CHAPTER - 11

GENERALIZATION OF HELLY-BRAY LEMMA

2.1. INTRODUCTION:

Let \( \{ X_n, n \geq 1 \} \) be a sequence of i.i.d. r.v.s. with a common d.f. \( F \). Let \( S_n = \sum_{k=1}^{n} X_k, \ n \geq 1 \). Let \( \{ A_n \} \) and \( \{ B_n \} \) be sequences of constants \( (B_n \to \infty \text{ as } n \to \infty) \). Set \( Z_n = B_n^{-1} S_n - A_n, \ n \geq 1 \).

When \( F \) belong to the domain of normal attraction of a stable law with exponent \( \alpha, 1 < \alpha < 2 \), Willis L. Owen [11] Obtained and estimate for \( E(|S_n|) \) and then this estimate is used to get a generalization of Helly-Bray Lemma or a moment convergence result.

In this chapter, when \( F \in \mathcal{DA}(\alpha), \ 1 < \alpha < 2 \), we obtain this moment convergence result of Willis L. Owen [11]. We assume throughout this chapter, that the limit law is stable with exponent \( \alpha, 1 < \alpha < 2 \). Consequently \( EX_4 \) exists. We assume with no loss of generality that
EX_i=0. Inturn Z_n becomes B_n S_n, n ≥ 1.

In next section we establish some needed lemmas and in section 2.3, we prove our main results.

2.2. NEEDED LEMMA:

**LEMMA 2.2.1:**
There exists positive constant N_0, s and δ_0 (δ_0 > 0) independent of n, such that, for t > s,

\[
\lim \sup_{n \to \infty} \mathbb{P} \left( \left| \frac{S_n}{B_n} \right| > t \right) \leq \frac{N_0}{t^a} - \delta_0 \quad (2.2.1)
\]

**Proof:**
We will prove this lemma by truncation. For fixed n and t > 1, define

\[ Y_i = \begin{cases} X_i, & \text{if } |X_i| \leq t B_n \\ 0, & \text{otherwise} \end{cases} \]

Let \( V_i = X_i - Y_i \), \( S_{i,n} = Y_1 + Y_2 + \ldots + Y_n \) and \( S_{2,n} = V_1 + V_2 + \ldots + V_n \).

Notice that

\[
\mathbb{P} \left( \left| \frac{S_n}{B_n} \right| > t \right) \leq \mathbb{P} \left( \left| \frac{S_{1,n}}{B_n} \right| > t \right) + \mathbb{P} \left( S_{2,n} \neq 0 \right) \quad (2.2.2)
\]
Now

\[ P \left( S_{2,n} \not\leq 0 \right) \leq n P \left( V_1 \not\leq 0 \right) \text{ and } P \left( V_1 \not\leq 0 \right) = P \left( |X_1| > tB_n \right) \]

By (1.4.1), we have,

\[ P \left( S_{2,n} \not\leq 0 \right) \leq n \frac{L(tB_n)}{t^\alpha B_n^\alpha} \frac{L(B_n)}{L(B_n)} \]

\[ \leq n \frac{L(B_n)}{B_n^\alpha} \frac{1}{t^\alpha} \frac{L(tB_n)}{L(B_n)} \]

Using Karamata's s.v. function, we get

\[ \frac{L(tB_n)}{L(B_n)} = \frac{a(tB_n)}{a(B_n)} \exp \left\{ t \frac{\varepsilon(y)}{y} dy - \frac{\varepsilon(y)}{y} dy \right\} \]

\[ = \frac{a(tB_n)}{a(B_n)} \exp \left\{ \frac{tB_n}{B_n} \frac{\varepsilon(y)}{y} dy \right\} \]

Since \( a(x) \to 1 \), as \( x \to \infty \) and \( \varepsilon(y) \to 0 \) as \( y \to \infty \). Hence there exists \( c_0 \) and \( \delta_0 \) such that

\[ \frac{a(tB_n)}{a(B_n)} \leq c_0 \text{ and } \frac{\varepsilon(y)}{y} \leq \delta_0, \quad \forall \ y \geq B_n. \]

\[ \therefore \frac{L(tB_n)}{L(B_n)} \leq c_0 \exp \left\{ \delta_0 \log t \right\} = c_0 t^{\delta_0}, \quad \forall \ t > 1. \quad (2.2.3) \]

Therefore using (1.4.2), there exists \( c_4 \left( > c_0 \right) \), such that

\[ P \left( S_{2,n} \not\leq 0 \right) \leq c_4 \frac{t^{\delta_0}}{t^\alpha} = \frac{c_4}{t^{\alpha - \delta_0}} \quad (2.2.4). \]

Now consider the first term in the right of (2.2.2).

Since \( EX_1 = 0 \).
\[ |EY_1| = \left| \int_{-tB_n}^{tB_n} x \, dF(x) \right| \leq \int_{-tB_n}^{tB_n} |x| \, dF(x) \]
\[ \leq -tB_n \int_{-\infty}^{tB_n} x \, dF(x) + \int_{tB_n}^{\infty} x \, dF(x) \]
\[ (2.2.5) \]

Consider, integrating by parts, we get
\[ \int_{-tB_n}^{tB_n} x \, dF(x) = x \, F(x) \bigg|_{-tB_n}^{tB_n} - \int_{-\infty}^{\infty} F(x) \, dx. \]

Let \( x = -y \Rightarrow dx = -dy \) in the second integral.
\[ = -tB_n \, F(-tB_n) + \int_{-\infty}^{\infty} F(-y) \, dy. \]
\[ (2.2.6) \]

Again integrating, the second integral of (2.2.5), by parts, we get
\[ \int_{tB_n}^{\infty} x \, dF(x) = -\int_{tB_n}^{\infty} x(1-F(x)) \bigg|_{tB_n}^{\infty} + \int_{tB_n}^{\infty} (1-F(x)) \, dx. \]
\[ = tB_n \, (1-F(x)) + \int_{tB_n}^{\infty} (1-F(x)) \, dx \]
\[ (2.2.7) \]

Adding (2.2.6) and (2.2.7), we get (2.2.5) as
\[ |EY_1| \leq tB_n \, F(-tB_n) + \int_{-\infty}^{tB_n} F(-y) \, dy + tB_n \left[ 1-F(tB_n) \right] + \int_{tB_n}^{\infty} \left[ 1-F(x) \right] \, dx. \]
Again using (1.4.1), one gets that

\[ |EY_1| \leq tB_n \frac{L(tB_n)}{t^{\alpha-1}B_n^{\alpha-1}} + \int_{tB_n}^{\infty} \frac{L(x)}{x^{\alpha}} \ dx. \]

By using asymptotic properties of regularly varying functions (see [2], page 272) and integrating parts, we get

\[ |EY_1| \leq c_2 \frac{L(tB_n)}{t^{\alpha-1}B_n^{\alpha-1}}, \]

where \( c_2 > 0 \) is some constant.

Now

\[ |E(S_{tB_n})| \leq \frac{n}{B_n} |EY_1| = \frac{n}{B_n} c_1 \frac{L(tB_n)}{t^{\alpha-1}B_n^{\alpha-1}} \frac{L(tB_n)}{L(tB_n)}. \]

By (2.2.3) and (1.4.2), we have

\[ |E(S_{tB_n})| \leq \frac{c_3}{t^{\alpha-1}B_n^\alpha}, \] where \( c_3 > 0 \) is a constant.

Since \( \alpha > 1 \) and \( t > 1 \), hence \( |E(S_{tB_n})| \leq c_3. \)

\[ \Rightarrow -c_3 \leq E(S_{tB_n}) \leq c_3. \]
Consider
\[ \frac{S_{1:n}}{B_n} > t \quad \text{or} \quad \frac{S_{1:n}}{B_n} < -t \]

\[ \implies \frac{S_{1:n}}{B_n} - E\left(\frac{S_{1:n}}{B_n}\right) > t - E\left(\frac{S_{1:n}}{B_n}\right) \geq t - (c_3) \]
or
\[ \frac{S_{1:n}}{B_n} - E\left(\frac{S_{1:n}}{B_n}\right) < -t - E\left(\frac{S_{1:n}}{B_n}\right) < -t + c_3. \]

\[ \implies \left\{ \left| \frac{S_{1:n}}{B_n} \right| > t \right\} = \left\{ \frac{S_{1:n}}{B_n} - E\left(\frac{S_{1:n}}{B_n}\right) < -t + c_3 \quad \text{or} \quad \frac{S_{1:n}}{B_n} - E\left(\frac{S_{1:n}}{B_n}\right) > t + c_3 \right\} \]

\[ \leq \left\{ \left| \frac{S_{1:n}}{B_n} - E\left(\frac{S_{1:n}}{B_n}\right) \right| > (t - c_3) \right\} \]

Let \( s = 1 + c_3 \). Henceforth, we can take \( t > s \). Then

\[ P\left( \left| \frac{S_{1:n}}{B_n} \right| > t \right) \leq P\left( \left| \frac{S_{1:n}}{B_n} - E\left(\frac{S_{1:n}}{B_n}\right) \right| > (t - c_3) \right) \quad (2.2.8) \]

From Chebyshev's inequality, we get

\[ P\left( \left| \frac{S_{1:n}}{B_n} - nE(Y_i) \right| > (t - c_3) \right) \leq \frac{\text{Var}(\frac{S_{1:n}}{B_n})}{(t - c_3)^2} \]

\[ \leq \frac{n\text{Var}(Y_i)}{(t - c_3)^2B_n^2} \quad (2.2.9) \]

Since \( \text{Var}(Y_i) \leq EY_i^2 = \int_{-tB_n}^{tB_n} y^2 \, dF(y) = \int_{-tB_n}^{0} y^2 \, dF(y) + \int_{0}^{tB_n} y^2 \, dF(y) \cdot 0 \)

\[ \leq \int_{-tB_n}^{0} y^2 \, dF(y) + \int_{0}^{tB_n} y^2 \, d(1 - F(y)) \]

\[ \leq \int_{-tB_n}^{0} y^2 \, dF(y) + \int_{-tB_n}^{0} y^2 \, dF(y) \]

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Using integration by parts, we get

\[
\leq y^2 F(y) \left|_{-t_B}^{t_B} \right. - 2 \int_{-t_B}^{t_B} yF(y) dy - y^2 (1-F(y)) \left|_{0}^{t_B} \right. + 2 \int_{0}^{t_B} y(1-F(y)) dy.
\]

Let \( y = -x \Rightarrow dy = -dx. \)

\[
\leq -t_B^2 \left( 1-F(-t_B) \right) - 2 \int_{-t_B}^{0} y(-x)^a (-dx) - t_B^2 \left( 1-F(t_B) \right)
\]

\[
+ 2 \int_{0}^{t_B} y(1-F(y)) dy.
\]

\[
\leq -t_B^2 \left( 1-F(-t_B) + F(-t_B) \right) + 2 \int_{0}^{t_B} x \left( 1-F(x) + F(-x) \right) dx.
\]

By using (1.4.1), we have

\[
\text{Var}(Y_t) \leq -t_B^2 \frac{L(t_B)}{t_B^a} + 2 \int_{0}^{t_B} \frac{xL(x)}{x^a} dx.
\]

By asymptotic properties of regularly varying functions (see [2], page 272), one gets, for some \( c_4 > 0 \) constant,

\[
\text{Var}(Y_t) \leq c_4 \frac{L(t_B)}{t_B^a} \frac{L(t_B)}{t_B^{a-2}} \frac{L(t_B)}{L(t_n)} \quad (2.2.10)
\]

\[
\frac{n \text{Var}(Y_t)}{B_n^a} \leq c_4 \frac{nL(B_n)}{B_n^a} \cdot \frac{L(B_n)}{L(B_n)} \cdot \frac{1}{t^{a-1}}
\]

By (2.2.3) and (1.4.1) one gets that

\[
\frac{n \text{Var}(Y_t)}{B_n^a} \leq \frac{c_4}{t^{a-1}}.
\]

Which in turn implies that for \( t > s \),
\[
\frac{\text{Var} \left( \frac{S_{1:n}}{S_n} \right)}{(t-c_0)^2} \leq \frac{n \text{Var}(Y_i)}{B_n(t-c_0)^2} = \frac{c_4 t^{2-\alpha-\delta_0}}{t^2(1-\frac{3}{t})} \sim c_5 t^{-\alpha-\delta_0},
\]

Where \( c_5 = \frac{c_4}{(1-\frac{c_0}{t})^2} \).

Which implies

\[
\frac{\text{Var} \left( \frac{S_{1:n}}{S_n} \right)}{(t-c_0)^2} \leq \frac{c_5}{t^{\alpha-\delta_0}}.
\]  

(2.2.11)

By (2.2.8), (2.2.9) and (2.2.11), one gets that

\[
P \left( \left| \frac{S_{1:n}}{S_n} \right| > t \right) \leq \frac{c_d}{t^{\alpha-\delta_0}}.
\]  

(2.2.12)

Where \( c_d > 0 \) some constant. Let \( N_0 = c_0 + c_d \). Then by (2.2.4) and (2.2.12) together imply that

\[
P \left( \left| \frac{S_{1:n}}{S_n} \right| > t \right) \leq \frac{N_0}{t^{\alpha-\delta_0}}, \quad \forall t > 0.
\]

2.3. CONVERGENCE OF MOMENT ESTIMATES:

THEOREM 2.3.1

For each real number \( q \) with \( 0 < q < 0 \), there exists a finite positive real constant \( Q \), depending on \( q \) but independent of \( n \) such that

\[
E \left[ \left| Z_n \right|^q \right] \leq Q
\]  

(2.3.1).
In particular, there exists a constant $M$ independent of $n$ such that

$$E \left( \left| S_n \right| \right) \leq MB_n. \quad (2.3.2)$$

Where $Z_n = \frac{S_n}{B_n}$.

Proof:

We can observe that the result is true for $q=0$. So choose $q$ such that $0 < q < a$. Let $N_0$ and $\alpha$ be as in lemma 2.2.1. Then

$$E \left( \left| \frac{S_n}{B_n} \right|^q \right) = \int_0^\infty x^q \, dp \left( \left| \frac{S_n}{B_n} \leq x \right) \right) + \int_0^\infty x^q \, dp \left( \left| \frac{S_n}{B_n} \leq x \right) \right)$$

Now

$$\int_0^\infty x^q \, dp \left( \left| \frac{S_n}{B_n} \leq x \right) \right) \leq \int_0^\infty x^q \, dp \left( \left| \frac{S_n}{B_n} \leq x \right) \right) = s^q.$$

Similarly

$$\int_0^\infty x^q \, dp \left( \left| \frac{S_n}{B_n} \right| > x \right) \leq \int_0^\infty x^q \, d \left( 1 - P \left( \left| \frac{S_n}{B_n} \right| > x \right) \right)$$

Integrating by parts and applying lemma 2.2.1, one gets
\[
\int_{-\infty}^{\infty} x^q \, dP \left( \left\{ \frac{S_n}{B_n} \leq x \right\} \right) \leq -x^q \, P \left( \left\{ \frac{S_n}{B_n} > x \right\} \right)_{x}^{\infty} + q \int_{x}^{\infty} x^{q-t} \, P \left( \left\{ \frac{S_n}{B_n} > x \right\} \right) \, dx.
\]

\[
\leq x^q \, P \left( \left\{ \frac{S_n}{B_n} > x \right\} \right) + q \int_{x}^{\infty} \frac{N_0}{x^\alpha - \delta_0} \, dx
\]

\[
\leq x^q + qN_0 \int_{x}^{\infty} \frac{dx}{x^\alpha - \delta_0 - q + 1}
\]

\[
\leq x^q + qN_0 \cdot \frac{x^{q+\delta_0 - \alpha}}{\alpha - q - \delta_0} \tag{2.3.4}
\]

Adding (2.3.3) and (2.3.4), one gets,

\[
E \left( \left| Z_n \right|^q \right) = E \left( \left| \frac{S_n}{B_n} \right|^q \right) \leq x^q + q^q + qN_0 \cdot \frac{x^{q+\delta_0 - \alpha}}{\alpha - q - \delta_0}
\]

\[
\leq 2x^q + qN_0 \cdot \frac{x^{q+\delta_0 - \alpha}}{\alpha - \delta_0 - q}.
\]

Let \( Q_1 = 2x^q + qN_0 \cdot \frac{x^{q+\delta_0 - \alpha}}{\alpha - \delta_0 - q} \), then,

\[
E \left( \left| Z_n \right|^q \right) \leq Q_1. \tag{2.3.5}
\]

Let \( Q_2 = \max_n E \left( \left| Z_n \right|^q \right) \).

\[
Q = \max \left\{ Q_1, Q_2 \right\}.
\]

Now (2.3.1) follows from (2.3.5) and (2.3.6).

If we take \( q = 1 \) then (2.3.2) is immediate.

Hence proof of the theorem is complete.
**THEOREM 2.3.2:**
\[
\lim_{n \to \infty} E \left( \left| \frac{S_n}{B_n} \right|^q \right) = E \left( \left| Y \right|^q \right) \tag{2.3.7}
\]

Moreover, for all \( q \) with \( 0 < q < \alpha \), we have
\[
\lim_{n \to \infty} E \left( \left| \frac{S_n}{B_n} \right|^q \right) = E \left( \left| Y \right|^q \right) \tag{2.3.8}
\]

**Proof:**

One can notice that the result is clear, if \( q = 0 \). So let \( q > 0 \) and choose \( s \), such that \( 0 < q < s < \alpha \).

Let \( u = \frac{s}{q} \) and \( G(t) = |t|^u \). Then \( \frac{G(t)}{t} \to \infty \) as \( t \to \infty \).

By the convergence theorem, Loève, M [5], Page 183, we know that for (2.3.8) to hold, \( \left| Z_n \right|^q \) should be uniformly integrable. Also theorem 22, Meyer, Paul A [7], shows that \( \left| Z_n \right|^q \) is uniformly integrable, whenever
\[
\sup_n E \left( \left| Z_n \right|^q \right) < \infty.
\]

But observe that
\[
E \left( \left| Z_n \right|^q \right) = E \left( \left| Y \right|^q \right)
\]

From the above theorem 2.3.1, we have
\[
\sup_n E \left( \left| Z_n \right|^q \right) < Q.
\]

Hence \( \left| Z_n \right|^q \) is uniformly integrable and in turn (2.3.8) is established.
In particular, when $q=1$, by noting that the uniform integrability of $|Z_n|$ implies that of $Z_n$ and again appealing to the convergence theorem, Loève, M [5], Page 183, (2.3.7) gets established. Hence proof of the theorem is complete.