CHAPTER 1
1.1 Introduction to stochastic storage models:

Stochastic storage models occupy an important place in modern research because many real life problems coming across in day to day life can be solved through stochastic modelling. The main feature of such problems is that the parameters involved in the model are functions of time element. In other words the parameter also changes along with time t. This makes the problem more complicated and to obtain solutions under probabilistic models fails, because in probabilistic models the parameters under consideration are assumed as constants. This can be explained elaborately with the following example.

Let us consider the well known poisson Distribution with the probability density function

\[ f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \lambda > 0, x = 0, 1, 2, \ldots \alpha, (1.1.1) \]

where \( \lambda \) is the parameter of the distribution which may be estimated from the sample data as

\[ \hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{N}, (1.1.2) \]
where $\hat{\lambda}$ denotes the estimate of the $\lambda$ and $f_i$ denotes the number of days on which $x_i$ ($i = 1, 2, \ldots, n$) the number of calls received and $N = \sum_{i=1}^{n} f_i$ the number of days on which the data is collected.

The above example is a typical case suitable for static theory which will be useful to answer such questions like the average number of telephone calls per day (or) to obtain confidence intervals to the average number of telephone calls and so on. While this information is useful in its own way the other important and practical questions from the viewpoint of management like the average number of telephone calls coming in a specified period of time say between 10 AM to 11 AM, the law governing the random interval between two telephone calls cannot be answered with the help of (1.1.1) and related theory. Such type of questions can be answered only when it is assumed that the parameter $\lambda$ is a function of $t$ (time), which makes the model a stochastic one. In other words a stochastic model is a mathematical model, where the parameters involved in it are assumed as functions of other continuous variable usually it is time in many real life problems arising in Biology,
Medicine, Inventory, Banking and so on. In Physics and Population Studies it is usually assumed as energy and age respectively.

Thus stochastic storage models are playing a predominant role to obtain solutions for the problems arising in many diverse fields like Management, Industry, Biology, Medicine, Banking and so on.

Erlangian (1909-20) was the originator of the stochastic models through his works on queueing theory. Later P.A.P Moran (1954) applied these models to dam theory and Arrow and et-al (1958) applied to inventory and reliability theory. Gani (1957) and Prabhu (1964) gave some hints to bring all the above four apparently different models namely queueing, dam, reliability and inventory under the same roof with the common title as 'stochastic storage models'. Any stochastic storage model is basically governed by the following five parameters namely:
1) The input law
2) The output law
3) Number of servers or channels
4) Queue discipline and
5) Capacity of the waiting room/reserve room/dam

Since any stochastic storage model is mainly based on the above five parameters, Kendall (1948) has given a notation M/M/1/FIFO/∞, where the first letter represents the law governing input is negative exponential and the second letter represents the law governing output is also negative exponential, the third letter stands for the number of channels/servers, whereas fourth and fifth items represent queue discipline and capacity of waiting room respectively. Here the queue discipline is first-in first-out (FIFO) and the capacity of waiting room is infinite. The letter 'M' stands for the important property of the negative exponential distribution namely "Markovian" (or) "Memoryless" property. Thus by imposing various conditions on the above five parameters we obtain different kinds of stochastic storage models. Some of the commonly occurring stochastic storage models are:
1) M/ M/1/FIFO/K (Where K is finite)
2) M/G/1/FIFO/\alpha
3) M/G/1/FIFO/K
4) G/M/1/FIFO/\alpha
5) G/M/1/FIFO/K
6) M/M/n/FIFO/\alpha
7) M/M/n/FIFO/K
8) M/M/1/LIFO/\alpha
9) M/M/1/LIFO/K and so on.

In the above stochastic storage models 'G' represents a general distribution and LIFO represents the queue discipline last-in first-out.

Sarma (1983) proposed a stochastic storage model applicable in banking system in his doctoral thesis and obtained both analytic and explicit solutions of banking system. The model considered by him is briefly explained in the following section for a ready reference.

1.2. Stochastic Banking Model:

Let the stochastic variable X(w,t) represents the money reserve \( \omega \) available with the system at any arbitrary
time $t$, where $\omega \in \Omega$ and $t \in T$. Here $\Omega$ is called the "State space" and $T$ is called "Index set". For brevity $X(\omega, t)$ is usually represented as $X(t)$. Thus $X(t)$ is the amount available with the banking system at time epoch $t > 0$.

The reserve level $X(t)$ is the combined effect of two independent random variables namely:

1) Amount of withdrawals, denoted by the r. v., "v" and is governed by a known probability law with the probability density function (pdf) $g(.)$ and

2) Inter-withdrawal times denoted by the r. v., "u" and is governed by a known probability law with the pdf $h(.)$.

The input fed in to the system is assumed to be a function of $t$ (time), which increases linearly at a unit rate along with time $t$. It is assumed that the random amount of withdrawals depletes the reserve level $X(t)$ instantaneously at random epochs of withdrawals. A typical realization of a stochastic banking model with the above assumptions is given in Fig. (1.2.1). After proposing the above stochastic banking model, Sarma (1983) obtained solutions first for the stochastic banking model with
infinite capacity of the reserve room and later obtained the solutions for a stochastic banking model with finite capacity.

1.3 Motivation:

Sarma (1992) in his M.Phil thesis proposed a stochastic banking model on similar lines as explained in section (1.2) and obtained analytic solutions for $M/E_n/1/FIFO/\alpha$ and $E_m/M/1/FIFO/\alpha$ stochastic banking models where $E(.)$ represents Erlangian distribution with $(.)$ stages. The main objectives of his dissertation are:

1) To generalise the results obtained by Sarma (1983) and

2) To obtain the solutions for those stochastic banking models, where the inter withdrawal times "$u$" or the amount of withdrawals "$v$" follow a non Markovian law. With this motivation, Sarma (1992) considered an Erlangian distribution with $K$ stages, which does not posses Markovian property and can be viewed as a generalisation to Negative exponential distribution, when $K=1$. 
This motivated us to work further on these lines to consider many other situations, where the Markovian law does not hold good. For instance Gamma distribution can also be considered as a generalisation to Negative exponential distribution and does not possesses memory less property. Further like poisson distribution the mean and variance of Gamma distribution are equal and this can be considered as a continuous distribution, where as poisson distribution is a discrete one.

Thus the main objectives of this dissertation are;
1) To generalise the results obtained by Sarma (1983) and
2) To obtain solutions for the stochastic banking models, when the inter-withdrawal times (u) or the amount of withdrawals (v) follow Gamma distribution with parameters \( \lambda \) and \( K \), where \( \lambda \) represents the mean, which equals to variance and \( K \) represents the different stages of the Gamma variate under consideration. Since the main focus of this dissertation is to demonstrate the utility of Gamma distribution in stochastic banking models, the assumption, relationships, definition, properties and applications of this distribution are discussed elaborately in the following section.
1.4 Some details of Gamma distribution:

As already pointed out the main theme of this dissertation is to obtain solutions for non-markovian stochastic banking models, and since Gamma distribution is one of the distributions having non-markovian nature, it is discussed in detail as follows:

The basic assumptions underlying Gamma distribution are given by:

1.4.1 Assumptions of Gamma distribution:

1) It is assumed that a customer has to pass through \( K \) different phases or stages to complete his service.

2) It is assumed that the duration of time spent by the customer in each phase or stage follow a negative exponential distribution with the density function

\[
 f(t) = \lambda e^{-\lambda t}, \quad t > 0, \quad \lambda > 0. \tag{1.4.2}
\]

3) Arrival of customers into the system is assumed to follow poisson distribution with the density function

\[
 p(n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad \lambda > 0, \quad n = 0, 1, 2, \ldots \alpha. \tag{1.4.3}
\]
4) It is assumed that the queue discipline is FIFO and one customer is allowed into the service channel at a time. When he completes his service in all the K phases (or) stages, then the next customer is allowed for the service.

Under the above assumptions a Gamma variate X is defined as follows.

**Definition (1.4.4)**

A continuous r.v., X having the following pdf is said to have a Gamma distribution with the parameters λ and k

\[ f(x) = \frac{\lambda^k e^{-\lambda x} x^{k-1}}{(k-1)!} \quad \text{if} \quad 0 < x < \infty, \lambda > 0, k > 0. \]  

Typical realizations of a Gamma variate for different values of \( \lambda \) and \( K \) are depicted graphically in the Figure (1.4.6).

It is very important to note that a Gamma variate X, with the above pdf (1.4.5) has very interesting relationships with the other familiar distributions, as follows:
Fig 1.4.6. The Gamma Density $f_k(x)$ for some values of $\lambda, \kappa$. 

<table>
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<th>$\lambda$</th>
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(1.4.7) When $\lambda = 1$ (1.4.5) reduces to $e^{-x} \frac{x^{k-1}}{(k-1)!}$, which is the pdf of Gamma distribution with parameter $k$.

(1.4.8) When $K = 1$ (1.4.5) reduces to $\lambda e^{-x}$, which is the pdf of Negative exponential distribution. Hence Gamma distribution can be viewed as a generalisation of Negative exponential distribution.

(1.4.9) When $\lambda = K$ (1.4.5) becomes $\frac{(\lambda K)^k - \lambda K x^{k-1}}{(k-1)!} e^{-\lambda K x^{k-1}}$, which is the pdf of Erlangian distribution.

(1.4.10) If $\chi^2$ is a chi-square variate with $n$ degrees of freedom then $\chi^2/2$ is a Gamma variate with parameter $n/2$. Thus if we take $\chi^2/2 = x$ the pdf of $\chi^2$ distribution becomes the pdf of Gamma distribution, i.e., when $\lambda = 1/2$ and $K = n/2$ ($n$ is a positive integer) then (1.4.5) becomes (for $x > 0$)

$$\frac{X^{n/2-1} e^{-x/2}}{2^{n/2} \sqrt{\pi n/2}}$$

which is the pdf of $\chi^2$ distribution with $n$ df.

(1.4.11) If $X$ is a standard normal variate then $X^2/2$ is a Gamma variate with parameter $1/2$ (vide Medhi (1982))
Further Gamma distribution with parameter $K$, has the following important properties.

(1.4.12) Properties of one parameter Gamma distribution with parameter $K$:

(1) Mean $= K$.
(2) Variance $= K$.
(3) $\mu_3 = 2K$.
(4) $\mu_4 = 6K + 3K^2$.
(5) $\beta_1 = 4/K$.
(6) $\beta_2 = 3 + 6/K$.
(7) Moment generating function $= (1-t)^{-K}$.
(8) Characteristic function $= (1-it)^{-K}$.
(9) Cumulant generating function $= K[t + t^2/2 + t^3/3 + \ldots]$.
(10) The Laplace Transform $= 1/(s+1)^K$.
(11) The sum of independent Gamma variates is also a Gamma variate. Therefore Gamma distribution has additive property.
(12) As $K \to \infty$ Gamma distribution tends to Normal distribution.

For a two parameter Gamma distribution with the parameters $\lambda$ and $K$, we have the following properties.
Properties of two parameter Gamma distribution with parameters $\lambda$ and $K$:

1. Mean $= \frac{K}{\lambda}$.
2. Variance $= \frac{K}{\lambda^2}$.
3. Moment Generating Function $= (1 - \frac{t}{\lambda})^{-K}$.
4. Characteristic Function $= (1 - it/\lambda)^{-K}$.
5. The Laplace Transform $= (\frac{\lambda}{\lambda + s})^k$.

The above discussed Gamma distribution has very important practical applications. Some of the real life problems, where a Gamma distribution can fruitfully be applied are given below.

Applications of Gamma distribution.

The Gamma distribution described above, often furnishes a good fit for observations that are intrinsically non-negative and of fairly wide range. Among other applications it has proved useful in the analysis of weekly sales data and in connection with certain inventory models.
The Gamma distribution has a useful role to play in banking models since the customer to complete his service has to pass through different phases like issuing of tokens, passing the cheque after making suitable entries, receiving the cash at cash counter and so on. The customers arriving in to the bank are assumed to follow poisson distribution and hence the waiting time follows Gamma distribution. Similarly amounts withdrawn can also be considered to follow a Gamma distribution because the major loans (for instance, loans for constructing houses, cinema theaters, factories, industries and so on) are released in different phases. Thus application of Gamma distribution in stochastic banking system is considered in this dissertation.

1.5 Review of earlier literature:

As the chief objective of this dissertation is to obtain solutions of a stochastic banking model when the inter-withdrawal times or amount of withdrawals follow Gamma distribution, here in this section a purposeful review of earlier literature is given relevent to stochastic banking models, which also can serve as a ready reference to derive these results as special case of our results obtained by us.
in this dissertation. Thus the results obtained by Sarna (1983) for M/M/1/FIFO/\infty stochastic banking model are briefly explained as follows.

After introducing a stochastic banking model (vide 1.2) to obtain solutions Sarna (1983) first introduced two conditional probability density functions namely \( M(x,y,t) \) and \( M_1(x,y,t) \), which govern the reserve level \( X(t) \) at any arbitrary time \( t \), which are defined as follows.

**Definition (1.5.1)**

\[ M(x,y,t) \overset{\text{def}}{=} \lim_{\Delta \to 0^+} \frac{1}{\Delta} \int_{x}^{x+\Delta} \int_{0}^{u} \mathbb{P} \left[ 0 < x(t) \leq x + \Delta, x(u) > 0, u \in (0,t) \right] dx \]

\[ x(0-) > x(0) = x(0+) = y \] /\Delta. \hspace{1cm} (1.5.2)

**Definition (1.5.3)**

\[ M_1(x,y,t) \overset{\text{def}}{=} \lim_{\Delta \to 0^+} \frac{1}{\Delta} \int_{x}^{x+\Delta} \int_{0}^{u} \mathbb{P} \left[ 0 < x(t) \leq x + \Delta, x(u) > 0, u \in (0,t) \right] dx \]

\[ x(t-) > x(t) = x(t+) \] /\Delta. \hspace{1cm} (1.5.4)
A critical comparison between $M(x,y,t)$ and $M_1(x,y,t)$ reveals that (i) There is a withdrawal at epoch $t$ in $M_1(x,y,t)$ whereas it need not be in $M(x,y,t)$, because of the presence of end condition $[x(t-) > x(t) = X(t+)]$ in (1.5.4).

(ii) The origin is an epoch of demand in both $M(x,y,t)$ and $M_1(x,y,t)$ and the reserve level after the demand is known and is equal to $y$ because of the presence of the condition $X(0-) > x(0) = x(0+) = Y.(known)$

In other words origin is a point of regeneration in both $M(x,y,t)$ and $M_1(x,y,t)$ whereas epoch $t$ is a point of regeneration in $M_1(x,y,t)$ only. A regenerative point can be defined as follows:

Definition (1.5.5)

Let $x(t)$ denote a stochastic variate (for simplicity $w$ is omitted). An epoch $T$ is said to be a point of regeneration if

$$\text{dist } [x(t)|x(T)] = \text{dist } [x(t)|x(\tau) \text{ for all } \tau < T]$$

for all $t > T$.  \hfill (1.5.6)
After introducing the two c.p d.f.s namely $M(x, y, t)$ and $N(x, y, t)$, integral equations are formed by using imbedded regenerative process technique by identifying suitable regenerative points. The integral equations given by Sarma(1983) are as follows:

$$M(x, y, t) = U(x-y) \delta(x-y-t) \int_{t}^{\infty} h(u) \, du$$

$$+ \int_{0}^{t} h(u) \, du \int_{0}^{y+u} g(v) M(x, y+u-v, t-u) \, dv \quad (1.5.7)$$

and $M_i(x, y, t) = U(x-y) \delta(x-y-t) h(t) + \int_{0}^{t} h(u) \, du$

$$\int_{0}^{y+u} g(v) M_i(x, y+u-v, t-u) \, dv, \quad (1.5.8)$$

where $U(.)$ and $\delta(.)$ are heavy -side -unit -step function and Dirac-delta function respectively.

After introducing the above two integral equations for $M(x, y, t)$ and $M_i(x, y, t)$ he obtained analytic solutions for $M/M/1/FIFO/\alpha$ model by using double Laplace transform (d.l.t) technique which is defined as follows:
Definition (1.5.9)

The d.l.t. $F^* (s, p)$ of $f(y, t)$ is defined as

$$F^*(s, p) \overset{\text{def.}}{=} \int_0^\infty e^{-sy} \int_0^\infty e^{-pt} f(y, t) \, dt \, dy$$

Re. $s > 0$, Re. $p > 0$.

(1.5.10)

The d.l.t. $H^* (x, s, p)$ of $H(x, y, t)$ for $M/M/1/FIFO/\alpha$ model obtained by Sarma (1983) is given by:

$$H^* (x, s, p) = \frac{(s+\mu)}{-s^2 - s(\mu - p - \lambda) + \mu p} \left[ e^{-sx} - e^{-(\lambda + p)x} \right]$$

$$- \frac{\lambda \mu}{(\lambda + p + \mu)} H^* (x, \lambda + p, p) \right). \quad (1.5.11)$$

Similarly, the d.l.t. $H^*_1 (x, s, p)$ of $H(x, y, t)$ for $M/M/1/FIFO/\alpha$ model is given by:

$$H^*_1 (x, s, p) = \frac{\lambda(s+\mu)}{-s^2 - s(\mu - p - \lambda) + \mu p} \left[ e^{-sx} - e^{-(\lambda + p)x} \right]$$

$$- \frac{\mu}{(\lambda + p + \mu)} H^*_1 (x, \lambda + p, p) \right]. \quad (1.5.12)$$
A critical comparison of the results in (1.5.11) and (1.5.12) reveals that:

The non-homogeneous term in (1.5.12) has a multiplicative constant $\lambda$. This is because of the fact that in $M_i(x,y,t)$ there is a withdrawal at epoch $t$.

1.6 Chapter summaries:

Chapter I

The present chapter is an introductory one in nature, where stochastic storage models with particular reference to stochastic banking models are introduced first. Later one section is completely devoted to review the earlier literature relevant to the topic of this dissertation and derived the results obtained by Sarma (1983) for $M/M/1/FIFO/\alpha$ stochastic banking model. Two useful conditional probability density functions $M(x,y,t)$ and $M_i(x,y,t)$ introduced by Sarma (1983), integral equations formed by him, and the solutions of the functions are given as a ready reference.
Chapter II

In the second chapter, first, the role played by a banking system with particular reference to Indian Economy is given. After introducing the two conditional probability density functions $M(x,y,t)$ and $M_1(x,y,t)$ on similar lines of Sarma (1983) integral equations are formed. Analytic solutions for these two functions for $G_k/M/1/FIFO/\alpha$ stochastic banking model are obtained, where $G_k$ denotes a Gamma variate with $K$ stages. In other words here it is assumed that the random variable $u$ governing the inter-withdrawal times is assumed to follow a Gamma distribution with $K$ stages. Results obtained by Sarma (1983) are derived as special cases.

Chapter III

In this chapter a stochastic banking model is considered when the amount of withdrawals are drawn in different phases. Thus here it is assumed that the $r.v., \upsilon$ follows a Gamma distribution with $n$ stages. Analytic solutions for $M(x,y,t)$ and $M_1(x,y,t)$ are obtained for $M/G_n/1/FIFO/\alpha$ stochastic banking model. Earlier results
obtained by Sarma (1983) are derived as special cases. Finally, further scope of the work is also discussed in concluding section. The thesis is appended with various references coming across this dissertation.