no junction neighbor. Let $d(u_k) = k$ where $3 \leq k \leq n$, $k \neq i$ and $d(u_i) = i + 1$ and $d(u'_3) = 3$. Take $S_k = <N[u_k]>$ for all $3 \leq k \leq n$, $k \neq i$ and $S_i = <N[u_i] - u_j>$. Here $u_r = u'_3$. Let $w$ be the vertex adjacent to $u_r$. Take $S_1 = u_rw$ (see fig 5.28).

![Fig 5.28](image-url)
Then the induced subgraph $< E(L) - E(S_1 \cup S_3 \cup \ldots \cup S_n) > \cong P_2$, (since $L$ has no junction neighbor) which can be taken as $S_2$. Thus $L$ admits a CMSD.

**Conclusion and open problems**

In this chapter, we have found a sufficient condition for a Lobster $L$ to admit a CMSD $(S_1, S_2, \ldots, S_n)$. The necessary condition gives the upper bound $\text{diam}(L) \leq 2n - 1$. We have characterized Lobsters with diameter $2n - 1$, $2n - 2$ and $2n - 3$. For all other Lobsters with $n(n + 1)/2$ edges, the problem remains open.

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