CHAPTER-IV

FLOW IN A ROTATING POROUS ANNULUS GENERATED BY DIFFERENTIAL RATES OF SUCTION/INJECTION AT THE WALLS
Introduction

The theory of rotating flows bounded by one or two cylinders is a basic problem in fluid dynamics and is of great industrial importance and practical interest. Such flow models with or without suction at the boundaries are important to the study of the movement of natural air, oil and water through oil reservoirs and in chemical engineering for filtration and purification purposes.

A wide variety of research on flows bounded by cylindrical surfaces was inspired by the Taylor-Proudman theorem (Taylor [1]). When a solid body moves horizontally with low Rossby number in a fluid rotating about a vertical axis, the vertical cylinder circumscribing the body separates regions of dissimilar velocity distribution. The fluid inside the circumscribing cylinder moves with the body while the fluid outside the cylinder flows around it in a two dimensional pattern. This result was discovered experimentally by Taylor [1] and the phenomena are often called a Taylor column. Some of the features of the experiment are explained using the Taylor-Proudman theorem, which states that steady geostrophic motions are independent of distance along the rotation vector. Vertical boundary layers known as steady Stewartson layers arise in a slightly viscous rotating fluid as an indirect consequence of the strong constraint placed
on the dynamics of rotating fluids by the Taylor-Proudman theorem. They occur along a vertical boundary or as detached shear layers straddling the vertical vortex sheets deduced from the inviscid dynamics as determined by Stewartson [2]. In general, these boundary layers have a double structure consisting of an inner layer of thickness order $E^{1/3}$ and of an outer layer of thickness $E^{1/4}$ where $E$ is the Ekman number. Barcilon [3] has shown that the detailed structure of the Stewartson layer strongly depends upon the problem under consideration and that the $E^{1/3}$ and $E^{1/4}$ layers behave as separate entities which can arise in various combinations. Experimental work on the existence of Stewartson layers was confirmed by Baker [4]. Consequently these vertical boundary layers were the subject of vast research by many authors in rotating fluids. Perhaps, the most systematic approach was developed by Moore and Saffman [5] who used detailed asymptotic analysis for a number of difficult problems to show the interconnection between these different shear layers and between the separate layers and any Ekman layers that would form on a horizontal surface. Through careful matching from one region to another region, they were able to find complete solutions in each of the regions and gain a clear understanding of the structure and role of the layers in satisfying the basic dynamical balances. The theory developed so is valid for small Rossby numbers. The Rossby number is essentially a measure of the magnitude of the perturbation relative to a solid body rotation. The assumption of small Rossby number enabled the development of a linear theory. Bennetts and Hocking [6] have assumed small values for
Rossby number which are much less than $E^{1/4}$. One of the most striking researches in the field of rotating flows applied to rigidly rotating cylinders was due to Hide [7]. Hide studied the nature of an incompressible source-sink flow in a rapidly rotating annular region bounded radially by a pair of concentric cylinders using the well known technique originating in the work of Stewartson [2]. When the source and the sink are coaxial porous infinitely long cylinders and the fluid enters and leaves the cylinders at a constant rate, it is observed that a viscous boundary layer occurs only on the sink. Away from the sink, individual fluid particles conserve their angular momentum in the main body of the fluid. In the case when the annulus is bounded vertically by top and bottom end plates and there is a uniform injection and withdrawal along the inner and outer side walls respectively, it is observed that the fluid entering the inner cylinder is immediately deflected in a set of vertical shear layers towards the horizontal end walls. The fluid then traverses along the end wall Ekman boundary layers and is subsequently deflected into a set of shear layers on the outer boundary before withdrawn from the container. Consequently, the entire mass transport takes place through the boundary layer and the container walls. Hide also conducted experiments to demonstrate how this transport of fluid takes place using various distributions of sources and sinks. Conlisk and Walker [8] investigated source-sink flows in a rapidly rotating contained annulus of finite height for all possible types of axi-symmetric injection or withdrawal at the side walls. The principal contributions correspond to solution for injection or withdrawal through gaps in
the side walls having vertical dimension of order \( E^{1/2} \). The calculated fluid motions arise from total mass transport from the inner to the outer cylinder which is small and order \( E^{1/2} \). The solutions are obtained on the basis of linear theory using the method of matched asymptotic expansions and in some regions of the flow field, are constructed from an infinite series of images of the similarity solutions. It is observed that the principal modification for each different flow occurs in the \( E^{1/3} \) layer. Bennetts and Hocking [6] studied the source-sink flows of larger mass flow rates based on non-linear theory and obtained results which are in agreement with the experiments of Hide. Smith [9] has later observed that there are three basic stages in the transient formation of the linear Stewartson layers; an inviscid, geostrophic stage where the time \( t \) satisfies \( 1 \ll t \ll E^{1/3} \); a viscous stage where \( t = O(E^{-1/3}) \); and the spin-up stage where \( t = O(E^{-1/2}) \). After an Impulsive start to the motion, the Ekman layer along the horizontal surfaces is formed where \( t = O(1) \), effectively being steady after three or four rotations; elsewhere, the flow is initially potential, but is then continuously modified by geostrophic forces and as \( t \) increases, a shear layer of width \( O(t^{-1}) \) becomes identifiable, within which the velocities increase in magnitude. When \( t = O(E^{-1/3}) \), viscous forces are no longer negligible and the layer is consolidated with width of order \( O(E^{1/3}) \), the flow becoming steady there in the limit as \( E^{1/2} t \to \infty \). It is only at this stage that the \( E^{1/4} \) layer begins to form, through a combination of vortex stretching (such as the one formed during the spin-up) and
diffusion, which is completed as $E^{1/2}t \to \infty$. Thus Smith established that the $E^{1/3}$ layer is already formed before the $E^{1/4}$ layer even comes into existence.

The intention of this chapter is to consider a viscous incompressible rotating flow in an annulus bounded between two infinitely long coaxial cylinders. The whole region is filled by a porous medium of high permeability. It can be verified from the governing equations that the whole system with the porous medium and the internal fluid can keep a state of rigid body rotation initially with a uniform angular velocity. The disturbance over this state is created by a source sink flow consisting of a uniform suction/injection at the cylindrical boundaries with different rates. Even if the rate of flow at the boundaries is assumed to be the same constant, it will still be able to disturb the rigid rotation unlike in the case of the flow between two parallel infinite disks where a rigid body rotation of the entire system is still possible even under a constant suction/injection with the same normal velocity. To create a disturbance in such fluids, a differential rotation or a differential suction will be necessary. However, the situation is different in the case of rotating flows in circular annular regions. In this case, a rigid body rotation of the porous medium together with the internal fluid in the annular region is possible in the absence of any suction/injection at the boundaries. But when the suction/injection is applied at the two cylindrical boundaries with the same uniform constant rate at the surfaces, the system will not be able to keep rigid body rotation. These aspects are observed by Aruna Prasad.
and Venkatasiva Murthy [10]. These authors considered the transient flow of a viscous incompressible rotating fluid occupying a highly permeable medium between two infinitely long coaxial cylinders. The whole system was initially in a state of rigid body rotations. The disturbance over this state is created by a uniform injection at the inner cylinder and suction at the outer cylinder. The rate of flow is same at these boundaries. It is well known that this suction/injection mechanism disturbs the rigid body rotation even in the case of the pure fluid. Such a problem of the flow through a rotating porous strip is of some interest in the areas of filters, chemical process cooling towers and electrochemical reactors. They used the Brinkman’s law to represent the motion in the porous medium. The exact solution of this unsteady flow is obtained by Laplace transformation technique. They investigated how the unsteady flow evolves before it reaches a steady state as a consequence of the viscous Coriolis force balance. In the case when the permeability of the medium tends to infinity, they regained the corresponding problem of Hide [7]. The way in which the viscous boundary layer appearing on the sink gets modified due to the resistance of the porous medium was examined. The suppression of velocity in the annulus due to the resistance of the permeable medium results in a boundary layer type of flow near the inner cylinder (source) as well, and a nearly constant velocity in the interior. Such a behaviour is absent near the inner cylinder in the corresponding case of infinite permeability described by Hide [7].
The flows of rotating fluids in porous media are not explored in detail, more so in the case of flows in cylindrical annuli. We consider a viscous fluid occupying a highly permeable medium bounded between two infinitely long circular cylinders. It is assumed that the porosity of the medium is nearly unity and the fluid occupies almost all parts of it. Consequently, the viscous stress is taken in the same form as in the case of pure fluid. We use the Brinkman's law to represent the motion in the porous medium. The whole system was initially in a state of rigid body rotation. From the problem described by Aruna Prasad and Venkatasiva Murthy [10], it is evident that when the constant suction/injection velocities at the cylindrical boundaries are such that the rate of uniform suction at one of the boundaries equal the rate of uniform injection at the other, the flow is independent of the azimuthal and axial coordinates and is radial. If the rate of constant suction at one boundary is different from the rate of injection at the other boundary, the flow depends on the axial coordinate \( z \) and the axial velocity also exists in the flow. Since the governing equations become highly non-linear, the corresponding linear equations are solved for the steady flow by introducing the similarity solutions for the variables. The solution involves Bessel functions of orders 0 and 1. Numerical results are plotted graphically to describe the effect of permeability on the velocity components. The axial velocity arising due to the differential suction at the boundaries show certain interesting features in the flow. The effect of permeability of the medium on the rotational velocity is
brought out. The problem also serves to illustrate that a rigidly rotating flow can be disturbed from its state of steady rotation by mechanisms other than a differential rotation at the boundaries. Here is a case where the rigid rotation is disturbed by a differential suction at the cylindrical boundaries.

**Formulation of the problem**

We consider a viscous incompressible fluid between two permeable cylindrical boundaries \( r = a \) and \( r = b \) in cylindrical coordinate system \((r, \theta, z)\) with the \( z \)-axis as the common axis of the cylinders. The annular region is filled by a highly tenuous porous medium. The whole system consisting of the porous medium together with the fluid filling it, is in a state of rigid body rotation about the \( z \)-axis, with a uniform angular velocity \( \Omega \). If \((u, v, w)\) are the velocity components in the fluid, the equations of motion in this basic state are satisfied by \( u = 0 \), \( v = r \Omega \), \( w = 0 \) and \( p = p_0 + \frac{1}{2} \rho r^2 \Omega^2 \) where \( p_0 \) and \( \rho_0 \) are constant pressure and density. This shows that a rigid body rotation of the entire system together with the porous medium and the interior fluid is possible. The disturbance over this state of rigid rotation is created by imposing a uniform suction/injection at the boundaries \( r = a \) and \( r = b \) in such a way that the rate of fluid
injected at the inner cylinder $r = a$ is different from the rate of suction imposed at the outer cylinder $r = b$. Aruna Prasad and Venkatasiva Murthy [10] considered the problem where the similarity variables and the boundary conditions are introduced as follows:

$$u(r, t) = \lambda/r, \quad v(r, t) = r\Omega + f(r, t),$$

$$w(r, t) = 0, \quad p(r, t) = p_0 + \frac{1}{2} \rho r^2 \Omega^2 + P(r, t).$$

The initial and boundary conditions were taken as

$$u = 0, \quad v = r\Omega, \quad w = 0 \quad \text{and} \quad p = p_0 + \frac{1}{2} \rho r^2 \Omega^2 \quad \text{for} \quad t = 0.$$ 

$$u = \lambda/a, \quad v = r\Omega, \quad w = 0 \quad \text{for} \quad t > 0$$

$$u = \lambda/b, \quad v = r\Omega, \quad w = 0$$

The above boundary conditions imply that the rates of injection and suction at the cylinders are equal. In the present problem we assume that the rates of injection and suction at the two cylinders $r = a$ and $r = b$ are different. Consistent with the equation of conservation of mass for an incompressible fluid, namely $\nabla \cdot \vec{q} = 0$, we write,

$$u(r, t) = \varepsilon \frac{f(r)}{r},$$

$$w(r, t) = -\varepsilon z \frac{1}{r} \frac{\partial f}{\partial r} + h(r, t),$$

$$v(r, t) = r\Omega + \varepsilon g(r, t).$$
where $\varepsilon \ll 1$.

The initial and boundary conditions are

\[
\begin{align*}
  u &= 0, v = r\Omega, w = 0 \quad \text{at} \quad t = 0, \\
  u = \frac{\varepsilon a^2\Omega}{a}, \quad v = r\Omega, w = 0 \quad \text{for} \quad t > 0, \\
  u = \frac{2\varepsilon a^2\Omega}{b}, \quad v = r\Omega, w = 0
\end{align*}
\]

The differential rate of suction/injection prescribed at the boundaries $r = a$ and $r = b$ disturbs the rigid rotation and creates a flow along the axial direction also, given by

\[
w(r,t) = -\varepsilon z \frac{1}{r} \frac{\partial f}{\partial r} + h(r,t).
\]

This axial component of velocity is absent in the corresponding case of uniform rate of suction/injection at the boundaries proposed by Aruna Prasad and Venkatasiva Murthy [10]. We further assume that $\varepsilon$ is small.

Substituting these transformations in the governing equations and neglecting terms of order $\varepsilon^2$ in the equations of motion, the unsteady linearised equations of motion are:

\[
\frac{\partial p}{\partial r} = \rho r \Omega^2 - \rho \left( \frac{1}{r} \frac{\partial f}{\partial t} - 2\Omega g \right) + \mu \left( \frac{1}{r} \frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2} \frac{\partial f}{\partial r} \right) - \frac{\mu}{k} \frac{1}{r} f,
\]

(1)
\[ \rho \left( \frac{\partial g}{\partial t} + \frac{2\Omega}{r} f \right) = \mu \left( \frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} - \frac{g}{r^2} - \frac{\mu}{k} g \right), \quad (2) \]

\[ \frac{\partial p}{\partial z} = \varepsilon z \left\{ \frac{\rho}{r} \frac{\partial^2 f}{\partial r \partial t} - \mu \left[ \frac{1}{r^3} \frac{\partial f}{\partial r} - \frac{1}{r^2} \frac{\partial^2 f}{\partial r \partial t} + \frac{1}{r} \frac{\partial^3 f}{\partial r^3} - \frac{1}{kr} \frac{\partial f}{\partial r} \right] \right\} + \varepsilon \mu \left[ \frac{\partial^3 h}{\partial r^3} + \frac{1}{r} \frac{\partial h}{\partial r} - \frac{1}{k} h \right], \quad (3) \]

From equation (1) we obtain \[ \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial r} \right) = 0. \] Therefore \[ \frac{\partial}{\partial r} \left( \frac{\partial p}{\partial z} \right) = 0. \]

From equation (3) we obtain \[ \frac{\partial p}{\partial z} = \varepsilon \alpha(r, t) + \beta(r, t). \]

We therefore obtain \[ \alpha(r, t) = \text{constant} \quad \text{and} \quad \beta(r, t) = \text{constant}. \]

Assuming that \[ \frac{\partial p}{\partial z} = 0 \] at \( z = 0 \) we obtain \[ \beta(r, t) = 0. \]

We therefore write \[ \frac{\partial p}{\partial z} = -\varepsilon A z \] where \( A \) is an unknown constant.

Equation (1) will determine the pressure after obtaining the solution for \( f \) and \( g \). The steady equations of motion now reduce to

\[ r^2 \frac{\partial^3 f}{\partial r^3} - r \frac{\partial^2 f}{\partial r^2} + \left( 1 - \frac{r^2}{k} \right) \frac{\partial f}{\partial r} = Ar^3, \quad (4) \]

\[ r^2 \frac{\partial^2 g}{\partial r^2} + r \frac{\partial g}{\partial r} - \left( 1 + \frac{r^2}{k} \right) g = \frac{2\Omega}{v} \frac{r}{r} f, \quad (5) \]
\[ \frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} - \frac{1}{k} h = 0 \quad (6) \]

where \( A \) is an arbitrary constant.

If we write \( F = \frac{\partial f}{\partial r} \), we can write equation (4) in the form

\[ r^2 \frac{\partial^2 F}{\partial r^2} - r \frac{\partial F}{\partial r} + \left(1 - \frac{r^2}{k}\right) F = Ar^3 \quad (7) \]

The equations (5), (6) and (7) are to be solved using appropriate boundary conditions in terms of \( f \), \( g \) and \( h \).

\[ f = a^2 \Omega, \quad \frac{\partial f}{\partial r} = 0, \quad g = 0, \quad h = 0 \quad \text{at} \quad r = a, \]

\[ f = 2a^2 \Omega, \quad \frac{\partial f}{\partial r} = 0, \quad g = 0, \quad h = 0 \quad \text{at} \quad r = b. \]

We introduce the following non-dimensional variables and parameters.

\[ u^* = u/a\Omega, \quad v^* = v/a\Omega, \quad w^* = w/a\Omega \]

\[ f^* = f/a^2 \Omega, \quad g^* = g/a\Omega, \quad h^* = h/a\Omega \]

\[ r^* = r/a, \quad z^* = z/a, \quad d = b/a. \]

In terms of these new variables we obtain the following equations governing the motion, on dropping the superscript *.

\[ r^2 \frac{\partial^2 g}{\partial r^2} + r \frac{\partial g}{\partial r} - \left(1 + Hr^2\right) g = 2Rrf \quad (8) \]
\[ r^2 \frac{\partial^3 f}{\partial r^3} - r \frac{\partial^2 f}{\partial r^2} + (1 - Hr^2) \frac{\partial f}{\partial r} = Ar^3 \]  

Putting \( F = \frac{\partial f}{\partial r} \), we obtain

\[ r^2 \frac{\partial^2 F}{\partial r^2} - r \frac{\partial F}{\partial r} + (1 - Hr^2) F = Ar^3 \]  

\[ r^2 \frac{\partial^3 h}{\partial r^3} + r \frac{\partial h}{\partial r} - Hh = 0 \]

where \( H = \frac{a^2}{k} \) is a Permeability parameter and \( R = \frac{a^2 \Omega}{\nu} \) is a Reynolds number.

The boundary conditions in non-dimensional form are, on dropping the superscript *,

\[
\begin{align*}
  f &= 1, \quad \frac{\partial f}{\partial r} = F = 0, \quad g = 0, \quad h = 0 \quad at \quad r = 1 \\
  f &= 2, \quad \frac{\partial f}{\partial r} = F = 0, \quad g = 0, \quad h = 0 \quad at \quad r = d.
\end{align*}
\]

Solving Eq. (11) subject to the above conditions on \( h \) will give \( h = 0 \). The solutions for \( F \) satisfying the above boundary conditions is

\[ F(r) = \frac{\partial f}{\partial r} = C r J_0(i\sqrt{H} \ r) + D r Y_0(i\sqrt{H} \ r) - \frac{A}{H} r. \]  

On integrating the above equation and using the relations

\[ \int r J_0(xr) \, dr = \frac{r}{x} J_1(xr) + \text{constant,} \]
\[ \int r Y_0(xr) \, dr = \frac{r}{x} Y_1(xr) + \text{constant}, \]

we obtain the solution for \( f(r) \) as

\[ f(r) = C \frac{r}{i \sqrt{H}} J_1(i \sqrt{H} r) + D \frac{r}{i \sqrt{H}} Y_1(i \sqrt{H} r) - \frac{A}{H} \frac{r^2}{2} + B \quad (14) \]

where \( A, B, C \) and \( D \) are arbitrary constants.

Substituting for \( f(r) \) in equation (8) we obtain the differential equation governing \( g \) in the form

\[ r^2 \frac{\partial^2 g}{\partial r^2} + r \frac{\partial g}{\partial r} - (1 + Hr^2)g = 2R \left\{ C \frac{r^2}{i \sqrt{H}} J_1(i \sqrt{H} r) + D \frac{r^2}{i \sqrt{H}} Y_1(i \sqrt{H} r) - \frac{A}{H} \frac{r^3}{2} + Br \right\} \quad (15) \]

Solving the above equation, we obtain the solution for \( g \) in the following form.

\[ g(r) = E J_1(i \sqrt{H} r) + F Y_1(i \sqrt{H} r) + q(r) \quad (16) \]

where \( q(r) \) is a particular integral of equation (15) given by

\[ q(r) = \pi R \left\{ CY_1(ct) \int t J_1^2(ct) \, dt - CJ_1(cr) \int t J_1(ct) Y_1(ct) \, dt + Dy_1(cr) \int t J_1(ct) Y_1(ct) \, dt + \right. \]

\[ \left. - DJ_1(cr) \int t Y_1^2(ct) \, dt - \frac{cA}{2H} Y_1(cr) \int t^2 J_1(ct) \, dt + \frac{cA}{2H} J_1(cr) \int t^2 Y_1(ct) \, dt + \right. \]

\[ \left. + cBY_1(cr) \int J_1(ct) \, dt - cB J_1(cr) \int Y_1(ct) \, dt \right\}, \quad \text{where } c = i \sqrt{H}. \quad (17) \]
The integrals involved in the Eq. (17) can be evaluated using the following results on Bessel functions:

\[
\frac{d}{dz} J_0(z) = -J_1(z),
\]

\[
\frac{d}{dz} Y_0(z) = -Y_1(z),
\]

\[
\frac{d}{dz} [z^n J_0(z)] = z^n J_{n-1}(z),
\]

\[
\frac{d}{dz} [z^n Y_0(z)] = z^n Y_{n-1}(z),
\]

If \( C_v \) and \( D_v \) are cylindrical functions, Luke[13] gives the following formulae:

\[
\int tC_v(kt)C_v(lt) \, dt =
\]

\[
= \frac{r}{k^2 - l^2} \{ k C_{v+1}(kr) D_v(lr) - l C_v(kr) D_{v+1}(lr) \}, \quad k \neq l.
\]

\[
= \frac{r^2}{4} \{ 2C_v(kr)D_v(kr) - C_{v-1}(kr)D_{v+1}(kr) - C_{v+1}(kr)D_{v-1}(kr) \}, \quad k = l.
\]
Using these formulae, we obtain

\[ \int J_2^2 \cdot (ct) \, dt = \frac{r^2}{4} \left\{ 2J_1^2 \cdot (cr) - 2J_0 \cdot (cr)J_2 \cdot (cr) \right\} \]

\[ \int J_1^2 \cdot (ct)Y_1 \cdot (ct) \, dt = \frac{r^2}{4} \left\{ 2J_1 \cdot (cr)Y_1 \cdot (cr) - J_0 \cdot (cr)Y_2 \cdot (cr) - J_2 \cdot (cr)Y_0 \cdot (cr) \right\} \]

\[ \int J_1^2 \cdot (ct) \, dt = \frac{r^2}{4} \left\{ 2Y_1^2 \cdot (cr) - 2J_0 \cdot (cr)Y_2 \cdot (cr) \right\} \]

\[ \int J_0 \cdot (ct) \, dt = \frac{1}{c} \cdot r^2J_2 \cdot (cr) \]

\[ \int J_1 \cdot (ct) \, dt = \frac{1}{c} \cdot r^2Y_2 \cdot (cr) \]

\[ \int J_1 \cdot (ct) \, dt = \frac{1}{c} \cdot J_0 \cdot (cr) \]
\[ \int_{r} Y_1(\sigma t) dt = -\frac{1}{c} Y_0(\sigma r). \]

Using these formulae, we obtain,

\[ q(r) = \pi R \left\{ -\frac{r^2}{2} CY_1(\sigma r) J_0(\sigma r) J_2(\sigma r) - \right. \]

\[ - \frac{r^2}{4} [DY_1(\sigma r) - CJ_1(\sigma r)] \{ J_0(\sigma r) Y_2(\sigma r) + J_2(\sigma r) Y_0(\sigma r) \} + \]

\[ + \frac{r^2}{2} DJ_1(\sigma r) Y_0(\sigma r) Y_2(\sigma r) + \]

\[ + \frac{A}{2H} r^2 \{ J_1(\sigma r) Y_2(\sigma r) - Y_1(\sigma r) J_2(\sigma r) \} + \]

\[ + B [J_1(\sigma r) Y_0(\sigma r) - Y_1(\sigma r) J_0(\sigma r)] \}. \]

The coefficients \( A, B, C, D, E \) and \( F \) are given by

\[ X_1 = \frac{1}{2} (d^2 - 1) J_0(c) - \frac{1}{c} \{ d J_1(c) - J_1(c) \}, \]

\[ X_2 = \frac{1}{2} (d^2 - 1) Y_0(c) - \frac{1}{c} \{ d Y_1(c) - Y_1(c) \}, \]
\[ Y_1 = \frac{1}{2} d(d^2 - 1)J_0(cd) - \frac{d}{c} [d J_1(cd) - J_1(c)], \]

\[ Y_2 = \frac{1}{2} d(d^2 - 1)Y_0(cd) - \frac{d}{c} [d Y_1(cd) - Y_1(c)], \]

\[ C = \frac{dX_2 - Y_2}{X_1Y_2 - X_2Y_1}, \]

\[ D = \frac{Y_1 - dX_1}{X_1Y_2 - X_2Y_1}, \]

\[ A = H[CJ_0(c) + DY_0(c)], \]

\[ B = 1 + \frac{A}{2H} - \frac{C}{c} J_1(c) - \frac{D}{c} Y_1(c), \]

\[ E = \frac{q(d)Y_1(c) - q(1)Y_1(cd)}{J_1(c)Y_1(cd) - Y_1(c)J_1(cd)} \]

\[ F = \frac{q(1)J_1(cd) - q(d)J_1(c)}{J_1(c)Y_1(cd) - Y_1(c)J_1(cd)} \]
In the absence of porous medium, the solution can be obtained by taking the limit 
k \to \infty \ (H \to 0). \ It may also be obtained by putting \( H = 0 \) in the governing equations and solving them. The governing equations are

\[
\begin{align*}
\frac{r^2}{\partial r^3} f - r \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r} &= Ar^3 \\
\frac{r^2}{\partial r^2} g + r \frac{\partial g}{\partial r} - g &= 2Rrf
\end{align*}
\]

Putting \( F = \frac{\partial f}{\partial r} \) we obtain,

\[
\frac{r^2}{\partial r^2} F - r \frac{\partial F}{\partial r} + F = Ar^3
\]

The solution of these equations satisfying the boundary conditions

\[
\begin{align*}
f &= 1, \quad \frac{\partial f}{\partial r} = F = 0, \quad g = 0, \quad \text{at} \quad r = 1 \\
f &= 2, \quad \frac{\partial f}{\partial r} = F = 0, \quad g = 0, \quad \text{at} \quad r = d.
\end{align*}
\]

is given by
\[
f(r) = \frac{Cr^2}{2} + D\frac{r^2}{2} \left( \log r - \frac{1}{2} \right) + A\frac{r^4}{16} + B
\]

\[
g(r) = Er + F\frac{1}{r} + q(r)
\]

where \( q(r) \) is given by

\[
q(r) = C \frac{R}{8} r^3 - D \frac{5R}{32} r^3 + A \frac{R}{192} r^5 - BR\frac{r}{2} + \frac{DR}{8} r^3 \log r + BRr \log r,
\]

The values of \( A, B, C, D, E \) and \( F \) are given by

\[
A = \frac{16 \log d}{(d^2 - 1)[(d^2 - 1)(\log d + 1) - 2d^2 \log d]}
\]

\[
C = -A/4
\]

\[
D = \frac{-4}{(d^2 - 1)(\log d + 1) - 2d^2 \log d}
\]

\[
B = 1 - \frac{C}{2} + \frac{D}{4} - \frac{A}{16}
\]
\[ E = -F - q(1) \]

\[ F = \frac{d[dq(1) - q(d)]}{1 - d^2} \]

**Discussion of the results**

The response of velocity distribution for various values of the physical parameters has been graphically represented in Figs. 1 to 8. The numerical work is carried out taking moderate channel width \((d = 3)\) and also a larger channel width \((d = 5)\). The velocity distribution shows interesting properties when the channel width is large. It is observed that the suction/injection mechanism is instrumental in disturbing the state of rigid rotation and inducing a perturbation flow analogous to the classical Ekman flow that normally arises due to the differential rotation of a boundary of the container (in the case of pure fluid). While a suction/injection mechanism with equal rates at the boundaries will be able to disturb the uniform angular velocity in the annulus and induces a rotational perturbation (no axial velocity), the differential rates of suction/injection imposed at the two cylindrical boundaries in the present problem will also induce an additional axial velocity. This axial velocity linearly depends on the axial coordinate \(z\).
The axial velocity is plotted in Figs. 1 and 2 for various values of the permeability parameter \( H \). Figure 1 corresponds to the case where the ratio of the radii \( d = b/a = 3 \) and Fig. 2 corresponds to the case \( d = 5 \). It is observed that the magnitude of the radial velocity decrease as the medium becomes less permeable. This happens in a larger part of the annulus leaving a neighbourhood of the inner cylinder. The boundary layer behaviour is absent at the boundaries and the axial velocity varies slowly with the distance near the boundaries.

The axial velocity represented by \( f \) is shown in Figs. 3 and 4 for various values of the permeability parameter \( H \). For a given \( H \), the axial velocity is less in magnitude in the case \( d = 5 \) (Fig. 4) compared to the case \( d = 3 \) (Fig. 3). The behaviour of the profiles remain the same irrespective of the channel width. If the medium is less permeable, a boundary layer behaviour shows up both at the inner cylinder and at the outer cylinder (more so when \( d = 5 \)) whereas such a behaviour is totally absent in the case of pure fluid \( (H = 0) \). When \( H \) is sufficiently large, outside the boundary layers, the flow is linear (see curves (c) and (d) of Fig. 4). Thus the interior flow is linear when the channel width is sufficiently large. The rotational velocity is plotted in Figs. 5 and 6. It can be observed from Fig. 5 that when the channel width is not large, the rotational velocity does not have a boundary layer structure and the velocity decreases in magnitude as the medium becomes less permeable. But when the gap is sufficiently large (Fig. 6), the profile
remarkably changes to one of the boundary layer type. The boundary layer behaviour is more pronounced near the inner cylinder in comparison to the profile near the outer cylinder. This is in contrast to the axial velocity whose boundary layer behaviour is more pronounced at the outer cylinder rather than at the inner cylinder. As already remarked, the Ekman type behaviour is seen both in the axial velocity and in the rotational velocity, when $d$ is sufficiently large (Figs. 4 and 6). Figures 7 and 8 show the rotational velocity for different values of the Reynolds number (inverse of an Ekman number in rotational flows). As usually expected, the rotational velocity grows in magnitude as the Reynolds number $R$ increases. The boundary layer shows up only for sufficiently wide annulus.

It may be noted that the case of infinite permeability ($H = 0$) Hide [7] observed a boundary layer flow only at the sink because of the fact that the fluid particles drifting away from the source carry away the momentum and prevent any steep rise in the rotational velocity near the source and there is boundary layer only on the sink. The profiles outside the boundary layer are nearly linear. While this is the case in a pure fluid, Aruna Prasad [10] observed that in the case of equal rates of uniform suction/injection at the boundaries, the permeability of the medium in the annulus gives rise to a boundary layer behaviour near the source also, though not very pronounced in comparison with that at the sink. It is shown through Figs. 6 and 8 of this work that a differential suction (with
different rates at the source and sink) will induce a strong boundary layer flow at both the cylinders, more so near the source (inner cylinder). This is because of the pressure that builds up in the fluid due to the difference in the rates of suction and injection. This is an important aspect observed in this work in comparison with the results of Hide [7] and Aruna Prasad [10].
Figure 1:

The radial velocity plotted for various values of the permeability parameter $H$ when $d = 3$ and $R = 10$.

(a) $H = 0$ (b) $H = 5$ (c) $H = 20$ (d) $H = 50$. 
Figure 2:

Radial velocity plotted for various values of the permeability parameter $H$ when $d = 5$ and $R = 10$.

(a) $H = 0$ (b) $H = 5$ (c) $H = 20$ (d) $H = 50$. 
Figure 3:
The axial velocity plotted with the radial distance in the channel for various values of the permeability parameter $H$ when $d = 3$ and $R = 10$.

(a) $H = 0$  (b) $H = 5$  (c) $H = 20$  (d) $H = 50$. 
Figure 4:

The axial velocity plotted for various values of the permeability parameter $H$ when $d = 5$ and $R = 10$.

(a) $H = 0$  (b) $H = 5$  (c) $H = 20$  (d) $H = 50$. 
Figure 5:
The rotational velocity plotted with the radial distance in the annulus for various values of the permeability parameter $H$ when $d = 3$ and $R = 10$.

(a) $H = 0$  (b) $H = 5$  (c) $H = 20$  (d) $H = 50$. 
Figure 6:
Rotational velocity plotted for various values of the permeability parameter H when $d = 5$ and $R = 10$.
(a) $H = 5$  (b) $H = 20$  (c) $H = 50$. 
Figure 7:
The rotational velocity plotted for various values of the parameter R when $d = 3$ and $H = 10$.

(a) $R = 10$  (b) $R = 20$  (c) $R = 30$. 

The rotational velocity plotted for various values of the parameter R when $d = 3$ and $H = 10$.

(a) $R = 10$  (b) $R = 20$  (c) $R = 30$. 

Figure 8:
Rotational velocity plotted for various values of the Reynolds' number $R$ when $d = 5$ and $H = 10$.
(a) $R = 10$  (b) $R = 20$  (c) $R = 30$. 
References


