Chapter 6

A STABLE HIGH ACCURACY CUBIC SPLINE APPROXIMATION FOR TWO DIMENSIONAL NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS ON A GEOMETRIC MESH

6.1 Introduction

This chapter presents a nine point compact discretization of order two in y- and three in x-directions for the solution of two dimensional nonlinear elliptic boundary value problems on a non-uniform mesh using cubic spline approximations. We discuss the complete derivation strategy of the method in details and also discuss how our discretization is able to handle Poisson’s equation in polar coordinates. Convergence of the method has been established. Some physical examples and their numerical results are provided to justify the usefulness of the proposed method. The second order elliptic equations are obtained as the steady state solutions (as \( t \to \infty \)) of the parabolic and wave equations. Solutions of these equations are of great importance in many fields of science, such as electromagnetics, astronomy, heat transfer, fluid mechanics etc. because they may represent a temperature, electric or magnetic potential, and displacement for an elastic membrane. Simulations of steady heat flows or irrotational flows of an inviscid, incompressible fluid, pressure computations for either the flow through a porous medium or that associated with the flow of a viscous, incompressible fluid, and many others all involve solving elliptic equations. Fourth order compact finite difference scheme for the solution of linear elliptic equations on a constant mesh have been discussed by several authors (see [13, 59, 60, 100]). Jain et al [46-47] have first derived fourth order difference methods for the solution of the system of nonlinear elliptic equations on a constant mesh in Cartesian coordinates and obtained convergent solution for many physical models like Navier-Stokes equations of motion. Later, Mohanty et al [63, 71, 73] have extended their technique to obtain fourth order approximation for non linear elliptic equations in polar coordinates. Because of the instability not many numerical methods for the solution of elliptic equations on a geometric mesh (or, variable mesh) have been developed. Further, the use of cubic spline approximations for the solution of non linear differential equations play an important role in many physical models, especially on a non-uniform mesh. During last three decades many researchers have developed high order numerical methods with (or without) cubic spline approximations for the solution of nonlinear two point boundary value problems (see [9, 43, 44, 53, 68, 69, 83, 85]). Much recently, using cubic spline approximations Mohanty et al [74, 75, 77, 78] have derived high order stable numerical methods on both uniform and non-uniform mesh for the solution of non linear parabolic and hyperbolic partial differential equations and obtained convergent results. In our knowledge, no high order nine point compact numerical scheme of order two in y-direction and order three in x-direction on a non-uniform mesh for the solution
of second order nonlinear elliptic partial differential equation has been discussed in the literature so far.

In this chapter, using nine-point compact stencil (see Fig. 6.1), we discuss a new stable method of order two in $y$- and order three in $x$-directions on a variable mesh based on cubic spline approximations for the solution of two-dimensional nonlinear elliptic boundary value problems. It has been experienced in the past that for problems in polar coordinates the solution for high order methods usually deteriorates in the vicinity of the singularities. We overcome this difficulty by modifying our method in such a way that the solution retains its order and accuracy everywhere in the vicinity of the singularity.

We begin by considering a two dimensional nonlinear elliptic partial differential equation of the form

$$\nabla^2 u = \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) u = f \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad (x, y) \in R$$

(6.1.1)

where region $R$ is the unit square, $R = (0,1) \times (0,1)$.

The corresponding Dirichlet boundary conditions are prescribed by

$$u(x,y) = \psi(x,y), \quad (x,y) \in \partial R$$

(6.1.2)

where $\partial R$ is the boundary of region $R$.

We assume that for $0 < x, y < 1$,

(i) $f \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$ is continuous,

(ii) the partial derivatives $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial u_x}, \frac{\partial f}{\partial u_y}$ exists and are continuous,

(iii) $\frac{\partial f}{\partial u} \geq 0$, $\left| \frac{\partial f}{\partial u_x} \right| \leq G$ and $\left| \frac{\partial f}{\partial u_y} \right| \leq H$

(6.1.3a)

(6.1.3b)

(6.1.3c)

where $G$ and $H$ are positive constants (see [46, 47, 63]). In addition we assume that $u(x,y)$ is differentiable as high order as possible under consideration.

Fig. 6.1. Schematic representation of a single computational cell
6.2 The cubic spline approximation based numerical scheme

We consider a rectangular grid $G_R$ on the solution region $[0,1] \times [0,1]$ with variable grid spacings $h_i$ and constant step length $k$ in the directions $x$- and $y$- respectively, $h_i$ and $k$ being positive numbers. Let $x_i - x_{i-1} = h_i$ and $x_{i+1} - x_i = h_{i+1}$ be the variable mesh size and $\sigma_i = (h_{i+1}/h_i) > 0$ the mesh ratio parameter. Then the mesh points $(x_l, y_m)$ denoted by $(l,m)$ are defined as $x_l = x_0 + \sum_{k=1}^{l} h_k$ and $y_m = mk$, $l = 0,1, ..., N + 1$, $m = 0,1, ..., M + 1$, where $0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1$ and $N$ and $M$ are positive integers such that $(M + 1)k = 1$, as shown in Fig. 6.1.

Further, let $U_{l,m}$ and $u_{l,m}$ be the exact and approximate solution values of $u(x,y)$, respectively at the grid point $(x_l, y_m)$.

For the derivation of the cubic spline finite difference method for the solution of partial differential equation (6.1.1), we follow the ideas given by Jain and Aziz [43]. We use cubic spline approximations in $x$-direction and second order finite difference approximation in $y$-direction.

At the grid point $(x_l, y_m)$, we denote:

$$U_{pq} = \frac{\partial^p u}{\partial x^p \partial y^q}(x_l, y_m)$$

Let $S_m(x)$ be the cubic spline interpolating polynomial of the function $u(x,y)$ at the grid point $(x_l, y_m)$, and is given by

$$S_m(x) = \frac{(x_l-x)^3}{6h_l} M_{l-1,m} + \frac{(x-x_{l-1})^3}{6h_l} M_{l,m} + \left(u_{l-1,m} - \frac{h_l^2}{6} M_{l-1,m}\right) \left(\frac{x_l-x}{h_l}\right)$$

$$+ \left(u_{l,m} - \frac{h_l^2}{6} M_{l,m}\right) \left(\frac{x-x_{l-1}}{h_l}\right), \quad x_{l-1} \leq x \leq x_l;$$

$$l = 1,2, ..., N + 1, m = 0,1, ..., M + 1 \quad (6.2.1)$$

which satisfies at $m$th-line parallel to $x$-axis the following properties:

(i) $S_m(x)$ coincides with a polynomial of degree three on each subinterval $[x_{l-1}, x_l]$, 

$$l=1,2, ..., N+1, m = 0,1, ..., M+1;$$

(ii) $S_m(x) \in C^2[0, 1]$, and

(iii) $S_m(x_l) = u_{l,m} \quad l=0,1, ..., N + 1, m=0,1, ..., M + 1$.

The derivatives of the cubic spline function $S_m(x)$ are given by

$$S'_m(x) = -\frac{(x_l-x)^2}{2h_l} M_{l-1,m} + \frac{(x-x_{l-1})^2}{2h_l} M_{l,m} + \frac{u_{l,m} - u_{l-1,m}}{h_l} - \frac{h_l}{6} \left[M_{l,m} - M_{l-1,m}\right].$$
\[ S'_m(x) = \frac{(x-x_l)^2}{2h_{l+1}} M_{l,m} + \frac{(x-x_{l+1})^2}{2h_{l+1}} M_{l+1,m} + \frac{h_{l+1} - h_l}{h_{l+1}} [M_{l+1,m} - M_{l,m}], \quad x \in [x_{l-1}, x_l] \quad (6.2.2) \]

\[ S''_m(x) = \frac{1}{h_l} M_{l-1,m} + \frac{1}{h_l} [M_{l,m} + 2M_{l-1,m}], \quad x \in [x_l, x_{l+1}] \quad (6.2.3) \]

and where we have

\[
M_{l,m} = S'''_m(x_l) = U_{xxl,m} = -U_{yyl,m} + f(x_l, y_m, U_{l,m}, U_{xl,m}, U_{yl,m}),
\]

\[ l = 0, 1, ..., N + 1; m = 0, 1, ..., M + 1 \]

\[
S'_m(x_l) = U_{x_l,m} = \frac{U_{l,m} - U_{l-1,m}}{h_l} + \frac{h_{l+1}}{6} [M_{l-1,m} + 2M_{l,m}], \quad x \in [x_{l-1}, x_l] \quad (6.2.5a) \]

\[
S'_m(x_{l+1}) = U_{x_l+1,m} = \frac{U_{l+1,m} - U_{l,m}}{h_{l+1}} - \frac{h_{l+1}}{6} [M_{l+1,m} + 2M_{l,m}], \quad x \in [x_l, x_{l+1}] \quad (6.2.5b) \]

From the continuity condition we may obtain the consistency relation for \( S'_m(x_l) \).

Furthermore, from equation (6.2.2), (6.2.3),

\[ S'_m(x_{l-1}) = U_{x_{l-1},m} = \frac{U_{l,m} - U_{l-1,m}}{h_l} - \frac{h_{l+1}}{6} [M_{l,m} + 2M_{l-1,m}] \quad (6.2.6a) \]

and \[ S'_m(x_{l+1}) = U_{x_{l+1},m} = \frac{U_{l+1,m} - U_{l,m}}{h_{l+1}} + \frac{h_{l+1}}{6} [M_{l,m} + 2M_{l+1,m}] \quad (6.2.6b) \]

These important properties of the cubic spline function \( S_m(x) \) have been used in building up our numerical scheme (see [43]).

At the grid point \( (x_l, y_m) \), let us denote

\[
P_l = \sigma_i^2 + \sigma_i - 1, \\
Q_l = (\sigma_i + 1)(\sigma_i^2 + 3\sigma_i + 1), \\
R_l = \sigma_i(1 + \sigma_i - \sigma_i^2), \\
S_l = \sigma_i(\sigma_i + 1) \quad (6.2.7)\]

We require the following approximations.

\[ \bar{U}_{y_l,m} = (U_{l,m+1} - U_{l,m-1})/2k \quad (6.2.8a) \]

\[ \bar{U}_{y_{l+1},m} = (U_{l+1,m+1} - U_{l+1,m-1})/2k \quad (6.2.8b) \]

\[ \bar{U}_{y_{l-1},m} = (U_{l-1,m+1} - U_{l-1,m-1})/2k \quad (6.2.8c) \]

\[ \bar{U}_{yy,l,m} = (U_{l,m+1} - 2U_{l,m} + U_{l,m-1})/k^2 \quad (6.2.8d) \]
Further, let

\[\bar{U}_{x_l+1,m} = \frac{U_{l+1,m} - U_{l,m}}{h_l} + \frac{\sigma_l h_l}{6} [\bar{M}_{l,m} + 2\bar{M}_{l+1,m}] \] (6.2.10a)

\[\bar{U}_{x_l-1,m} = \frac{U_{l,m} - U_{l-1,m}}{h_l} - \frac{\sigma_l h_l}{6} [2\bar{M}_{l-1,m} + \bar{M}_{l,m}] \] (6.2.10b)

\[\bar{U}_{x_l,m} = \bar{U}_{x_l,m} - \frac{\sigma_l h_l}{6(1+\sigma_l)} [\bar{M}_{l+1,m} - \bar{M}_{l-1,m}] \] (6.2.10c)

where, we have

\[\bar{M}_{l,m} = -\bar{U}_{y_l,m} + \bar{F}_{l,m} \] (6.2.11a)

\[\bar{M}_{l+1,m} = -\bar{U}_{y_l+1,m} + \bar{F}_{l+1,m} \] (6.2.11b)

\[\bar{M}_{l-1,m} = -\bar{U}_{y_l-1,m} + \bar{F}_{l-1,m} \] (6.2.11c)

\[\bar{F}_{l,m} = f(x_l, y_m, U_{l,m}, \bar{U}_{x_l,m}, \bar{U}_{y_l,m}) \] (6.2.11d)

\[\bar{F}_{l+1,m} = f(x_{l+1}, y_m, U_{l+1,m}, \bar{U}_{x_l+1,m}, \bar{U}_{y_l+1,m}) \] (6.2.11e)

\[\bar{F}_{l-1,m} = f(x_{l-1}, y_m, U_{l-1,m}, \bar{U}_{x_l-1,m}, \bar{U}_{y_l-1,m}) \] (6.2.11f)

\[\bar{F}_{l+1,m} = f(x_{l+1}, y_m, U_{l+1,m}, \bar{U}_{x_l+1,m}, \bar{U}_{y_l+1,m}) \] (6.2.11g)

\[\bar{F}_{l-1,m} = f(x_{l-1}, y_m, U_{l-1,m}, \bar{U}_{x_l-1,m}, \bar{U}_{y_l-1,m}) \] (6.2.11h)

and \[\bar{F}_{l,m} = f(x_l, y_m, U_{l,m}, \bar{U}_{x_l,m}, \bar{U}_{y_l,m}) \] (6.2.11i)

Then at each internal mesh point \((x_l, y_m)\), \(l = 1, 2, ..., N, m = 1, 2, ..., M\) the nonlinear elliptic partial differential equation (6.1.1) is discretized by means of the formula

\[L_u \equiv U_{l+1,m} - (1 + \sigma_l)U_{l,m} + \sigma_l U_{l-1,m}
= -\frac{h_l^2}{12} \left[ P_l U_{y_{l+1},m} + Q_l U_{y_{l+1},m} + R_l U_{y_{l-1},m} \right] \]
where the local truncation error is
\[
\tilde{T}_{l,m} = O\left(k^2 h^2_l + k^2 h^3_l + h^5_l\right).
\]

The proposed cubic spline method is of accuracy of \(O(k^2 + k^2 h_l + h^3_l)\) for the solution of nonlinear elliptic partial differential equation (6.1.1). Note that the coefficients in (6.2.12) are positive if \(\sigma_i^2 + \sigma_i - 1 > 0\) and \(1 + \sigma_i - \sigma_i^2 > 0\), that is, for convergence of the numerical scheme it is essential that the parameters \(\sigma_i\) satisfy
\[
\sqrt{5} - 1 < 2\sigma_i < \sqrt{5} + 1.
\]

### 6.3 Derivation strategy for the scheme

We begin our derivation of the method by writing, at the grid point \((x_l, y_m)\), the given differential equation (6.1.1) as
\[
U_{xx,l,m} + U_{yy,l,m} = f(x_l, y_m, U_{l,m}, U_{x,l,m}, U_{y,l,m}) \equiv F_{l,m} \text{ (say)} \quad (6.3.1)
\]

It is then easy to verify using Taylor series expansion about the grid point \((x_l, y_m)\), from equation (6.3.1) we obtain
\[
L_u = -\frac{h^2_l}{12} [P_l U_{yy,l+1,m} + Q_l U_{yy,l,m} + R_l U_{y,yl-1,m}]
+ \frac{h^2_l}{12} [P_l F_{l+1,m} + Q_l F_{l,m} + R_l F_{l-1,m}]
+ O\left(k^2 h^2_l + k^2 h^3_l + h^5_l\right); \quad l = 1, 2, \ldots, N, m = 1, 2, \ldots, M \quad (6.3.2)
\]

Let us denote \(\alpha_{l,m} = \left(\frac{\partial f}{\partial u_x}\right)_{l,m}\)

Define,
\[
\bar{U}_{x,l,m} = \left(U_{l+1,m} - (1 - \sigma_i^2)U_{l,m} - \sigma_i^2 U_{l-1,m}\right)/(h_l S_i)
= \frac{1}{h_l \sigma_i(1 + \sigma_i)} \left[U_{l,m} + h_{l+1} U_{l+1,m} + \frac{h_{l+1}^2}{6} U_{xx,m} + \frac{h_{l+1}^2}{24} U_{xxx,m} + \ldots
- (1 - \sigma_i^2)U_{l,m} - \sigma_i^2 (U_{l,m} - h_l U_{x,m} + \frac{h^2_l}{2} U_{xx,m} - \frac{h^3_l}{6} U_{xxx,m} + \frac{h^4_l}{24} U_{xxxx,m} - \ldots)\right]
= \frac{1}{h_l \sigma_i(1 + \sigma_i)} \left[(\sigma_i h_l + \sigma_i^2 h_l) U_{x,m} + \frac{\sigma_i^3 h^3_l}{6} U_{xxx,m} + \frac{\alpha^2_i h^4_l}{24} U_{xxxx,m}
+ O(h^5_l)\right]
\]

\[
\Rightarrow \bar{U}_{x,l,m} = U_{x,l,m} + \frac{\sigma_i h^2_l}{6} U_{xxt,l,m} + O(h^3_l) \quad (6.3.3)
\]

\[
\bar{U}_{x,l+1,m} = \left((1 + 2\sigma_i)U_{l+1,m} - (1 + \sigma_i^2)U_{l,m} + \sigma_i^2 U_{l-1,m}\right)/(h_l S_i)
\]
\[
\begin{align*}
\frac{1}{h_l\sigma_l(1+\sigma_l)}[\sigma_l(1+\sigma_l)h_lU_x + \sigma_l^2(1+\sigma_l)h_l^2U_{xx} + \frac{(\sigma_l^2+2\sigma_l^4-\sigma_l^2)}{6}h_l^3U_{xxx} + O(h_l^4)] \\
= U_{xl,m} + \sigma_lh_lU_{xx} + \frac{\sigma_l}{6(1+\sigma_l)}(\sigma_l + 2\sigma_l^2 - 1)h_l^2U_{xxx} + O(h_l^3) \\
= U_{xl+1,m} - \frac{\sigma_l}{6} (1 + \sigma_l) h_l^2 U_{xxx} + O(h_l^3) \quad (6.3.4)
\end{align*}
\]

\[U_{xl-1,m} = (-U_{l+1,m} + (1 + \sigma_l)^2 U_{l,m} - \sigma_l(2 + \sigma_l)U_{l-1,m})/(h_lS_l)\]

\[\Rightarrow \bar{U}_{xl-1,m} - U_{xl-1,m} = \frac{1}{h_l\sigma_l(1+\sigma_l)}[-(U_{l,m} + \sigma_lh_lU_x + \frac{\sigma_l^2h_l^2}{2}U_{xx} + \frac{\sigma_l^3h_l^3}{6}U_{xxx} + \ldots) \\
+ (1 + \sigma_l^2 + 2\sigma_l)U_{l,m} - \sigma_l(2 + \sigma_l)(U_{l,m} - h_lU_x + \frac{h_l^2}{2}U_{xx} - \frac{h_l^3}{6}U_{xxx} + \ldots)] \\
- [U_{xl,m} - h_lU_{xx} + \frac{h_l^2}{2}U_{xxx} - \frac{h_l^3}{6}U_{xxxx} + \ldots] \\
= -\frac{(\sigma_l^2+2\sigma_l+1)}{6(1+\sigma_l)}h_l^2U_{xxx} + O(h_l^3) \\
= -\frac{(1+\sigma_l)}{6}h_l^2U_{xxx} + O(h_l^3) \quad (6.3.5)
\]

\[\Rightarrow \bar{U}_{xl-1,m} = U_{xl-1,m} - \frac{(1+\sigma_l)}{6}h_l^2U_{xxx} + O(h_l^3) \quad (6.3.5)\]

\[\bar{U}_{yl,m} = \left(U_{l,m+1} - U_{l,m-1}\right)/2k \]

\[= \frac{1}{2k} [U_{l,m} + kU_{yl,m} + \frac{k^2}{2}U_{yy} + \frac{k^3}{6}U_{yyy} + \ldots \\
- (U_{l,m} - kU_{yl,m} + \frac{k^2}{2}U_{yy} - \frac{k^3}{6}U_{yyy} + \ldots)] \\
= U_{yl,m} + \frac{k^2}{6}U_{yyy} + O(k^4) \quad (6.3.6)\]

\[\bar{U}_{yl+1,m} = \left(U_{l+1,m+1} - U_{l+1,m-1}\right)/2k \]

\[= U_{yl+1,m} + \frac{k^2}{6}U_{yyy} + O(k^2h_l) \quad (6.3.7)\]

\[\bar{U}_{yl-1,m} = \left(U_{l-1,m+1} - U_{l-1,m-1}\right)/2k \]

\[= U_{yl-1,m} + \frac{k^2}{6}U_{yyy} + O(-k^2h_l) \quad (6.3.8)\]

\[\bar{U}_{yy,l,m} = \left(U_{l,m+1} - 2U_{l,m} + U_{l,m-1}\right)/k^2 \]

\[= U_{yy,l,m} + \frac{k^2}{12}U_{yyyy} + O(k^4) \quad (6.3.9)\]

\[\bar{U}_{yyyy,l+1,m} = \left(U_{l+1,m+1} - 2U_{l+1,m} + U_{l+1,m-1}\right)/k^2\]
With the help of the approximations (6.3.3)-(6.3.8), from (6.2.11d)-(6.2.11f), we obtain

\[
\tilde{F}_{1,m} = f(x_i, y_m, U_{l+1,m}, \bar{U}_{xli+1,m}, \bar{U}_{yl+1,m}) = f(x_i, y_m, U_{l+1,m}, \bar{U}_{xli+1,m}, \bar{U}_{yl+1,m})
\]

\[
= f(x_i, y_m, \bar{U}_{l+1,m}, U_{xli+1,m}, \frac{h_t^2}{6} U_{xxxl+1,m} + O(h_t^3), U_{yl+1,m} + h_t^2 U_{yyy} + O(k^3))
\]

\[
= \tilde{F}_{l+1,m} - \frac{h_t^2}{6} S_{3l} \bar{U}_{30} \alpha_{l,m} + O(k^2 + k^2 h_t + h_t^3) \quad (6.3.12a)
\]

\[
\tilde{F}_{l+1,m} = f(x_i+1, y_m, U_{l+1,m}, \bar{U}_{xli+1,m}, \bar{U}_{yl+1,m}) = f(x_i+1, y_m, U_{l+1,m}, \bar{U}_{xli+1,m}, \bar{U}_{yl+1,m})
\]

\[
= f(x_i+1, y_m, \bar{U}_{l+1,m}, U_{xli+1,m}, -\frac{h_t^2}{6} (1 + \sigma_l) h_t^2 U_{xxx} + O(h_t^3),
\]

\[
U_{yl+1,m} + \frac{k^2}{12} U_{yyy} + O(k^2 h_t))
\]

\[
= \tilde{F}_{l+1,m} - (1 + \sigma_l) \frac{h_t^2}{6} U_{30} \alpha_{l+1,m} + O(k^2 + k^2 h_t + h_t^3)
\]

\[
= \tilde{F}_{l+1,m} - \frac{h_t^2}{6} S_{4l} \bar{U}_{30} \alpha_{l+1,m} + O(k^2 + k^2 h_t + h_t^3) \quad (6.3.12b)
\]

\[
\tilde{F}_{l-1,m} = f(x_{i-1}, y_m, U_{l-1,m}, \bar{U}_{xli-1,m}, \bar{U}_{yl-1,m}) = f(x_{i-1}, y_m, U_{l-1,m}, \bar{U}_{xli-1,m}, \bar{U}_{yl-1,m})
\]

\[
= f(x_{i-1}, y_m, U_{l-1,m}, \bar{U}_{xli-1,m} - \frac{1 + \sigma_l}{6} h_t^2 U_{xxx} + O(h_t^3),
\]

\[
U_{yl-1,m} + \frac{k^2}{12} U_{yyy} + O(-k^2 h_t))
\]

\[
= \tilde{F}_{l-1,m} - (1 + \sigma_l) \frac{h_t^2}{6} U_{30} \alpha_{l-1,m} + O(k^2 - k^2 h_t + h_t^3)
\]

\[
= \tilde{F}_{l-1,m} - (1 + \sigma_l) \frac{h_t^2}{6} U_{30} \alpha_{l-1,m} + O(k^2 - k^2 h_t + h_t^3) \quad (6.3.12c)
\]

Let,

\[
\tilde{M}_{l\pm1,m} = \bar{U}_{xli\pm1,m} = -\bar{U}_{yyli\pm1,m} + \tilde{F}_{l\pm1,m}
\]

\[
\tilde{M}_{l,m} = \bar{U}_{xlm} = -\bar{U}_{yylm} + \tilde{F}_{l,m}
\]

Now, \( U_{xx} = -U_{yy} + F \Rightarrow U_{xxx} = -U_{xy} + F_x \)

Using the cubic spline relations (6.2.10) we obtain

\[
\bar{U}_{xli+1,m} = \frac{1}{\sigma_l h_t} [U_{m} + \sigma_l h_t U_x + \sigma_l^2 h_t^2 U_{xx} + \sigma_l^3 h_t^3 U_{xxx} + \cdots - U_{l,m}]
\]

\[
+ \frac{\sigma_l h_t}{6} [-\bar{U}_{yyli,m} + \tilde{F}_{l,m} + 2(-\bar{U}_{yyli\pm1,m} + \tilde{F}_{l\pm1,m})]
\]

\[
= U_{xl,m} + \frac{\sigma_l h_t}{2} U_{xx} + \frac{\sigma_l^2 h_t^2}{6} U_{30} + O(h_t^3) + \frac{\sigma_l h_t}{6} [-(-U_{yyli,m} + \frac{k^2}{12} U_{04} + O(k^4))]
\]

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Similarly,

\[ \vec{U}_{x+1,m} = U_{x+1,m} - h_l U_{xx} + \frac{h_l^2}{2} U_{30} + O(k^2 + k^2 h_l + h_l^3) \]

Now, \( \vec{U}_{x+1,m} - U_{x+1,m} = U_{x+1,m} + \sigma_l h_l U_{xx} + \frac{\sigma_l h_l^2}{2} U_{30} + O(k^2 - k^2 h_l + h_l^3) \)

\[ \Rightarrow \vec{U}_{x+1,m} = U_{x+1,m} + O(k^2 - k^2 h_l + h_l^3) \] (6.3.13a)

Note that, we require \( O(k^2, k^2 h_l, h_l^3) \) approximation for \( U_{y+1,m} \).

Let us consider,

\[ U_{x,m} = \vec{U}_{x+1,m} + a h_l [F_{l+1,m} - F_{l-1,m}] + b h_l [\vec{U}_{y+1,m} - \vec{U}_{y-1,m}] \] (6.3.14)

where ‘a’ and ‘b’ are the free parameters to be determined.

With the help of the approximations (6.3.10)-(6.3.11), (6.3.12b)-(6.3.12c) and (6.3.3), from (6.3.14), we get

\[ U_{x,m} = U_{x,m} + \frac{\sigma_l h_l^2}{6} U_{xxx,x,m} + a h_l [F_{l+1,m} - F_{l-1,m}] + b h_l [U_{yy+1,m} - U_{yy-1,m}] \]

\[ = U_{x,m} + \frac{\sigma_l h_l^2}{6} U_{xxx,x,m} + \sigma_l U_{xx,x,m} + b(\sigma_l + 1) h_l^2 U_{xyy,x,m} \]

\[ = U_{x,m} + \frac{\sigma_l h_l^2}{6} U_{30} + a h_l^2 (\sigma_l + 1) (U_{xxx,x,m} + U_{xyy,x,m}) + b h_l^2 (\sigma_l + 1) U_{xyy,x,m} \]

\[ = U_{x,m} + \left[ \frac{\sigma_l}{6} + a(1 + \sigma_l) h_l^2 U_{30} + [a(1 + \sigma_l) + b(1 + \sigma_l)] h_l^2 U_{12} \right] \]

\[ + O(k^2 + k^2 h_l + h_l^3) \] (6.3.15)

Note that, we require \( O(k^2 + k^2 h_l + h_l^3) \) -approximation for \( U_{x,m} \).
Hence, the coefficient of $h_t^2$ must be zero in (6.3.15)

$$\Rightarrow \frac{a_4}{6} + a(1 + \sigma_t) = 0$$

and $$(a + b)(1 + \sigma_t) = 0$$

It can be evaluated that $a = -\frac{\sigma_t}{6(1+\sigma_t)}$ and $b = \frac{\sigma_t}{6(1+\sigma_t)}$.

Following above, equation (6.3.14) may now be written as

$$\bar{U}_{x,l,m} = U_{x,l,m} - \frac{\sigma_t}{6(1+\sigma_t)}h_t\left[\bar{F}_{l+1,m} - \bar{F}_{l-1,m}\right] + \frac{\sigma_t}{6(1+\sigma_t)}h_t\left[U_{y,y+l+1,m} - U_{y,y+l-1,m}\right]$$

Alternately, we may write the preceding equation in the form:

$$\bar{U}_{x,l,m} = U_{x,l,m} - \frac{\sigma_t h_t}{6(1+\sigma_t)}\left[M_{l+1,m} - M_{l-1,m}\right]$$

$$= U_{x,l,m} + O(k^2 + k^2 h_t + h_t^3) \quad (6.3.16)$$

Now, using approximations (6.3.6)-(6.3.8), (6.3.13a), (6.3.13b) and (6.3.16), from (6.2.11g)-(6.2.11i), we get

$$\bar{F}_{l+1,m} = f(x_{l+1}, y_m, U_{l,m}, \bar{U}_{x,l+1,m}, \bar{U}_{y,l+1,m})$$

$$= f(x_{l+1}, y_m, U_{l,m}, U_{x,l+1,m} + O(k^2 + k^2 h_t + h_t^3), U_{y,l+1,m} + O(k^2 + k^2 h_t))$$

$$= F_{l+1,m} + O(k^2 + k^2 h_t + h_t^3) \quad (6.3.17a)$$

$$\bar{F}_{l-1,m} = f(x_{l-1}, y_m, U_{l,m}, \bar{U}_{x,l-1,m}, \bar{U}_{y,l-1,m})$$

$$= f(x_{l-1}, y_m, U_{l,m}, U_{x,l-1,m} + O(k^2 + k^2 h_t + h_t^3), U_{y,l-1,m} + O(k^2 - k^2 h_t))$$

$$= F_{l-1,m} + O(k^2 + k^2 h_t + h_t^3) \quad (6.3.17b)$$

$$\bar{F}_{l,m} = f(x_l, y_m, U_{l,m}, \bar{U}_{x,l,m}, \bar{U}_{y,l,m})$$

$$= f(x_l, y_m, U_{l,m}, U_{x,l,m}, U_{y,l,m}) + O(k^2 + k^2 h_t + h_t^3)$$

$$= F_{l,m} + O(k^2 + k^2 h_t + h_t^3) \quad (6.3.17c)$$

Finally, using the approximations (6.3.17a)-(6.3.17c), from (6.2.12) and (6.3.14), we may obtain the local truncation error $\bar{F}_{l,m} = O(k^2 h_t^2 + k^2 h_t^3 + h_t^5)$.

Note that, the Dirichlet boundary conditions are given by (6.1.2). Incorporating the boundary conditions, we can write the cubic spline method (6.2.12) in a tri-block diagonal matrix form. If the differential equation (6.1.1) is linear, we can solve the linear system using Gauss-Seidel (tri-diagonal solver) method; in the nonlinear case, we can use Newton-Raphson iterative method to solve the nonlinear system (see [41, 50, 89, 99]).
6.4 Line Iterative Analysis

In this section, we discuss the convergence and performance of some line iterative methods with the geometric mesh cubic spline numerical scheme for the two dimensional linear elliptic partial differential equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = D(x) \frac{\partial u}{\partial x} + g(x, y), \quad 0 < x, y < 1 \quad (6.4.1)
\]

subject to appropriate Dirichlet boundary conditions prescribed. Assume that the functions \(D(x)\) and \(g(x, y)\) belong to \(C^2(R)\), that is, they have two derivatives that are continuous in \(R\).

We note that, for the nine diagonal matrix that arises due to discretization of two dimensional elliptic equation the line iterative method is equivalent to block iterative method. On applying formula (6.2.12) to the elliptic equation (6.4.1) and neglecting the local truncation error, we obtain the following difference scheme

\[
u_{l+1,m} - (1 + \sigma_l)u_{l,m} + \sigma_l u_{l-1,m} \\
+ \frac{h_i^2}{12} \left[ P_l u_{y, yl+1,m} + Q_l u_{y, yl,m} + R_l u_{y, yl-1,m} \right] \\
= \frac{h_i^2}{12} \left[ P_l D_{l+1} u_{x, xl+1,m} + Q_l D_{l} u_{x, xl,m} + R_l D_{l-1} u_{x, xl-1,m} \right] \\
+ \frac{h_i^2}{12} \left[ P_l g_{l+1,m} + Q_l g_{l,m} + R_l g_{l-1,m} \right], \quad l = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, M \quad (6.4.2)
\]

Note that, the scheme (6.4.2) is of \(O(k^2 + k^2 h_i + h_i^3)\). However, an important observation is that the scheme fails to realize at \(l = 1\), when the functions \(D(x)\) and/ or \(g(x, y)\) involve singular terms like \(\frac{1}{x}, \frac{1}{x^2}, \frac{1}{xy^3}\), etc. Consider for example, when \(D(x) = \frac{1}{x}\)

then \(D_{l-1} = \frac{1}{x_{l-1}}\) which becomes infinite at \(l = 1\) as \(x_0 = 0\). We overcome this difficulty by refining our scheme in such a way that the solution retains its order and accuracy everywhere in the solution region.

We consider the following approximations:

\[
D_{l+1} = D_{l+1} + \sigma_l h_i D_{l+1} + \frac{\sigma_l^2 h_i^2}{2} D_{20} + O(h_i^3) \equiv D_l^{**} + O(h_i^3) \quad (6.4.3a)
\]

\[
D_{l-1} = D_{l-1} - h_i D_{l-1} + \frac{h_i^2}{2} D_{20} - O(h_i^3) \equiv D_l^{**} + O(h_i^3) \quad (6.4.3b)
\]

\[
g_{l+1,m} = g_{l+1,m} + \sigma_l h_i g_{l+1,m} + \frac{\sigma_l^2 h_i^2}{2} g_{20} + O(h_i^3) \equiv g_{l,m} + O(h_i^3) \quad (6.4.3c)
\]

\[
g_{l-1,m} = g_{l-1,m} - h_i g_{l+1,m} + \frac{h_i^2}{2} g_{20} - O(h_i^3) \equiv g_{l,m} - O(h_i^3) \quad (6.4.3d)
\]

where \(g_{l,m} = g_{00} = g(x_l, y_m)\) etc

and \(D_l^{**} = D_{l+1} + \sigma_l h_i D_{l+1} + \frac{\sigma_l^2 h_i^2}{2} D_{20} \quad, \quad D_l = D_{l+1} - h_i D_{l+1} + \frac{h_i^2}{2} D_{20} \quad, \quad \ldots\) etc.
Now, substituting the approximations (6.4.3) in the difference scheme (6.4.2) and merging the higher order terms in local truncation error, we obtain the modified scheme.

\[ u_{l+1,m} - (1 + \sigma_l)u_{l,m} + \sigma_l u_{l-1,m} \]
\[ + \frac{h_l^2}{12} \left[ P_l \bar{u}_{y,yl+1,m} + Q_l \bar{u}_{y,yl,m} + R_l \bar{u}_{y,yl-1,m} \right] \]
\[ = \frac{h_l^2}{12} \left[ P_l D_l^i \bar{u}_{x,l+1,m} + Q_l D_l \bar{u}_{x,l,m} + R_l D_l^i \bar{u}_{x,l-1,m} \right] \]
\[ + \frac{h_l^2}{12} \left[ P_l g_{l,m}^* + Q_l g_{l,m} + R_l g_{l,m}^* \right], \quad l = 1,2,\ldots, N, m = 1,2,\ldots, M \] \hspace{1cm} (6.4.4)

Note that, the modified cubic spline scheme (6.4.4) is of \( O(k^2 + k^2 h_l + h_l^3) \) accurate and independent of the terms \( \frac{1}{x_{l+1}} \), hence applicable to both singular and non-singular elliptic partial differential equations of form (6.4.1).

The nine point modified cubic spline scheme (6.4.4) applied to (6.4.1) may then be written as

\[ \alpha_0 u_{l,m} + \alpha_1 u_{l+1,m} + \alpha_2 u_{l-1,m} + \alpha_3 u_{l,m-1} + \alpha_4 u_{l+1,m-1} + \alpha_5 u_{l-1,m-1} \]
\[ + \alpha_6 u_{l,m+1} + \alpha_7 u_{l+1,m+1} + \alpha_8 u_{l-1,m+1} = f_{l,m} \] \hspace{1cm} (6.4.5)

where we have

\[ \alpha_0 = -k^2 (1 + \sigma_l) + \frac{k^2 h_l}{12 \sigma_l} \left[ P_l D_l^i + (1 - \sigma_l) Q_l D_{00} - \sigma_l R_l (D_l^{**} - \frac{h_l^3}{6} D_{30}) \right] \]
\[ + \frac{k^2 h_l^2}{72} \left[ (1 - \sigma_l) P_l D_{00} D_l^i + P_l (1 + \sigma_l) \left( D_{00}^2 + \sigma_l^2 h_l^2 D_{10}^2 + 2 D_{00} (D_l^i - D_{00}) \right) \right] \]
\[ - \frac{(1 - \sigma_l)}{\sigma_l} R_l D_{00} D_l^{**} - Q_l D_{00} (D_l^i + D_l^{**}) - \frac{2(1 + \sigma_l)}{\sigma_l} R_l \left( D_{00}^2 + h_l^2 D_{10}^2 + 2 D_{00} (D_l^{**} - D_{00}) \right) \]
\[ + \frac{k^2 h_l^2 \sigma_l^2}{72} P_l D_{30} + \frac{h_l^3}{36} \left[ -P_l (D_{00} + \sigma_l h_l D_{10}) + R_l (D_{00} - h_l D_{10}) \right] - \frac{h_l^2}{6} Q_l ; \]

\[ \alpha_1 = k^2 - \frac{k^2 h_l}{12 \sigma_l} \left[ P_l D_l^i + \frac{2 \sigma_l}{(1 + \sigma_l)} Q_l D_{00} - \frac{h_l^3}{6} P_l + \frac{h_l^3}{36} \sigma_l [2 P_l (D_{00} + \sigma_l h_l D_{10}) + Q_l D_{00}] \right] \]
\[ + \frac{k^2 h_l^2}{72(1 + \sigma_l)} \left[ -P_l D_{00} D_l^{**} - P_l (1 + 2 \sigma_l) \left( D_{00}^2 + \sigma_l h_l^2 D_{10}^2 + 2 D_{00} (D_l^{**} - D_{00}) \right) \right] \]
\[ + \frac{2}{\sigma_l} R_l \left( D_{00}^2 + h_l^2 D_{10}^2 + 2 D_{00} (D_l^{**} - D_{00}) \right) + \frac{R_l D_{00} D_l^{**}}{\sigma_l} + \frac{Q_l D_{00}}{(1 + \sigma_l)} (D_l^i (1 + 2 \sigma_l) + D_l^{**}) \]
\[ - \frac{k^2 h_l^4 \sigma_l^2}{72} P_l D_{30} ; \]

\[ \alpha_2 = k^2 \sigma_l - \frac{h_l^3}{6} R_l + \frac{h_l^3}{36} \left[ 2 R_l (D_{00} - h_l D_{10}) - \frac{\sigma_l}{(1 + \sigma_l)} Q_l D_{00} \right] \]
\[ + \frac{k^2 h_l}{12} \left[ R_l \left( D_l^{**} - \frac{h_l^3}{6} D_{30} \right) + \frac{\sigma_l}{(1 + \sigma_l)} Q_l D_{00} \right] \]
\[ + \frac{k^2 h_l^2}{72(1 + \sigma_l)} \left[ -R_l (D_{00} D_l^{**} - \sigma_l^2 P_l (D_{00}^2 + \sigma_l h_l^2 D_{10}^2 + 2 D_{00} (D_l^{**} - D_{00}) \right) \]
\[ + 2 R_l (2 + \sigma_l) \left( D_{00}^2 + h_l^2 D_{10}^2 + 2 D_{00} (D_l^{**} - D_{00}) \right) + \frac{\sigma_l}{(1 + \sigma_l)} (D_l^i \sigma_l + D_l^{**} (2 + \sigma_l)) ; \]
The linear system \( (6.4.5) \) can then be solved by Block Gauss-Seidel iterative method (using tri-diagonal solver) whose iteration matrix \( G \) is given by

\[
G = (D + L)^{-1} \quad (6.4.6)
\]

where \( D = [\alpha_2, \alpha_0, \alpha_1] \), \( L = [\alpha_5, \alpha_3, \alpha_4] \) and \(-U = [\alpha_8, \alpha_6, \alpha_7]\) are tri-diagonal matrices.

Consider the 2 spatial dimensions Poisson’s equation in cylindrical polar coordinates in \( r-z \) plane

\[
\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \cosh \left( 2 \cosh \frac{1}{r} \sinh r \right) , \quad 0 < r,z < 1 \quad (6.4.7)
\]

Replacing variables \((x,y)\) by \((r,z)\) and setting \( D_{00} = -1/\tau_1, D_{10} = 1/\tau_1^2, D_{20} = -2/\tau_1^3 \) in \((6.4.4)\), we get \( O(k^2 + k^2 h_l + h_l^3) \) scheme for the solution of elliptic equation \((6.4.7)\).

Now we consider the steady state two-dimensional convection-diffusion equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \beta \frac{\partial u}{\partial x} , \quad 0 < x,y < 1 \quad (6.4.8)
\]

subject to the boundary condition \( u(x,y) = \psi(x,y) \), \((x,y) \in [0,1] \times [0,1]\).

Here \( \beta (=1/\epsilon , \epsilon \) being the perturbation factor) is a positive constant whose magnitude determines the ratio of convection to diffusion. This equation often appears in the description of transport phenomena. Numerical simulation of equation \((6.4.8)\) becomes increasingly difficult as the ratio of the convection to diffusion increases. Substituting \( D(x) = \beta \) and \( g(x,y) = 0 \) in the difference scheme \((6.4.4)\) and simplifying, we obtain a difference scheme of \( O(k^2 + k^2 h_l + h_l^3) \) accuracy for the solution of the above convection-diffusion equation \((6.4.8)\). Considering the special case of constant mesh by taking \( \sigma_l = 1 \) for \( l = 1(1)N \) and letting \( p = \frac{k}{h} \) we obtain a nine point cubic spline difference scheme as

\[
\gamma_0 u_{t,m} + \gamma_1 u_{t+1,m} + \gamma_2 u_{t-1,m} + \gamma_3 u_{t,m+1} + \gamma_4 u_{t,m-1} + \gamma_5 u_{t+1,m+1}
\]
where the coefficients \( \gamma_w, w = 0, 1, 2, \ldots, 8 \) are defined by

\[
\begin{align*}
\gamma_0 &= 24p^2 + 20 + 8p^2R^2, \\
\gamma_1 &= -[12p^2 - 2 - 12p^2R + 4p^2R^2 + 2R], \\
\gamma_2 &= -[12p^2 - 2 + 12p^2R + 4p^2R^2 - 2R], \\
\gamma_3 &= \gamma_4 = -10, \\
\gamma_5 &= \gamma_6 = -(1 - R), \\
\gamma_7 &= \gamma_8 = -(1 + R)
\end{align*}
\]

where \( R = \beta h / 2 \) is called the Cell Reynolds number.

The discretizations result in \( NM \) number of linear equations in \( NM \) unknowns. Incorporating the boundary conditions, the above system may be expressed in the matrix form as

\[
A u = b
\]

where \( A \) is a tri-block diagonal matrix of order \( (NM \times NM) \), \( u \) is the solution vector and \( b \) is the right hand side column vector arising from the boundary values of problem (6.4.8).

The coefficient matrix \( A \) has a block tri-diagonal structure,

\[
A = \text{tri}[L \ D \ -U]_{NM \times NM}
\]

with the submatrices \(-L, D\) and \(-U\) each of order \((N \times M)\) given by

\[
\begin{align*}
-L &= \text{tri}[\gamma_8 \ \gamma_4 \ \gamma_6] = -U, \\
D &= \text{tri}[\gamma_2 \ \gamma_0 \ \gamma_1]
\end{align*}
\]

We focus on line stationary iterative methods for solving the linear system of equations (6.4.10). The coefficient matrix \( A \) can be split as \( A = D - L - U \), where \( D \) is the block diagonal matrix of \( A \), \(-L\) is strictly block lower triangular part and \(-U\) strictly block upper triangular part of matrix \( A \). The iteration matrices of the block Jacobi and block Gauss-Seidel methods are described by

\[
G_j = D^{-1}(L + U) \quad \text{and} \quad G_{GS} = (D - L)^{-1}U
\]

The matrix \( A \) has block tri-diagonal form and hence is block consistently ordered [99].

It can be verified that \( \gamma_0 > 0 \) and \( \gamma_w < 0 \) for \( w = 1, 2, \ldots, 8 \) assuming the diffusion dominated case i.e. \( R \leq 1 \) and taking \( p \geq 1 / \sqrt{6} \). One can also easily verify that

\[
\gamma_0 = \sum_{w=1}^{8} |\gamma_w|
\]
which implies that the coefficient matrix $A$ generated from (6.4.9) is weakly diagonally dominant. Since $A$ is irreducible (as its directed graph is strongly connected), we conclude that it is an $M$-matrix and hence monotone [99].

Now, applying the Jacobi / Gauss-Seidel iteration method to the system of equations (6.4.10), we get the iterative scheme for $s = 0, 1, 2, ...$

\[
\begin{align*}
24p^2 \left(1 + \frac{R^2}{3}\right) + 20 \left[ (1-R)u_{l+1,m}^{(s+1)} + \left[ 12p^2 \left(1 - R + \frac{R^2}{3}\right) - 2(1-R) \right] u_{l+1,m}^{(s+1)} + (1-R)u_{l+1,m-1}^{(s+1)} + (1+R)u_{l-1,m+1}^{(s+1)} \right] \\
+ \left[ 12p^2 \left(1 + R + \frac{R^2}{3}\right) + 2(1+R) \right] u_{l-1,m}^{(s+1)} + (1+R)u_{l-1,m-1}^{(s+1)} + 10u_{l,m-1}^{(s+1)}
\end{align*}
\] (6.4.14)

For $\omega = 0$, (6.4.14) is called Jacobi iteration method and for $\omega = 1$, (6.4.14) is called Gauss-Seidel iteration method where $u_{l,m}^{(s+1)}$ and $u_{l,m}^{(s)}$ are the successive approximations for $u_{l,m}$ at $(s + 1)$th and $s$th iterations, respectively.

We examine the stability of the Jacobi iteration method (consider $\omega = 0$ in equation (6.4.14)) by studying the behaviour of the error equation. Let us assume that an error $\varepsilon_{l,m}^{(s)}$ exists at each mesh point $(x_l, y_m)$ at the $s$th iteration and is of the form

\[
\varepsilon_{l,m}^{(s)} = \xi^s A^l B^m \sin \left( \frac{\pi a l}{N+1} \right) \sin \left( \frac{\pi b m}{M+1} \right), \quad 1 \leq a \leq N, 1 \leq b \leq M
\] (6.4.15)

where $A$ and $B$ are arbitrary constants and $\xi$ is the propagating factor which determines the rate of growth or decay of the errors. The necessary and sufficient condition for the iterative method to be stable is

\[
|\xi| < 1, \quad 1 \leq a \leq N, 1 \leq b \leq M
\] (6.4.16)

We have the error equation:

\[
\begin{align*}
24p^2 \left(1 + \frac{R^2}{3}\right) + 20 \left[ \varepsilon_{l,m}^{(s+1)} = \left(1-R\right)\varepsilon_{l+1,m}^{(s)} + \varepsilon_{l+1,m-1}^{(s+1)} + \left[ 12p^2 \left(1 - R + \frac{R^2}{3}\right) - 2(1-R) \right] \varepsilon_{l+1,m}^{(s+1)} + (1-R)\varepsilon_{l+1,m-1}^{(s+1)} + (1+R)\varepsilon_{l-1,m+1}^{(s+1)} \right] \\
+ \left[ 12p^2 \left(1 + R + \frac{R^2}{3}\right) + 2(1+R) \right] \varepsilon_{l-1,m}^{(s+1)} + (1+R)\varepsilon_{l-1,m-1}^{(s+1)} + 10\varepsilon_{l,m-1}^{(s+1)}
\end{align*}
\] (6.4.17)

Substituting (6.4.15) in error equation (6.4.17), we obtain the characteristic equation for (6.4.14):

\[
\left(24p^2 + 20 + 8p^2 R^2\right)\xi \sin \left( \frac{\pi a l}{N+1} \right) \sin \left( \frac{\pi b m}{M+1} \right) = \left(1 - R\right)AB \sin \left( \frac{\pi a (l+1)}{N+1} \right) \sin \left( \frac{\pi b (m+1)}{M+1} \right)
\]

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We choose $A$ and $B$ such that the coefficients of $\cos\left(\frac{\pi a}{N+1}\right)$ and $\cos\left(\frac{\pi b m}{M+1}\right)$ in (6.4.18) are zero.

Thus we set,
\begin{align*}
10(B - B^{-1}) + (B - B^{-1}) \cos\left(\frac{\pi a}{N+1}\right) [A(1 - R) + A^{-1}(1 + R)] &= 0 \quad (6.4.19a) \\
(B + B^{-1}) \cos\left(\frac{\pi b m}{M+1}\right) [A(1 - R) - A^{-1}(1 + R)] + (12p^2 - 2 + 4p^2 R^2)(A - A^{-1}) - (12p^2 R - 2R)(A + A^{-1}) &= 0 \quad (6.4.19b) \\
and \quad (B - B^{-1})[A(1 - R) - A^{-1}(1 + R)] &= 0 \quad (6.4.19c)
\end{align*}

These are three equations in two unknowns $A, B$. 

\[ \Rightarrow (24p^2 + 20 + 8p^2 R^2)\xi \sin\left(\frac{\pi a}{N+1}\right) \sin\left(\frac{\pi b m}{M+1}\right) \]

\[ = \sin\left(\frac{\pi a}{N+1}\right) \sin\left(\frac{\pi b m}{M+1}\right) [10(B + B^{-1}) \cos\left(\frac{\pi b m}{M+1}\right) +
\]

\[ (B + B^{-1}) \cos\left(\frac{\pi a}{N+1}\right) \cos\left(\frac{\pi b m}{M+1}\right) [A(1 - R) + A^{-1}(1 + R)]
\]

\[ + \cos\left(\frac{\pi a}{N+1}\right) [(12p^2 - 2 + 4p^2 R^2)(A + A^{-1}) + (12p^2 R - 2R)(A^{-1} - A)]
\]

\[ + \sin\left(\frac{\pi a}{N+1}\right) \cos\left(\frac{\pi b m}{M+1}\right) [10(B - B^{-1}) \sin\left(\frac{\pi b m}{M+1}\right) +
\]

\[ (B - B^{-1}) \cos\left(\frac{\pi a}{N+1}\right) \sin\left(\frac{\pi b m}{M+1}\right) [A(1 - R) - A^{-1}(1 + R)]
\]

\[ + \cos\left(\frac{\pi a}{N+1}\right) \sin\left(\frac{\pi b m}{M+1}\right) [(B + B^{-1}) \cos\left(\frac{\pi a}{N+1}\right) \sin\left(\frac{\pi b m}{M+1}\right) [A(1 - R) - A^{-1}(1 + R)]
\]

\[ + \sin\left(\frac{\pi a}{N+1}\right) [(12p^2 - 2 + 4p^2 R^2)(A - A^{-1}) - (12p^2 R - 2R)(A + A^{-1})]
\]

\[ + \cos\left(\frac{\pi a}{N+1}\right) \cos\left(\frac{\pi b m}{M+1}\right) [(B - B^{-1}) \sin\left(\frac{\pi a}{N+1}\right) \sin\left(\frac{\pi b m}{M+1}\right) [A(1 - R) - A^{-1}(1 + R)]
\]

(6.4.18)
From equation (6.4.19c) we have either \((B - B^{-1}) = 0\) or \(A(1 - R) - A^{-1}(1 + R) = 0\). This implies either \(B = 1\) or \(A = \frac{1+R}{1-R}\).

Equations (6.4.19a) and (6.4.19c) are surely satisfied by \(B = 1\) so letting \(B = 1\) in equation (6.4.19b) we obtain

\[
2 \cos\left(\frac{\pi b}{M+1}\right) [A(1 - R) - A^{-1}(1 + R)] + (12p^2 - 2 + 4p^2 R^2 - 12p^2 R + 2R)A
- (12p^2 - 2 + 4p^2 R^2 + 12p^2 R - 2R)A^{-1} = 0
\]

which gives

\[
A = \left\{ \frac{(1 + R) \cos\left(\frac{\pi b}{M+1}\right) + 6p^2 - 1 + 2p^2 R^2 + 6p^2 R - R}{(1 - R) \cos\left(\frac{\pi b}{M+1}\right) + 6p^2 - 1 + 2p^2 R^2 - 6p^2 R + R} \right\}^{1/2}
\]

The propagating factor becomes upon substituting \(B = 1\), comparing left hand side and right hand side of (6.4.18)

\[
(24p^2 + 20 + 8p^2 R^2) \xi = 2 \cos\left(\frac{\pi b}{M+1}\right) [10 + \cos\left(\frac{\pi a}{N+1}\right) (A(1 - R) + A^{-1}(1 + R))] \\
+ \cos\left(\frac{\pi a}{N+1}\right) [(12p^2 - 2 + 4p^2 R^2 - 12p^2 R + 2R)A + \\
(12p^2 - 2 + 4p^2 R^2 + 12p^2 R - 2R)A^{-1}]
\]

Now the largest value of \(\cos\left(\frac{\pi a}{N+1}\right)\) and \(\cos\left(\frac{\pi b}{M+1}\right)\) occur when \(a = b = 1\).

\[
\Rightarrow (24p^2 + 20 + 8p^2 R^2) \xi = 2 \cos\left(\frac{\pi}{M+1}\right) [10 + \cos\left(\frac{\pi}{N+1}\right) (A(1 - R) + A^{-1}(1 + R))] \\
+ \cos\left(\frac{\pi}{N+1}\right) [(12p^2 - 2 + 4p^2 R^2 - 12p^2 R + 2R)A + \\
(12p^2 - 2 + 4p^2 R^2 + 12p^2 R - 2R)A^{-1}]
\]

\[
= 20 \cos\left(\frac{\pi}{M+1}\right) + 2 \cos\left(\frac{\pi}{N+1}\right) [(1 - R) \cos\left(\frac{\pi}{M+1}\right) + 6p^2 - 1 + 2p^2 R^2 - 6p^2 R + R]A \\
+ 2 \cos\left(\frac{\pi}{N+1}\right) A^{-1} [(1 + R) \cos\left(\frac{\pi}{M+1}\right) + 6p^2 - 1 + 2p^2 R^2 + 6p^2 R - R]
\]

Substituting the value of \(A\),

\[
\Rightarrow \xi = \frac{1}{(24p^2 + 20 + 8p^2 R^2)} [20 \cos\left(\frac{\pi}{M+1}\right) + \\
4 \cos\left(\frac{\pi}{N+1}\right) \sqrt{(1 - R) \cos\left(\frac{\pi}{M+1}\right) + 6p^2 - 1 + 2p^2 R^2 - 6p^2 R + R \times \\
\sqrt{(1 + R) \cos\left(\frac{\pi}{M+1}\right) + 6p^2 - 1 + 2p^2 R^2 + 6p^2 R - R}}}
\]

Simplifying above, the propagating factor \(\xi_j\) for the Jacobi iteration method is obtained as

\[
\xi_j = \frac{1}{5+6p^2\left(\frac{R^2}{1+R^2}\right)} [5 \cos\left(\frac{\pi}{M+1}\right) +
\]

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Thus, the Jacobi Iteration method is stable for those values of $R$ such that $|\xi_j| < 1$ and the rate of convergence of the Jacobi iteration method is given by

$$\nu_j = -\log |\xi_j|$$

Similarly, applying the Gauss-Siedel iteration method (taking $\omega = 1$ in (6.4.14)) to the system of equations (6.4.10) and assuming the error at each grid point $(x_i, y_m)$ at the $s^{th}$ iteration to be of the form (6.4.15), we obtain upon simplification of the characteristic equation

$$(24p^2 + 20 + 8p^2 R^2) \sin \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b m}{M+1} \right) \xi = \sin \left( \frac{\pi a l}{N+1} \right) \sin \left( \frac{\pi b m}{M+1} \right) \left[(1 - R)AB \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + (12p^2 - 2 - 12p^2 R + 2R + 4p^2 R^2)A \cos \left( \frac{\pi a}{N+1} \right) + (1 - R)\xi A^{-1} \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + (1 + R)A^{-1}B \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + \sin \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b m}{M+1} \right) \left[(1 - R)AB - (1 - R)\xi A^{-1} - (1 + R)\xi A^{-1}B^{-1} + (1 + R)A^{-1}B \cos \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b}{M+1} \right) + 10(B + \xi B^{-1}) \cos \left( \frac{\pi b}{M+1} \right) + (1 + R)\xi A^{-1}B \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + \cos \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b m}{M+1} \right) \left[(1 - R)AB + (1 - R)\xi A^{-1} - (1 + R)\xi A^{-1}B^{-1} - (1 + R)A^{-1}B \sin \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b}{M+1} \right) + \sin \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b m}{M+1} \right) \left[(1 - R)AB - (1 - R)\xi A^{-1} - (1 + R)\xi A^{-1}B^{-1} + (1 + R)A^{-1}B \sin \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b}{M+1} \right) + \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b m}{M+1} \right) \right] \right] \right]$$

Again choosing $A$ and $B$ such that the coefficients of $\cos \left( \frac{\pi a}{N+1} \right)$ and $\cos \left( \frac{\pi b m}{M+1} \right)$ are zero. Thus we must have

$$(24p^2 + 20 + 8p^2 R^2)\xi = [(1 - R)AB + (1 - R)\xi A^{-1} - (1 + R)\xi A^{-1}B^{-1} + (1 + R)A^{-1}B \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + (12p^2 - 2 - 12p^2 R + 2R + 4p^2 R^2)A \cos \left( \frac{\pi a}{N+1} \right) + (1 + R)A^{-1}B \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + 10(B + \xi B^{-1}) \cos \left( \frac{\pi b}{M+1} \right) + (1 + R)\xi A^{-1}B \cos \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b}{M+1} \right) + \cos \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b m}{M+1} \right) \left[(1 - R)AB - (1 - R)\xi A^{-1} - (1 + R)\xi A^{-1}B^{-1} - (1 + R)A^{-1}B \sin \left( \frac{\pi a}{N+1} \right) \sin \left( \frac{\pi b}{M+1} \right) + \sin \left( \frac{\pi a}{N+1} \right) \cos \left( \frac{\pi b m}{M+1} \right) \right] \right]$$

where,
\[(1-R)AB - (1-R)\xi AB^{-1} - (1+R)\xi A^{-1}B^{-1} + (1+R)A^{-1}B\cos\left(\frac{\pi a}{N+1}\right)\sin\left(\frac{nb}{M+1}\right)\]

\[+10[B - \xi B^{-1}]\sin\left(\frac{nb}{M+1}\right) = 0\]

\[\Rightarrow (B - \xi B^{-1})\left[10 + (1-R)A\cos\left(\frac{\pi a}{N+1}\right) + (1+R)A^{-1}\cos\left(\frac{\pi a}{N+1}\right)\right] = 0 \quad (6.4.23a)\]

\[\sin\left(\frac{\pi a}{N+1}\right)\left[10(12p^2 - 2 + 4p^2R^2 - 12p^2R + 2R)A - (12p^2 - 2 + 4p^2R^2 + 12p^2R\right)

-2R)\xi A^{-1}] + [(1-R)AB + (1-R)\xi AB^{-1} - (1+R)\xi A^{-1}B^{-1}

- (1+R)A^{-1}B] \sin\left(\frac{\pi a}{N+1}\right)\cos\left(\frac{nb}{M+1}\right) = 0\]

\[\Rightarrow [(1-R)A(B + \xi B^{-1}) - (1+R)A^{-1}(B + \xi B^{-1})] \sin\left(\frac{\pi a}{N+1}\right)\cos\left(\frac{nb}{M+1}\right)

+ [(12p^2 - 2 + 4p^2R^2 - 12p^2R + 2R)A - (12p^2 - 2 + 4p^2R^2 + 12p^2R\right)

- 2R)\xi A^{-1}] \sin\left(\frac{\pi a}{N+1}\right) = 0 \quad (6.4.23b)\]

and

\[[(1-R)A(B - B^{-1}\xi) - (1+R)A^{-1}(B - B^{-1})] \sin\left(\frac{\pi a}{N+1}\right)\sin\left(\frac{\pi b}{M+1}\right) = 0\]

\[\Rightarrow (B - B^{-1}\xi)\[(1-R)A - (1+R)A^{-1}\] = 0 \quad (6.4.23c)\]

or

\[((1-R)A - (1+R)A^{-1}] = 0 \quad \Rightarrow A = \frac{1+R}{1-R} \]

Substituting the value of \(B = \xi^{1/2}\) in (6.4.23b), we get

\[A = \left\{(1+R)\left[\xi^{1/2}\cos\left(\frac{\pi b}{M+1}\right) + 6p^2\xi\right] + (2p^2R^2 - R - 1)\xi\right\}^{1/2}

\left\{(1-R)\left[\xi^{1/2}\cos\left(\frac{\pi b}{M+1}\right) + 6p^2\right] + (2p^2R^2 + R - 1)\right\}\]

The corresponding propagation factor \(\xi_{GS}\) for \(a = b = 1\) is given by the root (maximum absolute) of the cubic equation

\[\eta^3 - 10\cos\left(\frac{\pi}{M+1}\right)\varphi + (1-R^2)6p^2 + (1-R)(2p^2R^2 - R - 1)\cos\left(\frac{\pi}{M+1}\right)\varphi^2\]

\[+\left[25(1-R^2)\right]\cos\left(\frac{\pi}{M+1}\right)\varphi^2 + ((1-R^2)36p^4 + (1-R)6p^2(2p^2R^2 - R - 1)

+ (1+R)6p^2 (2p^2R^2 + R - 1) + (2p^2R^2 - 2 - 1)\varphi^2]\eta - \cos\left(\frac{\pi}{M+1}\right)(1+R)

[(1-R)6p^2 + 2p^2R^2 + R - 1]\varphi^2 = 0 \quad (6.4.24)\]

where \(\eta = \xi_{GS}^{1/3}\), \(\varphi = \frac{\cos\left(\frac{\pi}{N+1}\right)}{6p^2+5+2p^2R^2}\) and \((N+1)h=1,(M+1)k=1\).

Thus, the Gauss-Siedel iteration method is stable for those values of \(R\) such that \(|\xi_{GS}| < 1\) and the rate of convergence of the Gauss-Siedel iteration method is given by

\[v_{GS} = -\log\xi_{GS}\]
\[ \log \eta^2 = -2 \log \eta \]

Consequently, the spectral radii \( \rho \) of the block Jacobi and block Gauss-Seidel matrices are related by

\[ \rho(G_{GS}) = \rho(G_J)^2 \]  

(6.4.25)

Hence, the associated iteration

\[ u^{(k+1)} = G u^{(k)} + c \]  

(6.4.26)

converges for any initial guess where, \( G \) is Jacobi or Gauss-Seidel iteration matrix.

### 6.5 Computational Implementation

We consider a lower order method, for comparison. Substituting the approximations (6.2.8a), (6.2.8d), (6.2.9a), (6.2.9d) in the differential equation (6.1.1), we obtain a central difference scheme of \( O(k^2 + h^2) \) of the form

\[ \bar{U}_{xt,m} + \bar{U}_{yt,m} = f(x_t,y_m,U_{t,m},\bar{U}_{xt,m},\bar{U}_{yt,m}) + O(k^2 + h^2) \]  

(6.5.1)

Numerical experiments are carried out to illustrate the viability of proposed method and to demonstrate computationally its convergence. We solve the following two dimensional elliptic boundary value problems on a geometric mesh both on rectangular and cylindrical polar coordinates whose exact solutions are known to us. The Dirichlet boundary conditions can be obtained using the exact solutions as a test procedure. We replace the solution domain \( R \) by a rectangular grid \( G_R \) where \( h_i = x_i - x_{i-1} \) and \( h_{j+1} = x_{j+1} - x_j \) is the variable mesh size in the \( x \)-direction and \( \sigma_i = \frac{h_{i+1}}{h_i} > 0, \quad l = 1,2, ..., N \) the mesh ratio parameter and \( y_m = mk, \) \( m = 0,1,...,M + 1, \) \( k > 0. \) Since, we have

\[ 1 = x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + ... + (x_1 - x_0) \]

\[ = h_{N+1} + h_N + ... + h_1 \]

\[ = h_1(1 + \sigma_1 + \sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_3 + ... + \sigma_1\sigma_2...\sigma_N) \]  

(6.5.2)

Thus,

\[ h_1 = \frac{1}{1 + \sigma_1 + \sigma_1\sigma_2 + ... + \sigma_1\sigma_2...\sigma_N} \]  

(6.5.3)

This determines the starting value of the first step length in \( x \) – direction and the subsequent step lengths are calculated by \( h_2 = \sigma_1 h_1, h_3 = \sigma_2 h_2, \) etc. Hence we determine each grid point \( (x_l, y_m) \) of the rectangular grid.

For the purpose of simplicity, we may consider \( \sigma_l = \sigma \) (a constant), for all \( l = 1,2, ..., N \), then \( h_1 \) reduces to

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\[ h_i = \frac{(1-\sigma)}{(1-\sigma^{x_{i+1}})}, \quad \sigma \neq 1 \]  

(6.5.4)

Therefore, by prescribing the total number of grid points in the \( x \) – direction, say to be, \( N+2 \) and the value of \( \sigma \), we can calculate \( h_i \) from the above relation and the remaining mesh points in \( x \)-direction is determined by \( h_{i+1} = \sigma h_i, l = l(N) \). We have compared our method with the corresponding lower order variable mesh difference scheme of \( O(k^2 + h_i) \) in terms of solution accuracy. In all cases, we have taken the initial guess \( u(x_i, y_m) = 0 \).

Stoping criteria: The iterations were terminated when the maximum norm of the changes to the solution was less than the tolerance of \( 10^{-10} \). All computations were carried out in double precision arithmetic using MATLAB. Graphs depicting exact and numerical solutions for selected parameters for each of the problems discussed have been included (see Figs. 6.2-6.4).

Example 6.1 (Convection-diffusion equation)

The problem is to solve (6.4.6) in the solution region \( 0 < x, y < 1 \) whose exact solution is

\[ u(x, y) = e^{\frac{\beta x}{2}} \sin \pi y \left( 2e^{\frac{\beta}{2}} \sinh \sigma x + \sinh \sigma (1-x) \right) / \sinh \sigma, \]  

where \( \sigma^2 = \pi^2 + \frac{\beta^2}{4} \).

Table 6.1 displays the maximum absolute errors for \( u \) for \( \sigma = 0.9 \). The graphs of exact and numerical solutions are plotted in Fig. 6.2(i) and 6.2(ii) for \( \sigma = 0.9 \) and \( \beta = 100 \).

Example 6.2 (Poisson’s equation in polar coordinates)

The problem is to solve (6.4.5) in the solution region \( 0 < r, z < 1 \). The exact solution of the partial differential equation is \( u(r, z) = \cos hr \coshz \). The maximum absolute errors for \( u \) for \( \sigma = 1.1 \) and \( \sigma = 1.15 \) are tabulated in Table 6.2. The graphs of exact and numerical solutions are plotted in Fig. 6.3(i) and 6.3(ii) for \( \sigma = 1.15 \).

Example 6.3 (Steady state Burgers’ Model Equation)

\[ \varepsilon \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = u \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) 
+ e^x \sin \left( \frac{\pi y}{2} \right) \left( \varepsilon \left( 1 - \frac{\pi^2}{4} \right) - e^x \left( \sin \left( \frac{\pi y}{2} \right) + \frac{\pi}{2} \cos \left( \frac{\pi y}{2} \right) \right) \right), \quad 0 < x, y < 1 \]  

(6.5.5)

where \( R_{e} = e^{-1} > 0 \) is called Reynolds number. The exact solution is given by \( u(x, y) = e^x \sin \left( \frac{\pi y}{2} \right) \). The maximum absolute errors for \( u \) are tabulated in Table 6.3(a) and Table 6.3(b) for various values of \( \varepsilon \) \((0 < \varepsilon \ll 1)\) for \( \sigma = 0.95 \) and \( \sigma = 1.03 \). The graphs of exact and numerical solutions are plotted in Fig. 6.4(i) and 6.4(ii) for \( \sigma = 1.03 \).
Table 6.1

Example 6.1: The maximum absolute errors ($\sigma = 0.9$)

<table>
<thead>
<tr>
<th>$\beta \rightarrow (N+1,M+1)$</th>
<th>Proposed Method (6.2.12)</th>
<th>Method (6.5.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>(30,30) cpu-BGS*</td>
<td>.2808(-03)</td>
<td>.4515(-03)</td>
</tr>
<tr>
<td></td>
<td>0.424358s</td>
<td>0.128150 s</td>
</tr>
<tr>
<td>(40,40) cpu-BGS</td>
<td>.1611(-03)</td>
<td>.1029(-03)</td>
</tr>
<tr>
<td></td>
<td>1.297527s</td>
<td>0.364116 s</td>
</tr>
<tr>
<td>(50,50) cpu-BGS</td>
<td>.1094(-03)</td>
<td>.5529(-04)</td>
</tr>
<tr>
<td></td>
<td>2.730789s</td>
<td>1.024950 s</td>
</tr>
<tr>
<td>(60,60) cpu-BGS</td>
<td>.8233(-04)</td>
<td>.4028(-04)</td>
</tr>
<tr>
<td></td>
<td>3.652260s</td>
<td>2.140070 s</td>
</tr>
<tr>
<td>(70,70) cpu-BGS</td>
<td>.6635(-04)</td>
<td>.3335(-04)</td>
</tr>
<tr>
<td></td>
<td>7.254613s</td>
<td>3.948505 s</td>
</tr>
<tr>
<td>(80,80) cpu-BGS</td>
<td>.5617(-04)</td>
<td>.2953(-04)</td>
</tr>
<tr>
<td></td>
<td>9.439466s</td>
<td>6.499839 s</td>
</tr>
</tbody>
</table>

*cpu-BGS: CPU time using Block-Gauss Seidel method
Fig 6.2 Convection Diffusion Equation
Exact Solution ($\sigma = 0.9, \beta = 100$)

Comparison of plots of solution of Example 6.1
Table 6.2

Example 6.2: The maximum absolute errors

<table>
<thead>
<tr>
<th>$(N + 1 = M + 1)$</th>
<th>$\sigma = 1.1$</th>
<th>$\sigma = 1.15$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Method (6.2.12)</td>
<td>Method (6.5.1)</td>
</tr>
<tr>
<td>10</td>
<td>.8746(-03)</td>
<td>.5910(-03)</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>0.107377 secs.</td>
<td>0.115649 secs.</td>
</tr>
<tr>
<td>20</td>
<td>.1136(-03)</td>
<td>.2781(-03)</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>0.688182 secs.</td>
<td>0.625148 secs.</td>
</tr>
<tr>
<td>30</td>
<td>2.668(-04)</td>
<td>1.894614 secs.</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>2.507027 secs.</td>
<td>2.822055 secs.</td>
</tr>
<tr>
<td>40</td>
<td>.1341(-04)</td>
<td>.1976(-03)</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>5.907806 secs.</td>
<td>5.111935 secs.</td>
</tr>
<tr>
<td>50</td>
<td>.9992(-05)</td>
<td>.1904(-03)</td>
</tr>
<tr>
<td>60</td>
<td>.8487(-05)</td>
<td>20.895574 secs.</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>23.962332 secs.</td>
<td>20.895574 secs.</td>
</tr>
<tr>
<td>70</td>
<td>.7674(-05)</td>
<td>.1875(-03)</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>34.893838 secs.</td>
<td>33.069921 secs.</td>
</tr>
<tr>
<td>80</td>
<td>.7182(-05)</td>
<td>.1856(-03)</td>
</tr>
<tr>
<td>cpu-BGS</td>
<td>58.305769 secs.</td>
<td>56.497552 secs.</td>
</tr>
</tbody>
</table>

*cpu-BGS*: CPU time using Block-Gauss-Seidel method.
Comparison of plots of solution of Example 6.2
### Table 6.3(a)

#### Example 6.3: The maximum absolute errors

<table>
<thead>
<tr>
<th>$N+1$</th>
<th>Method (6.2.12)</th>
<th>Method (6.5.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M+1$</td>
<td>$\sigma = 0.95$</td>
<td>$\sigma = 1.03$</td>
</tr>
<tr>
<td>$R_e = 50$</td>
<td>$R_e = 100$</td>
<td>$R_e = 50$</td>
</tr>
<tr>
<td>10 cputime</td>
<td>(.2508-02) .070546 secs.</td>
<td>(.2334-02) .061620 secs.</td>
</tr>
<tr>
<td>20 cputime</td>
<td>(.7817-03) .057496 secs.</td>
<td>(.6856-03) .121613 secs.</td>
</tr>
<tr>
<td>30 cputime</td>
<td>(.3231-03) .134507 secs.</td>
<td>(.2975-03) .217330 secs.</td>
</tr>
<tr>
<td>40 cputime</td>
<td>(.1778-03) .444110 secs.</td>
<td>(.1707-03) .451878 secs.</td>
</tr>
<tr>
<td>50 cputime</td>
<td>(.1138-03) .957390 secs.</td>
<td>(.1104-03) .931345 secs.</td>
</tr>
<tr>
<td>60 cputime</td>
<td>(.7908-04) 1.934783 secs.</td>
<td>(.7718-04) 2.125581 secs.</td>
</tr>
<tr>
<td>70 cputime</td>
<td>(.5807-04) 5.365489 secs.</td>
<td>(.5690-04) 3.114234 secs.</td>
</tr>
<tr>
<td>80 cputime</td>
<td>(.4445-04) 3.451604 secs.</td>
<td>(.4356-04) 4.246450 secs.</td>
</tr>
</tbody>
</table>

### Table 6.3(b)

#### Example 6.3: The maximum absolute errors ($k = \frac{20}{(N+1)^2}$)

<table>
<thead>
<tr>
<th>$(N_1,M_1)$</th>
<th>Method (6.2.12)</th>
<th>Method (6.5.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(N+1,M+1)$</td>
<td>$\sigma = 0.95$</td>
<td>$\sigma = 1.03$</td>
</tr>
<tr>
<td>$R_e = 10$</td>
<td>$R_e = 10^2$</td>
<td>$R_e = 10$</td>
</tr>
<tr>
<td>(10,5) cputime</td>
<td>(.1026-01) .055332 secs.</td>
<td>(.1035-01) 0.058428 secs.</td>
</tr>
<tr>
<td>(20,20) cputime</td>
<td>(.5893-03) .215578 secs.</td>
<td>(.5893-03) .214911 secs.</td>
</tr>
<tr>
<td>(40,80) cputime</td>
<td>(.3683-04) .310133 secs.</td>
<td>(.3672-04) .358052 secs.</td>
</tr>
<tr>
<td>(80,320) cputime</td>
<td>(.2158-05) .76641793 secs.</td>
<td>(.2176-05) 2.7237067 secs.</td>
</tr>
</tbody>
</table>
Comparison of plots of solution of Example 6.3
6.6 Concluding Remarks

The available numerical methods for the solution of two dimensional nonlinear elliptic boundary value problems on a non-uniform mesh are of first order accurate only and from application point of view in most cases the method is unstable. In this chapter, we have developed a new stable high order nine point compact scheme of $O(k^2 + k^2 h + h^3)$ based on cubic spline approximations for the solution of two dimensional nonlinear elliptic boundary value problems. The proposed method is successfully applied to Poisson’s equation in cylindrical polar coordinates and two-dimensional Burgers’ equation with high Reynolds number. The numerical results confirm that the proposed method produces oscillation free solutions for high Reynolds number, whereas the corresponding lower method (6.5.1) becomes unstable.