CHAPTER IV

MHD thermal convection flow through a porous medium in a vertical channel induced by travelling thermal waves
Thermal convection problem in porous media occurs in a broad spectrum of the disciplines ranging from Chemical Engineering to Geophysics. Applications include heat insulations by fibrous materials, spreading of pollutants, convection of Earth's mantle, and large cross-section of fundamental research has been carried out by several authors in the recent times. The unsteadiness in the flow may be due to time dependent free stream oscillations or oscillatory flux or the time dependent boundary movement. Likewise, the time dependent convection flows may be generated due to heat transfer in an oscillatory fluid flow bounded by walls maintained at non-uniform temperature or boundaries maintained at periodically varying temperatures. Such convection flows generated by a periodic boundary thermal wave has received attention in the recent years due to its applications in...
the design of oil or gas fired boilers and a few other physical phenomena. It has been shown that a travelling thermal wave can generate a mean shear flow within a layer of fluid and also gives rise to a significant secondary flow in the field. This analysis of convection flows generated due to these travelling thermal waves has been studied by Whitehead (8), Nanda and Purushothaman (3), Vajravelu and Debnath (6), Vajravelu (7), Ganapathy and Purushothaman (1). In some of these papers the boundaries are chosen to be non-uniform. In the said investigation by Vajravelu and Debnath (6) the convection flow is generated by travelling thermal waves prescribed at the wavy walls of the channel. The perturbation technique is used to obtain the mean and the perturbed flow under longwave approximation for four different possible configurations of the wavy channel. Nanda and Purushothaman (3) have discussed a similar problem related to flow through a vertical cylindrical pipe and the same was extended to flow through porous medium by Ganapathy and Purushothaman (1). The problem has also been extended to hydromagnetics by Vajravelu (7), who showed that the
magnetic parameter has strong effect on the stream function and the temperature distribution in the presence of a heat source. However, this study of convection flows due to imposed thermal boundary waves has not been done in horizontal channel flows bounded by uniform or non-uniform horizontal waves. Recently Ravindra (4) has discussed the unsteady mixed convection effects on the flow of an incompressible viscous, electrically conducting fluid through a porous medium in a vertical channel.

Keeping the above mentioned facts in view we would like to discuss the unsteady hydromagnetic thermal convection flow due to the imposed travelling thermal boundary waves through a vertical channel bounded by flat walls. The effect of free convection on the flow has been discussed by solving the governing unsteady non-linear equations under perturbation scheme. The shear stress and the average Nusselt number on the boundaries have been evaluated for different variations of the governing parameters.
4.2. FORMULATION OF THE PROBLEM

We consider the motion of an incompressible, viscous, electrically conducting fluid through a porous medium in a vertical channel bounded by flat walls. Choosing a rectangular cartesian system O(x,y) the channel walls are at $y=\pm L$ with x-axis in the vertical direction. An uniform magnetic field of strength $H_0$ is applied transverse to the walls. The thermal buoyancy in the flow field is created by a travelling thermal wave imposed on the boundary wall at $y=L$ while the boundary at $y=-L$ is maintained at constant temperature $T_1$. The Boussinesq approximation is used so that the density variation will be retained only in the buoyancy force. The viscous-dissipation is neglected in comparison to the heat flow by conduction and convection.
The equations governing the flow, and heat transfer are as follows.

Equation of continuity

\[ u_x + v_y = 0 \] (4.2.1)

Equations of linear momentum

\[ \rho \frac{d}{dt} (u_x u_x + u_y + v_y u_y) = -p_x + \mu \frac{d^2}{dy^2} u_x - \rho g \frac{h}{k} u - \rho \mu \frac{h^2}{k} u \] (4.2.2)

\[ \rho \frac{d}{dt} (v_x u_x + v_y + v_y v_y) = -p_y + \mu \frac{d^2}{dy^2} v_y - \rho \mu \frac{h^2}{k} v \] (4.2.3)

Equation of Energy

\[ \rho c_p \left( T_x + u_T + v_T \right) = k \frac{d^2}{dy^2} T + Q \] (4.2.4)

Equation of state

\[ \rho = \rho_0 \left[ 1 - \beta (T - T_0) \right] \] (4.2.5)

where \( u, v \) are the velocity components along \( 0(x, y) \) directions respectively, \( T \) is the temperature, \( \rho \) is the
pressure, \( \rho \) is the density, \( \mu \) is the coefficient of kinematic viscosity, \( k \) is the permeability coefficient, \( k_1 \) is the coefficient of thermal conductivity, \( \rho_0, T_0 \) are the equilibrium density and temperature respectively, \( C_p \) is the specific heat at constant pressure, \( \beta \) is the coefficient of thermal expansion, \( Q \) is the strength of the heat source, \( \sigma \) is the electrical conductivity and \( \mu_e \) is the magnetic permeability. The flow is maintained by a constant volume flux for which a characteristic velocity is defined as

\[
q = \frac{1}{L} \int_{-L}^{L} u \, dy
\]  

(4.2.5)

The boundary conditions for the velocity, temperature fields are

\[
\begin{align*}
  u &= 0, \quad v = 0, \quad T = T_1 \quad \text{on } y = -L \\
  u &= 0, \quad v = 0, \quad T = T_e + \Delta T_e \sin(mx + nt) \quad \text{on } y = L
\end{align*}
\]  

(4.2.7)
where
\[ \Delta T = T_1 - T_0 \] and \( \sin(mx + nt) \) is the imposed travelling thermal wave.

In view of the continuity equation (4.2.1) we define the stream function \( \psi \) as

\[
\begin{align*}
\psi &= \frac{\psi}{a} \\
\frac{\psi}{a} &= -\psi_x, \quad v = \psi_y
\end{align*}
\]  

(4.2.6)

Eliminating pressure \( p \) in equation (4.2.2) and (4.2.3) and using (4.2.3) the equation governing the flow in terms of \( \psi \) are

\[
\left[ \begin{array}{c}
\psi_y^2 \psi - \psi_x \\
\psi_x \end{array} \right] = \nu \psi_y^4 - \beta g T - \psi_y \psi_x^2
\]

\[
- \frac{\sigma \mu^2 R^2}{\rho e^2} \psi_y
\]

(4.2.9)

Introducing the non-dimensional variables

\[
\kappa' = \kappa, \quad y' = y/L, \quad t' = t \nu a, \quad \psi' = \psi/\psi_L, \quad \theta' = \frac{T - T_0}{\Delta T}
\]  

(4.2.10)
the governing equations in the non-dimensional form (after dropping the dashes) are

\[
\frac{\partial}{\partial t} \left( \delta \left( \nabla^2 \psi \right) \right) + \nabla \cdot \left( \rho \nabla \psi \right) = \nabla \cdot \left( \sigma \nabla \nabla \psi \right) = \frac{\partial}{\partial x} \left( \sigma \nabla \psi \right)
\]

(4.2.11)

\[
\delta \left( \frac{\partial}{\partial t} \left( \rho \psi \right) \right) \frac{\partial}{\partial x} \left( \rho \psi \right) = \nabla \cdot \left( \sigma \nabla \psi \right)
\]

(4.2.12)

\[
\delta \frac{\partial^2}{\partial x^2} = \delta \left( \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \right)
\]

where

- \( \delta \) = \text{aspect ratio}
- \( G = \frac{\beta g \Delta T \delta L^3}{\nu^2} \) = Grashof number
- \( M^2 = \frac{\sigma \nu^2 \mu^2 \gamma^2}{\nu^2} \) = Hartmann number
- \( P = \frac{\mu c_p}{k_1} \) = Prandtl number
- \( \alpha = \frac{Q L^2}{k_1} \) = Heat source parameter
The corresponding boundary conditions are

\[ \psi(1) - \psi(0) = -1 \]

\[ \psi_y = 0, \quad \psi_x = 0, \quad \theta = 1, \quad \text{at } y = 1 \]

\[ \psi_y = 0, \quad \psi_x = 0, \quad \theta = \sin(k + \gamma t) \quad \text{at } y = 1 \]

(4.2.12)

4.3. ANÁLISIS DE LA CORRIENTE

Assuming the aspect ratio \( \delta \) to be small we take

\[ \psi(x, y, t) = \psi_0(x, y, t) + \delta \psi_1(x, y, t) + \delta^2 \psi_2(x, y, t) + \ldots \]

\[ \theta(x, y, t) = \theta_0(x, y, t) + \delta \theta_1(x, y, t) + \delta^2 \theta_2(x, y, t) + \ldots \] (4.3.1)
On substitution (4.3.1) in (4.2.11)-(4.2.12) and separating the like powers of $\delta$ the equations and respective conditions to the zeroth order are

\[ \psi_0, yyy - (d_1^2 + m^2) \psi_0, yy = -\alpha y + \delta a \]  \hspace{1cm} (4.3.2)

\[ \theta_0, yy = -\alpha \]  \hspace{1cm} (4.3.3)

with

\[ \psi_0(+1) - \psi_0(-1) = -1. \]

\[ \psi_0, y = 0, \quad \psi_0, \kappa = 0, \quad \theta_0 = 1, \quad \text{at } y = -1 \]

\[ \psi_0, y = 0, \quad \psi_0, \kappa = 0, \quad \theta_0 = \sin(x + yt) \quad \text{at } y = 1 \]  \hspace{1cm} (4.3.4)

and to the first order are

\[ \psi_1, yyy - (d_1^2 + m^2) \psi_1, yy = 6 \theta_1, y + \psi_0, y \psi_0, yy \psi_0, \kappa, \psi_0, yyy \]  \hspace{1cm} (4.3.5)

\[ \theta_1, yy = F(\psi_0, \kappa, \theta_0, \psi_0, \kappa) \]  \hspace{1cm} (4.3.6)

with

\[ \psi_1 = 0, \quad \psi_1, y = 0, \quad \psi_1, \kappa = 0, \quad \theta_1 = 1, \quad \text{at } y = -1 \]

\[ \psi_1 = 0, \quad \psi_1, y = 0, \quad \psi_1, \kappa = 0, \quad \theta_1 = 0, \quad \text{at } y = 1 \]  \hspace{1cm} (4.3.7)
Solving the equations (4.3.2) and (4.3.3) subject to the boundary conditions (4.3.4) we obtain

\[
\begin{align*}
\phi &= \frac{\pi}{2} (1-y^2) + \frac{1}{2} (1-y) + \frac{\sin(k+y)}{2} (1+y) \\
\psi_0 &= b_0 + b_2 y + b_3 y^2 + b_5 y^3 + b_6 y^4 + b_7 y^5 + b_8 y^6 + \ldots \\
\psi_1 &= a_0 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 + a_7 y^7 + a_8 y^8 + \ldots \\
\psi_2 &= c_0 y + c_2 y^2 + c_4 y^4 + \ldots \\
\phi(y) &= b_1 y + b_3 y^2 + b_5 y^3 + b_7 y^4 + b_9 y^5 + b_{11} y^6 + \ldots \\
&+ b_{21} y^{21} + \ldots
\end{align*}
\]

where

\[
\phi(y) = b_1 y + b_3 y^2 + b_5 y^3 + b_7 y^4 + b_9 y^5 + b_{11} y^6 + \ldots \\
+ b_{21} y^{21} + \ldots
\]
The shear stress on the channel walls is given by

$$\tau = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{y=\pm L}$$

which in the non-dimensional form reduces to

$$\tau' = \frac{\mu \psi}{\delta} = (\delta^2 \psi_{xx} - \psi_{yy}) = \left[ -\psi_0, yy - \delta \psi_1, yy + \delta^2 (\psi_0, xx - \psi_2, yy + \ldots \right]_{y=\pm 1}$$

and the corresponding expressions are

$$\langle \tau \rangle_{y=1} = -(g_2 + g_3 + \delta^2)$$

$$\langle \tau \rangle_{y=-1} = -(g_2 + g_4 + \delta^2)$$

The rate of heat transfer (Nusselt number Nu) is given by

$$q = -K_1 (T_y)_{y=\pm L}$$
which in the non-dimensional form reduces to

\[
\text{Nu} = \frac{-qL}{\Delta T_e} = (8 y)^{\gamma} y = 1 = (8_0, y + \delta, 0, 1, y)^{\gamma} y = 1
\]

and the corresponding expressions are

\[
(Nu)_{y=1} = \frac{8_0 \sin(x + \gamma t) + \delta \omega_0}{8_m - \sin(x + \gamma t)}
\]

\[
(Nu)_{y=-1} = \frac{8_0 \sin(x + \gamma t) - \delta \omega_0}{8_m - 1}
\]

where \(8_m = 8 + \sin(x + \gamma t) + \delta \omega_0\).
4.6. DISCUSSION OF THE NUMERICAL RESULTS

The profiles for velocity and temperature fields for different values of the governing parameters $\theta, \gamma, \sigma_1, M$ and $x$ are plotted in Figs. (12)-(18). When $\theta$ is positive, the boundaries are at a higher temperature than the equilibrium temperature and corresponds to the heating of the channel walls. When $\theta < 0$ the boundaries are at a lower temperature than the equilibrium temperature and hence corresponds to the cooling of the walls. In general, the flow is asymmetric. The actual axial flow $(u)$ is positive. The variation of the axial velocity $u$ with $\theta(x)$ is shown in the Fig. 1. It can be observed that for $\theta = 10^3$ the velocity $u$ changes from positive to negative.
Fig. 1

Variation of axial velocity $u$ with $\theta$

$\sigma_i = 2, M = 2, \gamma = 2, \alpha = 2, t = \pi/4, \kappa = \pi/4$

$a, b, c, d, e, f, g, h$

$6 \times 10^3$ $3 \times 10^3$ $5 \times 10^3$ $10^4$ $-10^3$ $-3 \times 10^3$ $-5 \times 10^3$ $-10^4$
Fig. 2

$u$ with $\sigma_1$

$G=2 \times 10^3$, $M=2$, $\gamma=2$, $\alpha=2$, $\kappa=\pi/4$

$\sigma_1 \begin{array}{cccc}
2 & 4 & 6
\end{array}$
as we move away from the left boundary (y=-1) to the right boundary (y=1) thereby exhibiting reversal flow in the in the right region (0≤y≤1) and for B=−10^3 it is confined to the left region (−1≤y≤0). This reversal flow disappears for higher |B|≥3×10^3. In the heating case the maximum u is attained in the left region near the boundary at y=−0.4 where as in the cooling case it is attained at y=0.6. However in either cases |u| rapidly increases with |B| in the entire flow field. An interesting point to be noted here is that in heating case the fluid in the left mid-region moves with greater axial velocity than the fluid in the right mid-region while for B<0 the fluid movement in the right region is faster than that in the left region. Figs. 2 and 3 gives the variation of u with reference to σ_1 and M. We observe that the reversal flow which appears near the right boundary for σ_1,2 shifts to the left boundary for higher σ_1 and its size increase with σ_1. In general for σ_1<6 the magnitude of u reduces in the left region and increases in the right region with an increase in σ_1. With reference to M we find a reversal flow near the right boundary for (weak magnetic field) M<1.5, disappears at M=3 and appears at the left boundary.
Fig. 3

\[ u \text{ with } M \]

\[ \theta = 2 \times 10^3, \; \gamma = 2, \; \alpha = 2, \; x = t = \pi/4 \]

\[ M \quad 0.2 \quad 1 \quad 1.5 \quad 3 \quad 5 \]
for M=5. An increase in the intensity of magnetic field the axial velocity increases near both the boundaries while in the region (-0.4≤y≤0.4) we find a retarding nature in u(Fig.3). Figs.4 and 5 indicates the variation of u with different y at a given axial position x as well as at different axial positions x for a given y. The behaviour in general in either of the cases u oscillatory with magnitude fluctuating in view of the periodic wall temperature. With the increase in the thermal wave velocity γ the reversal flow which appears near the right boundary for γ=2 spreads to the entire fluid region shrinks in size for γ=5 and enlarges towards the heat boundary for further increases in γ>20. Also for fixed γ we find the reversal flow near the right boundary for x=π/4 spreads to the entire fluid region for x=3π/4 and disappears for higher x. Fig.6 gives the variation of u with heat source parameter a. We find a reversal flow for near the right boundary for smaller intensities of the heat source parameter a≤2 and disappears with increase in the intensity of the heat source while the reversal flow enlarges to the entire flow field with an increase in the
Figure 7: Variation of transverse velocity $v$ with $G$.

Figure 6: $u$ with $N$.

$G = 10^{-2}$, $S = 2$, $d = 2$, $t = \pi/4$, $v = 2$

$N = 2\times10^{-3}$, $S = 2$, $d = 2$, $t = \pi/4$

$N = 2\times10^{-4}$, $S = 2$, $d = 2$, $t = \pi/4$

$N = 2\times10^{-5}$, $S = 2$, $d = 2$, $t = \pi/4$
Fig. 8

$\nu$ with $\sigma_1$

$G = 2 \times 10^3$, $M = 2$, $\gamma = 2$, $\alpha = 1/4$, $\alpha = 2$

$a$, $b$, $c$, $d$, $\sigma_1$, $2$, $4$, $6$

Fig. 9

$\nu$ with $M$

$G = 2 \times 10^3$, $\gamma = 2$, $\sigma_1 = 2$, $\alpha = 1/4$, $\alpha = 2$

$a$, $b$, $c$, $d$, $m$, $0.5$, $1$, $1.5$, $3$
Fig. 10

$\nu$ with $\gamma$

$B=2 \times 10^3$, $M=2$, $\alpha=2$, $\pi=\pi/4$, $\sigma_t=2$

a b c d

$\gamma$ 2 5 10 20
The secondary velocity $v$ is plotted in Figs. (7)-(12) for different variations in the governing parameters. It is found that the profiles for $v$ are asymmetric bell shaped curves with maximum attained in the mid-region for $S>0$ and $S<0$. The secondary velocity ($v$) is directed towards the boundary both in the heating and cooling cases except for $S=10^{-3}$ in which case it is directed towards the mid-plane. The magnitude of $v$ increases with increase in $|S|$ (Fig. 7). It is observed that the magnitude of $u$ in the heating case is always greater than that in the cooling case. The variation of $v$ with $\sigma_1$ and $M$ can be observed from Figs. 8 and 9. For all variations in $\sigma_1$ and $M$ the velocity profiles are always directed to the boundary. As $\sigma_1$ increases we find that the profiles of $v$ tendency to symmetric with maximum value in the mid-region. The fluid moves lesser velocity in the lesser permeable medium in the region $-0.6<y<0.6$ while it moves with greater velocity near the boundaries. But for further increase in $\sigma_1$ the velocity $v$ increases in the
Fig. 11
v with $\alpha$

$g = 2 \times 10^3$, $\gamma = 2$, $\sigma = 2$, $\sigma_1 = 2$, $M = 2$, $\kappa = \Pi/4$

$a$, $b$, $c$, $d$, $e$, $f$

$\alpha = 2, 5, 10, -2, -5, -10$

Fig. 12
v with $\kappa$

$g = 2 \times 10^3$, $\gamma = 2$, $\sigma = 2$, $M = 2$, $\sigma_1 = 2$

$a$, $b$, $c$, $d$, $e$, $f$

$\kappa = \Pi/4, \Pi/2, 3\Pi/4, 5\Pi/4, \Pi/4, 2\Pi$
entire fluid region. An increase in $M$ retards $v$ in the entire fluid region although it has no effect on its phase with the increase in the magnetic field intensity the point of maximum $v$ shifts towards the left boundary. The effect of the thermal wave velocity on $v$ may be observed from Fig.10. Which includes profiles at different axial positions for a given wave velocity and vice-versa. As in the case of $u$ the magnitude of $v$ periodically varies with variation in $\kappa$ or $\gamma$. Also we find that the transition of its direction from towards the boundary to towards the mid-plane as we move along the axial direction. Thus transition takes place at $x=5\pi/4$ (Fig.11). A similar observation may be made with reference to $\gamma$ (Fig.10). Fig.12 shows the variation of $v$ with $\alpha$. It is found that for all values of $\alpha(<0)$, $v$ is always directed towards the boundary. An increase in the intensity of the heat source parameter $\alpha$, the secondary velocity $v$ increases in the left half and decreases in the right half of the channel. A reversed effect is observed with increase in the strength of the heat sink ($\alpha>0$).
Fig. 13
Variation of temperature with G

Fig. 14
Variation of temperature B with G

σ = 2, M = 2, γ = 2, α = 2

σ = 1, M = 2, γ = 2, α = 2
The behaviour of the temperature distribution \( \theta \) for Prandtl number \( P=0.71 \) can be observed in Figs. (13)-(18). The perturbed temperature in general is positive and hence contribute to the enhancement of the actual temperature in the field. Fig. 12 depicts the behaviour of \( \theta \) for different \( G(\xi,0) \). The temperature profiles exhibit that for all \( G(\xi,0) \) the temperature gradually falls from its maximum on the left boundary to a minimum attained on the right boundary as long as the argument \((x+yt)\) of the travelling thermal wave is \( \pi/4 \). The magnitude of \( \theta \) increases in the left half and decreases in the upper of the channel with an increase \( G(\xi,0) \). For \( G(0,0) \) decreases in the left region and increases in the right half. The temperature in the heating case is less than that in the cooling case in the lower half while the reverse is true in the upper half. From Figs. 14 and 15 find that an increase in \( \sigma_1 \) or \( M \) reduces \( \theta \) in the upper half while it exhibits an increasing tendency in the lower half. The influence of the boundary thermal wave on the temperature in the flow field may be observed from Fig. 16. We notice that for a
Fig. 17

θ with α

G = 2 \times 10^3, \alpha = 2, M = 2, x =

a b c d e f

α 2 5 10 -2 -5 -10

Fig. 18

θ with κ

κ \pi/4 \pi/2 \pi/4 \pi/4 \pi/4 \pi/4 α

Opening
given $\gamma$ as we move along the axial direction up to $x=\pi/4$. The temperature decreases in the region $(-0.8 \leq y \leq 0.2)$ while exhibiting an oscillatory tendency near the right thermally varying boundary. This phenomenon periodically recurs in a period of $2\pi$. The temperature at any point in the right mid-half is less than that at the corresponding position in the left mid-half. At a fixed axial position an increase in $\gamma$. The temperature exhibits an oscillatory nature with the temperature in greater half less than that in the right half for any given $\gamma$. From Fig. 17 we find that the temperature is greater than that of the equilibrium temperature for $\alpha>0$ and it increases all points in the fluid region with increase in heat source parameter $\alpha$. In the case of heat sink the temperature is lower than that of the equilibrium temperature. With an increase in the intensity of heat since $\alpha>0.5$ the temperature increases in the region $(-0.6 \leq y \leq 0.6)$ while if reduces in its magnitudes near the boundaries. For higher values of $|\alpha|$ we find an enhancement in its magnitude. The temperature at any point in the left half is always greater than that at the
**TABLE 1**

**SHEAR STRESS AT Y=1**

**WITH X+γL=π/4**

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<th>II</th>
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TABLE 2

SHEAR STRESS AT Y = -1

σ₁ = 2, M = 2, α = 2

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<td>10⁴</td>
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<td>23.558</td>
<td>26.430</td>
</tr>
<tr>
<td>-10³</td>
<td>3.721</td>
<td>0.156</td>
<td>-3.003</td>
</tr>
<tr>
<td>-3 x 10³</td>
<td>4.637</td>
<td>-6.067</td>
<td>-17.923</td>
</tr>
<tr>
<td>-5 x 10³</td>
<td>4.963</td>
<td>-12.895</td>
<td>-33.630</td>
</tr>
<tr>
<td>-10⁴</td>
<td>5.196</td>
<td>-32.617</td>
<td>-78.950</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k+γt) π/4</td>
<td>π/2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>π/4</td>
<td>3π/4</td>
<td>5π/4</td>
</tr>
</tbody>
</table>
### TABLE 3

**SHEAR STRESS AT Y=1**

*WITH X+Y=π/4*

<table>
<thead>
<tr>
<th>G</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>2.233</td>
<td>13.721</td>
<td>77.284</td>
<td>-0.930</td>
<td>-1.704</td>
<td>-1.236</td>
<td>6.059</td>
<td>10.328</td>
</tr>
<tr>
<td>$10^4$</td>
<td>-21.500</td>
<td>5.933</td>
<td>78.797</td>
<td>-51.670</td>
<td>-65.822</td>
<td>-77.046</td>
<td>52.559</td>
<td>108.105</td>
</tr>
<tr>
<td>$-10^3$</td>
<td>3.489</td>
<td>13.446</td>
<td>76.094</td>
<td>2.470</td>
<td>2.606</td>
<td>6.495</td>
<td>-0.519</td>
<td>-3.525</td>
</tr>
<tr>
<td>$-3\times10^3$</td>
<td>3.284</td>
<td>12.442</td>
<td>74.514</td>
<td>3.007</td>
<td>3.303</td>
<td>10.913</td>
<td>-6.889</td>
<td>-14.517</td>
</tr>
<tr>
<td>$-5\times10^3$</td>
<td>0.785</td>
<td>10.709</td>
<td>72.541</td>
<td>0.681</td>
<td>0.386</td>
<td>12.916</td>
<td>-12.246</td>
<td>-22.645</td>
</tr>
</tbody>
</table>

\[ \begin{array}{cccccccc} 
\sigma_1 & 2 & 3 & 4 & 2 & 2 & 2 & 2 \\
M & 2 & 2 & 2 & 1 & 0.5 & 2 & 2 \\
\alpha & 2 & 2 & 2 & 2 & 2 & -2 & -5 \\
\end{array} \]
Table 4
Shear Stress at Y=1
With c1=2, m=2, a=2

<table>
<thead>
<tr>
<th>G</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^3)</td>
<td>2.233</td>
<td>5.737</td>
<td>8.080</td>
<td>9.858</td>
</tr>
<tr>
<td>(3 \times 10^3)</td>
<td>-3.484</td>
<td>9.690</td>
<td>19.233</td>
<td>19.653</td>
</tr>
<tr>
<td>(5 \times 10^3)</td>
<td>-4.662</td>
<td>11.731</td>
<td>22.835</td>
<td>26.682</td>
</tr>
<tr>
<td>(10^4)</td>
<td>-21.500</td>
<td>18.475</td>
<td>28.683</td>
<td>24.764</td>
</tr>
<tr>
<td>(-10^3)</td>
<td>3.489</td>
<td>-0.125</td>
<td>-3.623</td>
<td>-3.706</td>
</tr>
<tr>
<td>(-3 \times 10^3)</td>
<td>3.284</td>
<td>-7.898</td>
<td>18.877</td>
<td>-20.040</td>
</tr>
<tr>
<td>(-5 \times 10^3)</td>
<td>0.785</td>
<td>-17.583</td>
<td>-36.682</td>
<td>-40.143</td>
</tr>
<tr>
<td>(-10^4)</td>
<td>0.785</td>
<td>-50.153</td>
<td>-92.352</td>
<td>-106.888</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x+y)</th>
<th>(\pi/4)</th>
<th>(\pi/2)</th>
<th>(3\pi/4)</th>
<th>(\pi/4)</th>
</tr>
</thead>
</table>

\(\pi/4\)  \(\pi/2\)  \(3\pi/4\)  \(\pi/4\)
corresponding point in the right half of the channel (Fig. 18).

The shear stress and the average Nusselt number (Nu) on the boundaries \( y=\pm 1 \) have been evaluated for different \( \Theta, \sigma_1, M, \alpha \) and \( \gamma \) and are given in tables (1)-(8). The shear stress on the wall \( y=-1 \) increases in magnitude with increase in \( \Theta \) in the heating and cooling cases at \( \sigma_1=2 \). But for \( \sigma_1=3 \) it decreases in magnitude and for higher value of \( \sigma_1 \) it increases in heating case and decreases in cooling case. Also it increases with \( \sigma_1 \) and \( \gamma \) at both the boundaries. With reference to \( M \) we find that \( \tau \) increases in its magnitude at \( y=-1 \) for all values of \( M \) while at \( y=+1 \) we find an increasing tendency in \( \tau \) with increase in \( M \) through smaller values and at \( M=2 \) it decreases/increases according the wall is heated/cooled. We note that an increase in the intensity of heat source/sink enhances \( \tau \) at \( y=-1 \) while it decreases/increases with increases in the strength of heat source/sink for \( \Theta \geq 0 \) at \( y=+1 \) (Table 3 and 4).
### Table 5

**Average Nusselt Number on Y = -1**

*with x+y = π/4*

<table>
<thead>
<tr>
<th>G</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^3</td>
<td>2.143</td>
<td>2.069</td>
<td>1.926</td>
<td>2.174</td>
<td>2.181</td>
<td>1.970</td>
<td>1.600</td>
<td>1.753</td>
</tr>
<tr>
<td>3×10^3</td>
<td>1.850</td>
<td>1.640</td>
<td>1.207</td>
<td>1.942</td>
<td>1.961</td>
<td>1.687</td>
<td>1.336</td>
<td>1.482</td>
</tr>
<tr>
<td>5×10^3</td>
<td>1.562</td>
<td>1.214</td>
<td>0.502</td>
<td>1.715</td>
<td>1.746</td>
<td>1.407</td>
<td>1.074</td>
<td>1.213</td>
</tr>
<tr>
<td>10^4</td>
<td>0.857</td>
<td>0.167</td>
<td>0.241</td>
<td>1.165</td>
<td>1.228</td>
<td>0.721</td>
<td>0.432</td>
<td>0.556</td>
</tr>
<tr>
<td>-10^3</td>
<td>2.439</td>
<td>2.502</td>
<td>2.630</td>
<td>2.411</td>
<td>2.405</td>
<td>2.257</td>
<td>1.866</td>
<td>2.028</td>
</tr>
<tr>
<td>-3×10^3</td>
<td>2.739</td>
<td>2.940</td>
<td>3.348</td>
<td>2.651</td>
<td>2.634</td>
<td>2.546</td>
<td>2.136</td>
<td>2.306</td>
</tr>
<tr>
<td>-5×10^3</td>
<td>3.044</td>
<td>3.381</td>
<td>4.070</td>
<td>2.897</td>
<td>2.867</td>
<td>2.840</td>
<td>2.408</td>
<td>2.587</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_i</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>M</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>α</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>-2</td>
<td>-5</td>
</tr>
</tbody>
</table>
### Table 8

**Average Nusselt Number on Y-1**

\( \sigma_1=2, \ M=2, \ \alpha=2 \)

<table>
<thead>
<tr>
<th>( G )</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^3 )</td>
<td>2.143</td>
<td>1.865</td>
<td>2.122</td>
<td>-1.360</td>
</tr>
<tr>
<td>( 3 \times 10^3 )</td>
<td>1.950</td>
<td>1.636</td>
<td>1.826</td>
<td>-0.757</td>
</tr>
<tr>
<td>( 5 \times 10^3 )</td>
<td>1.562</td>
<td>1.409</td>
<td>1.534</td>
<td>-0.141</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>0.857</td>
<td>0.854</td>
<td>0.819</td>
<td>1.459</td>
</tr>
<tr>
<td>(-10^3 )</td>
<td>2.439</td>
<td>2.097</td>
<td>2.421</td>
<td>-1.949</td>
</tr>
<tr>
<td>(-3 \times 10^3 )</td>
<td>2.739</td>
<td>2.331</td>
<td>2.723</td>
<td>-2.524</td>
</tr>
<tr>
<td>(-5 \times 10^3 )</td>
<td>3.044</td>
<td>2.567</td>
<td>3.028</td>
<td>-3.084</td>
</tr>
<tr>
<td>(-10^4 )</td>
<td>3.823</td>
<td>3.168</td>
<td>3.793</td>
<td>-4.404</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
</table>

\((x+y) \ \pi/4 \ \pi/2 \ 3\pi/4 \ 5\pi/4\)
### TABLE 7

**AVERAGE NUSSELT NUMBER ON Y=1**

*WITH X+Y=π/4*

<table>
<thead>
<tr>
<th>G</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^3</td>
<td>-1.311</td>
<td>-1.254</td>
<td>-1.137</td>
<td>-1.337</td>
<td>-1.342</td>
<td>-1.610</td>
<td>-2.773</td>
<td>-2.164</td>
</tr>
<tr>
<td>3x10^3</td>
<td>-1.091</td>
<td>-0.928</td>
<td>-0.595</td>
<td>-1.165</td>
<td>-1.180</td>
<td>-1.359</td>
<td>-2.399</td>
<td>-1.855</td>
</tr>
<tr>
<td>5x10^3</td>
<td>-0.873</td>
<td>-0.603</td>
<td>-0.455</td>
<td>-0.996</td>
<td>-1.021</td>
<td>-1.110</td>
<td>-2.030</td>
<td>-1.550</td>
</tr>
<tr>
<td>10^4</td>
<td>-0.339</td>
<td>0.197</td>
<td>-0.281</td>
<td>-0.583</td>
<td>-0.635</td>
<td>-0.500</td>
<td>-1.129</td>
<td>-0.805</td>
</tr>
<tr>
<td>-10^3</td>
<td>-1.533</td>
<td>-1.583</td>
<td>-1.601</td>
<td>-1.511</td>
<td>-1.506</td>
<td>-1.864</td>
<td>-3.152</td>
<td>-2.477</td>
</tr>
<tr>
<td>-3x10^3</td>
<td>-1.759</td>
<td>-1.914</td>
<td>-2.228</td>
<td>-1.866</td>
<td>-1.841</td>
<td>-2.390</td>
<td>-3.926</td>
<td>-3.116</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>ω_1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>M</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>α</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>-2</td>
<td>-5</td>
</tr>
</tbody>
</table>
### TABLE 8

**AVERAGE NUSSELT NUMBER ON Y=1**

\( \sigma_1=2, \, M=2, \, \alpha=2 \)

<table>
<thead>
<tr>
<th>( G )</th>
<th>( I )</th>
<th>( II )</th>
<th>( III )</th>
<th>( IV )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^3 )</td>
<td>-1.311</td>
<td>-1.067</td>
<td>-1.298</td>
<td>-2.424</td>
</tr>
<tr>
<td>( 3\times10^3 )</td>
<td>-1.091</td>
<td>-0.843</td>
<td>-1.073</td>
<td>-2.194</td>
</tr>
<tr>
<td>( 5\times10^3 )</td>
<td>-0.873</td>
<td>-0.622</td>
<td>-0.853</td>
<td>-1.965</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>-0.339</td>
<td>-0.080</td>
<td>-0.311</td>
<td>-1.390</td>
</tr>
<tr>
<td>( -10^3 )</td>
<td>-1.533</td>
<td>-1.294</td>
<td>-1.520</td>
<td>-2.654</td>
</tr>
<tr>
<td>( -3\times10^3 )</td>
<td>-1.759</td>
<td>-1.734</td>
<td>-1.972</td>
<td>-3.117</td>
</tr>
<tr>
<td>( -5\times10^3 )</td>
<td>-1.985</td>
<td>-1.754</td>
<td>-1.972</td>
<td>-3.117</td>
</tr>
<tr>
<td>( -10^4 )</td>
<td>-2.562</td>
<td>-2.341</td>
<td>-2.536</td>
<td>-3.689</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( I )</th>
<th>( II )</th>
<th>( III )</th>
<th>( IV )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x+y/t )</td>
<td>( \pi/4 )</td>
<td>( \pi/2 )</td>
<td>( 3\pi/4 )</td>
</tr>
</tbody>
</table>
The average Nusselt number measures the local rate of heat transfer across the boundary. We find from table 5 and 7 that the average Nusselt number (Nu) is negative at $y=+1$ and positive at $y=-1$ for variation in $G, \sigma_1, M, \alpha$ and $\gamma$. We notice that Nu at both the walls decreases in the heating case and increases in the cooling case except that for $\sigma_1=4$ the Nusselt number $y=+1$ increases in its magnitude in heating and cooling cases. As the intensity of the magnetic field increases we find a reduction/enhancement in Nu according as the walls $y=\pm 1$ are heated/coolated. An increases in the strength of the heat source/sink increases the Nusselt number at $y=-1$ and decreases in magnitude at $y=+1$. Also we notice that the Nusselt number at $y=+1$ increases while it exhibits an oscillatory nature at $y=-1$ as move along the axial direction in view of the periodic variation of the boundary temperature (Table 6 and 8).
APPENDIX

\[
\beta_1^2 = \sigma_1^2 + M^2
\]

\[
a_1 = \frac{(\sin(x+yt)-1)}{2} \quad ; \quad a_2 = \frac{a_6}{\beta_1}
\]

\[
a_3 = \frac{a_1^2}{2} \quad ; \quad a_6 = 2a_3/\beta_1\text{Sh}\beta_1
\]

\[
a_4 = -a_6\text{Ch}\beta_1
\]

\[
a_5 = a_2+1-\left(\frac{2a_2+0.5}{\text{Sh}\beta_1-\beta_1\text{Ch}\beta_1}\right)\text{Sh}\beta_1
\]

\[
a_6 = \frac{2a_3}{\beta_1\text{Sh}\beta_1} \quad ; \quad a_7 = \frac{2a_2+0.5}{\text{Sh}\beta_1-\beta_1\text{Ch}\beta_1}
\]

\[
a_8 = -a_1\text{Ch}\beta_1 \quad ; \quad a_9 = -a_1a_2-a_1\text{Sh}\beta_1
\]

\[
a_{10} = a_1a_2+a_1a_2^2 \quad ; \quad a_{11} = -a_2
\]

\[
a_{12} = -a_1a_6 \quad ; \quad a_{13} = -a_6
\]

\[
a_{14} = -a_7 \quad ; \quad a_{15} = -a_7
\]
\[ a_{16} = \frac{A_7}{2CH_1} + 3a_2 \quad ; \quad a_{17} = \frac{(-3a_2 - 2a_3 - A_7CH_1)}{2} \]

\[ a_{18} = -3a_2 - 2a_3 \quad ; \quad a_{19} = 3a_2 \]

\[ a_{20} = A_1 \cos(x + yt) \quad ; \quad a_{21} = a_1 \cos(x + yt)/2 \]

\[ a_{22} = a_3 \cos(x + yt) / \sin \beta_1 \quad ; \quad a_{23} = a_3 \cos(x + yt) / \sin \beta_1 \]

\[ a_{24} = a_8 - a_{16} \quad ; \quad a_{25} = a_9 - a_{17} \]

\[ a_{26} = a_{10} - a_{18} \quad ; \quad a_{27} = a_{11} - a_{19} \]

\[ a_{28} = a_{12} - a_{20} \quad ; \quad a_{29} = a_{13} - a_{21} \]

\[ a_{30} = a_{14} - a_{22} \quad ; \quad a_{31} = a_{15} - a_{23} \]

\[ a_{32} = P_{12}a_{24} / 2 \quad ; \quad a_{33} = P_{12}a_{29} / 6 \]

\[ a_{34} = P_{12}a_{26} / 24 \quad ; \quad a_{35} = P_{12}a_{27} / 120 \]

\[ a_{36} = P_{12}(a_{28}/\beta_1^2 - (2a_{31}/\beta_1^2)) \]

\[ a_{37} = P_{12}a_{29}/\beta_1^2 \quad ; \quad a_{38} = a_{39}/\beta_1^2 - 2a_{27}/\beta_1^2 \]

\[ a_{39} = a_{31}/\beta_1^2 \]
\[
\begin{align*}
\alpha_{40} &= -(a_{33} + a_{35} + a_{37} \Delta h_1 + a_{38} \Delta h_1) \\
\alpha_{41} &= -(a_{32} + a_{34} + a_{36} \Delta h_1 + a_{39} \Delta h_1) \\
\alpha_{42} &= -(6a_{2,3} \Delta \beta_1) \quad ; \quad \alpha_{43} = -\beta_1 a_{2,3} \Delta \beta_1 -(4a_{2,3} \Delta \beta_1) \\
\alpha_{44} &= -a_{3,2} \Delta \beta_1 \\
\alpha_{45} &= 3a_{2,2} \Delta \beta_1 \\
\alpha_{46} &= (2a_{2,2} \Delta \beta_1) (-\beta_1 \Delta \beta_1) - 1 \\
\alpha_{47} &= -a_{2,2} \Delta \beta_1 \Delta \beta_1 \\
\alpha_{48} &= 6a_{2,2} \Delta \beta_1 \Delta \beta_1 \\
\alpha_{49} &= A_{2,2} / 2 + a_{2,2} \Delta \beta_1 / 2 \\
\alpha_{50} &= a_{1,2} \Delta \beta_1 \Delta \beta_1 - 2a_{1,6} a_{1,7} / 2 \\
\alpha_{51} &= -6a_{2,2} = 0 \\
\alpha_{52} &= -6a_{2,2} = 0 \\
\alpha_{53} &= -6a_{2,6} = 0 \\
\alpha_{54} &= -6a_{2,6} = 0 \\
\alpha_{55} &= \beta_1^2 a_{1,7} = 0 \\
\alpha_{56} &= 6a_{2,7} = 0 
\end{align*}
\]
\[ a_{57} = -2a_{2}a_{2} \beta_{1}^{2} / \text{Sh} \beta_{1} = 0 \quad \text{and} \quad a_{58} = 2a_{2}a_{2} \beta_{1}^{2} / \text{Sh} \beta_{1} = 0 \]
\[ a_{59} = 2a_{3}a_{3} / \text{Sh} \beta_{1} \]
\[ a_{60} = 2a_{4}a_{4} / \text{Sh} \beta_{1} \]
\[ b_{1} = a_{40} + a_{39} \]
\[ b_{2} = 26a_{32} - 6a_{38} \text{Sh} \beta_{1} + a_{50} - a_{40} \]
\[ b_{3} = a_{41} \]
\[ b_{4} = 46a_{34} + a_{51} \]
\[ b_{5} = 6(\beta_{1}a_{38} - a_{37} + a_{32} - a_{45}) \]
\[ b_{6} = 66a_{59} + a_{53} - a_{46} \]
\[ b_{7} = -a_{47} \]
\[ b_{8} = a_{54} \]
\[ b_{9} = -Ga_{39} + a_{55} + a_{55} - a_{42} \]
\[ b_{10} = 66a_{37} + a_{55} - a_{42} \]
\[ b_{11} = a_{57} \]
\[ b_{12} = a_{58} \]
\[ b_{13} = a_{59} - a_{48} \]
\[ b_{14} = -\frac{2b_{3}}{\beta_{1}} \]
\[ b_{15} = \frac{b_{1}}{\beta_{1}} \]
\[ b_{16} = -\frac{b_{4}}{\beta_{1}} \]
\[ b_{17} = -\frac{b_{2}}{b_{1}} - \frac{b_{4}}{\beta_{1}} \]
\[ b_{18} = -\frac{b_{4}}{\beta_{1}} \]
\[ b_{19} = -\frac{5b_{6}}{2\beta_{1}} + \frac{3b_{8}}{2\beta_{1}} + \frac{b_{9}}{2\beta_{1}} \]
\[ b_{20} = -\frac{b_7}{2\beta_1} + \frac{b_{10}}{2\beta_1} + \frac{5b_{11}}{8\beta_1} \]

\[ b_{21} = -\frac{5b_6}{4\beta_1} \]

\[ b_{22} = \frac{b_{11}}{8\beta_1} \]

\[ b_{23} = \frac{b_5}{2\beta_1} + \frac{13b_7}{2\beta_1} - \frac{5b_{10}}{4\beta_1} + \frac{23b_{11}}{8\beta_1} \]

\[ b_{24} = \frac{b_6}{2\beta_1} + \frac{5b_8}{8\beta_1} \]

\[ b_{25} = \frac{b_7}{3\beta_1} + \frac{15b_{11}}{12\beta_1} \]

\[ b_{26} = \frac{1}{\beta_1} \]

\[ b_{27} = \frac{b_{13}}{4\beta_1} \]

\[ b_{28} = b_{29} \beta_1 \]

\[ b_{29} = -3a_2 - b_3 \beta_1 \beta_1 \]

\[ b_{30} = \frac{2a_3}{\beta_1 \operatorname{Sh} \beta_1} \]

\[ b_{31} = \frac{2a_2 + 0.5q}{\operatorname{Sh} \beta_1 - \beta_1 \beta_1} \]

\[ C_2 = -C_4 \operatorname{Sh} \beta_1 - \frac{Q'(+1) + Q'(-1)}{2} \]
\[ C_3 = - \frac{[Q'(+1) + Q'(-1)]}{2} \]

\[ C_4 = C_2^+ \frac{Q'(+1) + Q'(-1)}{2\theta \beta} \]

\[ g_1 = \frac{2\alpha_3}{\theta \beta} \quad \text{and} \quad g_3 = 6\beta_{17} + 2\theta_{18}^+ \beta_1 (b_{19} (\beta_1 \theta \beta + \theta \beta_1) + b_{21} (\beta_1 \theta \beta + 3\theta \beta) + \]

\[ \quad + b_{19} \theta \beta_1 + b_{21} (3\theta_1 \theta \beta_1 + 2\theta \beta_1) + \beta_1 b_{24} (\beta_1 \theta \beta_1 + \]

\[ \quad + 2\theta \beta_1 + 2b_{24} (\beta_1 \theta \beta_1 + \theta \beta_1) + 4\beta_1^2 b_{27} \theta \beta_1 + \]

\[ \quad + b_{26} (1 - \frac{\beta_1 \theta \beta_1}{\theta \beta_1}) + \beta_1 (2b_{26} \theta \beta_1) + 4\beta_2 \]

\[ \quad + (b_{26} b_{27}^+ \beta_1) (\beta_1 \theta \beta_1 + \theta \beta_1 \frac{\beta_1 \theta \beta_1}{\theta \beta_1}) + \]

\[ \quad + 6\beta_{25} + 3\beta_1 b_{26} (2b_1 \theta \beta_1 + \beta_1 \theta \beta_1 \frac{\beta_1 \theta \beta_1}{\theta \beta_1}) \]
\[ g_4 = -6b_1 t + 26b_2 b_1 \beta_1 (1 + \beta_1 (\beta_1 \text{Ch}_2 \beta_1 + \text{Sh}_2 \beta_1) - b_2 \beta_1 (\beta_1 \text{Ch}_2 \beta_1 + 3\text{Sh}_2 \beta_1) - \\
- b_1 \beta_1 \text{Sh}_2 \beta_1 + b_2 (3\beta_1 \text{Sh}_2 \beta_1 + 2\text{Ch}_2 \beta_1) - \beta_1 b_24 (\beta_1 \text{Sh}_2 \beta_1) + \\
+ 2\text{Ch}_2 \beta_1 - 2b_24 (\beta_1 \text{Ch}_2 \beta_1 + \text{Sh}_2 \beta_1) + 4\beta_1 b_27 \text{Sh}_2 \beta_1 + \\
+ b_16 (1 - \frac{\beta_1 \text{Ch}_2 \beta_1}{\text{Sh}_2 \beta_1}) + \beta_1 b_2 \text{Sh}_2 \beta_1 - 4\beta_2 2^2 (2b_2 \beta_1 + \\
+ \beta_1 b_23) (\beta_1 \text{Sh}_2 \beta_1 + \text{Ch}_2 \beta_1 - \frac{\beta_1 \text{Ch}_2 \beta_1}{\text{Sh}_2 \beta_1}) + (4\beta_2 2^2 + \\
+ \beta_1 b_25) (3\text{Ch}_2 \beta_1 + \text{Sh}_2 \beta_1 - \frac{\beta_1 \text{Ch}_2 \beta_1}{\text{Sh}_2 \beta_1}) + \\
+ 6b_25 + 2\beta_1 b_26 (\beta_1 \text{Ch}_2 \beta_1 + \beta_1 \text{Sh}_2 \beta_1 - \frac{\beta_1 \text{Ch}_2 \beta_1}{\text{Sh}_2 \beta_1}) \\
\] 

\[ g_5 = -a - \frac{1}{2} \]

\[ g_6 = 2a \cdot 32 + 4a \cdot 34 + a \cdot 36 \beta_1 \text{Sh}_2 \beta_1 + a \cdot 38 (\beta_1 \text{Ch}_2 \beta_1 + \text{Sh}_2 \beta_1) + 2a \cdot 33 + \\
+ 4a \cdot 35 + a \cdot 37 (\beta_1 \text{Sh}_2 \beta_1 + \text{Ch}_2 \beta_1) + a \cdot 38 (\beta_1 \text{Ch}_2 \beta_1 - \text{Sh}_2 \beta_1) \\
\]

\[ g_7 = \frac{a}{2} + \frac{1}{4} \]

\[ g_8 = \frac{2}{3} a \cdot 32 - \frac{4a \cdot 34}{5} + a \cdot 38 (\frac{\text{Sh}_2 \beta_1}{\beta_1} - \text{Ch}_2 \beta_1) + a \cdot 39 (\frac{\text{Ch}_2 \beta_1}{\beta_1} - \text{Sh}_2 \beta_1) \]
REFERENCES


