CHAPTER - II

ANALYTIC SOLUTIONS OF A STOCHASTIC BANKING MODEL WITH INFINITE CAPACITY
It is a well known fact that Banking system in any country is like a nervous system in the body. A healthy and effective economy of any nation is mainly built on the firm foundation of wealthy Banking system. Particularly in Developing Countries like India, Banking system has to play very predominant role in every activity of the nation relating to fiscal matters. Thus studies relating to Banking Systems with particular reference to the money reserve available with the system at any arbitrary time 't' has naturally attracted by many researchers. Thus the work in this direction occupies an important place in modern research because of the fact that Banking System is the backbone for
any nation's economy. Nation's economy is very sensitive that it is affected much by the policies adopted by Banking System mainly on Investments and capital formation. For instance, changes in the pattern of interest rates produce a large effect on the borrowings as well as on the Investment patterns. Investment again has a useful role in capital formation with important concomitant effects changing the complexion of the economy. Thus by suitably manipulating interest rates it is also possible to promote investments and also control the borrowings which increase capital reserves. Rich capital reserves are essential for fruitful investments which inturn result in heavy returns either from Industrial sector or from Agricultural sector or from both.

Thus arises the natural motivating interest to study the reserve level available with the Banking System on a perspective basis. With this motivation Sarma, K. L. A. P. in 1983 proposed a Stochastic Banking Model and obtained analytic as well as explicit results in his Doctoral thesis. The model proposed by him is already explained in (1.3) of Chapter-I. In order to obtain solutions he assumed that the inter-withdrawal times follow a negative exponential
distribution. A typical realisation of the model considered by him is given in Fig (2.3).

As already explained in the earlier chapter that the main object of the this dissertation is to obtain the results of a Stochastic Banking Model when the inter-withdrawal times or amount of withdrawals or both follows a mixture of exponential distributions. In the following section we will obtain first the results when the inter-withdrawal times denoted by the r.v., 'u' have the density function h(u) which is a mixture of two exponential distributions.

2.2 ANALYTIC SOLUTION OF M(x,y,t) FOR ME/M/1/FIFO/∞ STOCHASTIC BANKING MODEL

First we proceed to obtain analytic solution for M(x,y,t) defined in (1.3.2). For this an integral equation is formed by Sarma, K.L.A. in 1983 and is given as follows:

\[ M(x,y,t) = H(x-y) \delta(x-y-t) \int_{0}^{\infty} h(u) du \]

\[ + \int_{0}^{t} \int_{0}^{t+u} h(u) du \int_{0}^{y+u} g(v) M(x,y+u-v,t-u) dv \]

\[ \cdots (2.2.1) \]
FIG. 2.3 THE STORAGE MODEL (Infinite capacity)
Here it is assumed that the density function $h(u)$ governing the inter-withdrawal times follows a mixture of exponential distributions. Hence we have:

$$h(u) = qa e^{-au} + (1-q)be^{-bu}, \quad 0 \leq q \leq 1, \quad a > b > 0, \quad 0 < u < \infty. \quad \ldots (2.2.2)$$

**Theorem (2.2.3):**

The d.l.t $M^*(x,s,p)$ for $M(x,y,t)$ is given by

$$M^*(x,s,p) = \left[ \frac{(s+\mu)}{(s+\mu)(a+p-s)(b+p-s)} - qa\mu(b+p-s) - (1-q)b\mu(a+p-s) \right] \left\{ q(b+p-s)[e^{-sx} - e^{-(a+p)x}] \frac{a\mu}{(a+p+\mu)} M^*(x,a+p,p) \right. \right. $$

$$+ \left. (1-q)(a+p-s)[e^{-sx} - e^{-(b+p)x}] \frac{b\mu}{(b+p+\mu)} M^*(x,b+p,p) \right\} \right. \left. \right. \right. \ldots (2.2.4)$$

**Proof:**

The d.l.t $M^*(x,s,p)$ of $M(x,y,t)$ is given by

$$M^*(x,s,p) = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} M(x,y,t) dt \quad \ldots (2.2.5)$$

Re. $s > 0$, Re. $p > 0$.

Substituting (2.2.1) in (2.2.5) we have:

$$M^*(x,s,p) = A + B, \quad \ldots (2.2.6)$$

where

$$A = \int_0^\infty \int_0^\infty e^{-sy} e^{-pt} H(x-y) \delta(x-y-t) \int h(u) du \ dt \ dy \quad \ldots (2.2.7)$$
and
\[ B = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} \left[ \int_0^t h(u) du \int_0^{y+u} g(v) M(x, y+u-v, t-u) dv \right] dt \ dy. \]

Now substituting (2.2.2) in (2.2.7) and after some calculations we have:
\[ A = q \left[ \frac{e^{-sx} - e^{-(a+p)x}}{(a+p-s)} \right] + (1-q) \left[ \frac{e^{-sx} - e^{-(b+p)x}}{(b+p-s)} \right]. \]

Now substituting (2.2.2) and \( g(v) = \mu e^{-\mu v} \), if \( 0 < v < \infty, \mu > 0 \)
\[ = 0, \text{ otherwise} \]
in (2.2.8) and after some calculations we have:
\[ B = \frac{qa \mu}{(a+p-s)} \left[ \frac{M^*(x, s, p)}{(s+\mu)} - \frac{M^*(x, a+p, p)}{(a+p+\mu)} \right] \\
+ \frac{(1-q)b \mu}{(b+p-s)} \left[ \frac{M^*(x, s, p)}{(s+\mu)} - \frac{M^*(x, b+p, p)}{(b+p+\mu)} \right]. \]

Now adding (2.2.9) and (2.2.11) and after some simplifications we obtain (2.2.4).

Hence the Proof.

**Theorem (2.2.12):**

The time dependent solution of \( M(x, y, t) \) is well determined in terms of \( M^*(x, s, p) \) using (2.2.4).
Proof:

First we observe that the R.H.S of (2.2.4) involves two unknown constants namely \( M^*(x,a+p,p) \) and \( M^*(x,b+p,p) \). \( M^*(x,s,p) \) is completely determined only after the evaluation of these two unknown constants. We next notice that \( M^*(x,s,p) \) is analytic in ‘s’ in the right half of the plane, i.e. \( \text{Re. } s > 0 \). Then we observe that the denominator of (2.2.4) namely

\[
[(s+\mu)(a+p-s)(b+p-s)-qa\mu(b+p-s)-(1-q)b\mu(a+p-s)]
\]

is a polynomial of order 3 in ‘s’ argument and hence it has three zeroes of which one zero with negative real part and the remaining two zeroes with positive real parts. Let these zeroes with positive real parts are denoted by \( \delta_1 \) and \( \delta_2 \) respectively. Since \( M^*(x,s,p) \) is analytic in ‘s’ for \( \text{Re. } s > 0 \), the numerator of (2.2.4) also must vanish at \( \delta_1 \) and \( \delta_2 \). Thus equating the numerator on the R.H.S of (2.2.4) to zero twice, we obtain two non-homogeneous linear equations involving the two unknown constants. Solving these two equations the unknown constants \( M^*(x,a+p,p) \) and \( M^*(x,b+p,p) \) can be determined, so that \( M^*(x,s,p) \) is completely known. By taking Inverse Laplace Transform of \( M^*(x,s,p) \) with respect to the arguments \( s \) & \( p \) successively \( M^*(x,s,p) \) is determined.

Hence the Proof.
Now we proceed to obtain the analytic solution of \( M(x,y,t) \) for M/ME/1/FIFO/\( \infty \) Stochastic Banking Model in the following section.

### 2.3 ANALYTIC SOLUTION OF M(x,y,t) FOR M/ME/1/FIFO/\( \infty \) STOCHASTIC BANKING MODEL

Here it is assumed that the density function \( g(v) \) governing the amount of withdrawals follows a mixture of two exponential distributions. Hence we have:

\[
g(v) = q^* a^* e^{-a^* v} + (1-q^*) b^* e^{-b^* v}, \; 0 \leq q^* \leq 1, \; a^* > b^* > 0 \quad 0 < v < \infty \quad \ldots (2.3.1)
\]

**Theorem (2.3.2):**

The d.l.t \( M^*(x,s,p) \) for \( M(x,y,t) \) is given by

\[
M^*(x,s,p) = \left[ \frac{(s+a^')(s+b')}{(\lambda+p-s)(s+a^')(s+b')-\lambda[q^* a^' (s+b^') + (1-q^*) b^' (s+a^')]} \right] \\
\left\{ e^{-sx} - e^{-(\lambda+p)x} - \frac{\lambda}{(\lambda+p+a^')(\lambda+p+b^')} \left[ q^* a^' (\lambda+p+b^') + (1-q^*) b^' (\lambda+p+a^') \right] M^*(x,\lambda+p,p) \right\} \\
\ldots (2.3.3)
\]
Proof:

The d.l.t $M^*(x,s,p)$ of $M(x,y,t)$ is given by

$$M^*(x,s,p) = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} M(x,y,t) dt$$ \hspace{1cm} ...(2.3.4)

Substituting (2.2.1) in (2.3.4) we have:

$$M^*(x,s,p) = A + B , \hspace{1cm} ...(2.3.5)$$

where

$$A = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} h(x-y) \delta(x-y-t) \int_0^\infty h(u) du \ dt \ dy$$ \hspace{1cm} ...(2.3.6)

and

$$B = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} \left[ \int_0^t h(u) du \int_0^y g(v) M(x,y+u-v,t-u) dv \right] dt \ dy.$$ \hspace{1cm} ...(2.3.7)

Now substituting

$$h(u) = \lambda e^{-\lambda u}, \text{ if } 0 < u < \infty, \ \lambda > 0$$

= 0, otherwise \hspace{1cm} ...(2.3.8)

in (2.3.6) and after some calculations we have:

$$A = \left[ \frac{e^{-sx} - e^{-(\lambda+p)x}}{(\lambda+p-s)} \right] . \hspace{1cm} ...(2.3.9)$$

Now substituting (2.3.1) and (2.3.8) in (2.3.7) and after some calculations we have:

$$B = \frac{\lambda}{(\lambda+p-s)} \left[ q' a' (s+b') + (1-q') b' (s+a') \right] M^*(x,s,p)$$

$$- \frac{\lambda}{(\lambda+p-s)} \left[ q' a' (\lambda+p+b') + (1-q') b' (\lambda+p+a') \right] M^*(x,\lambda+p,p). \hspace{1cm} ...(2.3.10)$$
Now adding (2.3.9) and (2.3.10) and after some simplifications we obtain (2.3.3).

Hence the Proof.

**Theorem (2.3.11):**

The time dependent solution of \( M(x,y,t) \) is well determined in terms of \( M^*(x,s,p) \) using (2.3.3).

**Proof:**

First we observe that the R.H.S of (2.3.3) involves an unknown constant namely \( M^*(x,\lambda+p,p) \). \( M^*(x,s,p) \) is completely determined only after the evaluation of this unknown constant. We next notice that \( M^*(x,s,p) \) is analytic in 's' in the right half of the plane, \( \text{Re.} s > 0 \). We then observe that the denominator of (2.3.3) namely

\[
\{(\lambda+p-s)(s+a')(s+b')-\lambda[q'a'(s+b')+(1-q)b'(s+a')]\}
\]

is a polynomial of order 3 in 's' argument and hence it has three zeroes of which one zero with positive real part and the remaining two zeroes are negative real parts. Let the zero with positive real part be denoted by \( \delta \). Since \( M^*(x,s,p) \) is analytic 's' for \( \text{Re.} s > 0 \). The numerator of (2.3.3) also must vanish at \( \delta \). Thus equating the numerator on the R.H.S of (2.3.3) to zero, we obtain the
non-homogeneous linear equation involving one unknown constant. By this equation \( M^*(x, \lambda + p, p) \) can be determined. So, that \( M^*(x, s, p) \) is completely known. By taking Inverse Laplace Transform of \( M^*(x, s, p) \) with respect to the arguments 's' and 'p' successively \( M(x,y,t) \) is determined.

Hence the Proof.

Now we proceed to obtain the analytic solution of \( M(x,y,t) \) for ME/ME/1/FIFO/\( \infty \) Stochastic Banking Model in the following section.

2.4 ANALYTIC SOLUTION OF \( M(x,y,t) \) FOR ME/ME/1/FIFO/\( \infty \) STOCHASTIC BANKING MODEL

Here it is assumed that the density function \( h(u) \) and \( g(v) \) governing the inter-withdrawal times and amount of withdrawals follow mixture of exponential distributions. Hence we have \( h(u) \) and \( g(v) \) follow mixture of two exponential distributions defined in (2.2.2) and (2.3.1) respectively.
**Theorem (2.4.1):**

The d.l.t \( M^*(x,s,p) \) for \( M(x,y,t) \) is given by

\[
M^*(x,s,p) = \left[ \frac{(s+a)^*(s+b)^*}{(a+p-s)(b+p-s)(s+a^*)(s+b^*) - qa(1-q^*)b^*(a+p-s)(s+b^*)} \right] \]

\[
q(b+p-s)(e^{-sx} - e^{-(a+p)x}) + (1-q)(a+p-s)(e^{-sx} - e^{-(b+p)x})
\]

\[
- \left[ \frac{qa(1-q^*)a^*(b+p-s)}{(a+p+a^*)} + \frac{qa(1-q^*)b^*(b+p-s)}{(a+p+b^*)} \right] M^*(x,a+p,p)
\]

\[
- \left[ \frac{(1-q)bq^*a^*(a+p-s)}{(b+p+a^*)} + \frac{(1-q)b(1-q^*)b^*(a+p-s)}{(b+p+b^*)} \right] M^*(x,b+p,p)
\]

\...(2.4.2)

**Proof:**

The d.l.t \( M^*(x,s,p) \) of \( M(x,y,t) \) is given by

\[
M^*(x,s,p) = \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} M(x,y,t) \, dt \, dy \quad \text{Re.} \, s > 0, \, \text{Re.} \, p > 0.
\]

Substituting (2.2.1) in (2.4.3) we have:

\[
M^*(x,s,p) = A + B , \quad \text{...(2.4.4)}
\]

where

\[
A = \int_0^\infty e^{-sx} \int_0^\infty e^{-pt} H(x-y) \, \delta(x-y-t) \int_0^\infty h(u) \, du \, dt \, dy \quad \text{...(2.4.5)}
\]

and

\[
B = \int_0^\infty e^{-sx} \int_0^\infty e^{-pt} \left[ \int_0^t h(u) \, du \int_0^{y+u} g(v) M(x,y+u-v,t-u) \, dv \right] \, dt \, dy . \quad \text{...(2.4.6)}
\]
Now substituting (2.2.2) in (2.4.5) and after some calculations we have:
\[ A = q \left[ \frac{e^{-sx} - e^{-(a+p)x}}{(a+p-s)} \right] + (1-q) \left[ \frac{e^{-sx} - e^{-(b+p)x}}{(b+p-s)} \right] . \]

... (2.4.7)

Now substituting (2.2.2) and (2.3.1) in (2.4.6) and after some calculations we have:
\[ B = \frac{qaq'a'}{(a+p-s)(a+p+a'^*)} + \frac{qa(1-q')b'}{(a+p-s)(s+b'^*)} + \frac{(1-q) bq'a'}{(b+p-s)(s+a'^*)} \]
\[ + \frac{(1-q)b(1-q')b'}{(b+p-s)(s+b'^*)} \] \[ M^*(x,s,p) \]
\[ - \left[ \frac{qaq'a'}{(a+p-s)(a+p+a'^*)} + \frac{qa(1-q')b'}{(a+p-s)(a+p+b'^*)} \right] M^*(x,a+p,p) \]
\[ - \left[ \frac{(1-q)b q'a'}{(b+p-s)(b+p+a'^*)} + \frac{(1-q)b(1-q')b'}{(b+p-s)(b+p+b'^*)} \right] M^*(x,b+p,p) . \]

... (2.4.8)

Now adding (2.4.7) and (2.4.8) and after some calculations we obtain (2.4.2).

Hence the Proof.

**Theorem (2.4.9):**

The time dependent solution of \( M(x,y,t) \) is well determined in terms of \( M^*(x,s,p) \) using (2.4.2).

**Proof:**

First we observe that the R.H.S of (2.4.2) involves two unknown constants namely \( M^*(x,a+p,p) \) and \( M^*(x,b+p,p) \).
$\text{M}^*(x,s,p)\text{ is completely determined only after the evaluation of these two unknown constants. We next notice that }$ $\text{M}^*(x,s,p)\text{ is analytic in 's' in the right half of the plane, i.e., } \text{Re.} s>0\text{, then we observe that the denominator of (2.4.2) namely}$

$$\begin{align*}
\{ (a+p-s)(b+p-s)(s+a')(s+b')-qa' a'(b+p-s)(s+b') \} \\
\{ -qa(1-q') b' (b+p-s)(s+a')-(1-q) bq' a'(a+p-s)(s+b') \} \\
-(1-q) b(1-q') b'(a+p-s)(s+a')
\end{align*}$$

is a polynomial of order 4 in 's' argument and hence it has four zeroes of which two zeroes with negative real parts and the remaining two zeroes with positive real parts. Let these zeroes with positive real part are denoted by $\delta_1$ and $\delta_2$. Thus equating the numerator on the R.H.S of (2.4.2) to zero twice, we obtain two non-homogeneous linear equations involving two unknown constants. Solving these two equations the unknown constants $\text{M}^*(x,a+p,p)$ and $\text{M}^*(x,b+p,p)$ can be determined, so that $\text{M}^*(x,s,p)$ is completely known. By taking Inverse Laplace Transform of $\text{M}^*(x,s,p)$ with respect to the arguments $s$ & $p$, $M(x,y,t)$ is determined.

Hence the Proof.
In this chapter we have obtained analytic solutions for a Stochastic Banking Model when the inter-withdrawal times (or) amount of withdrawals or both follows a mixture of exponential distributions. The mixture of exponential distributions can be considered as a generalisation of negative exponential distribution. Because if \( q=1 \) and \( a=\lambda \) is substituted in (2.2.2) we obtain density function of a negative exponential distribution, given in (2.3.8). This can also be obtained by substituting \( q=0 \) and \( b=\lambda \). Now we proceed to derive the earlier results obtained by us and obtained by Sarma, K.L.A.P in 1883 in the following table.
### 2.5.1 TABLE OF RECOVERY

<table>
<thead>
<tr>
<th>S.No.</th>
<th>From</th>
<th>Substitution</th>
<th>To</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2.2.4)</td>
<td>$q=1, \ a=\lambda$</td>
<td>$(1.3.6)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(or) $q=0, \ b=\lambda$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(2.3.3)</td>
<td>$q^<em>=1, \ a^</em>=\mu$</td>
<td>$(1.3.6)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(or) $q^<em>=0, \ b^</em>=\mu$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(2.4.2)</td>
<td>$q^<em>=1, \ a^</em>=\mu$</td>
<td>$(2.2.4)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(or) $q^<em>=0, \ b^</em>=\mu$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(2.4.2)</td>
<td>$q=1, \ a=\lambda$</td>
<td>$(2.3.3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(or) $q=0, \ b=\lambda$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(2.4.2)</td>
<td>$q=1, a=\lambda, q^<em>=1, a^</em>=\mu$</td>
<td>$(1.3.6)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(or) $q=0, b=\lambda, q^<em>=0, b^</em>=\mu$</td>
<td></td>
</tr>
</tbody>
</table>

### 2.5.2 REMARK

In the above table, in equation number (2.2.4), if we substitute $q=1, \ a=\lambda$ we derive the result $(1.3.6)$ obtained by Sarma, K.L.A. P1** in 1983. Similarly the other results can also be derived, as explained in the above table.