CHAPTER II

FOURIER-PLANCHEREL TRANSFORM
FOR VECTOR-VALUED FUNCTIONS
AND BOEHMIANS
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The classical Plancherel theorem asserts that the Fourier-Plancherel transform is a Hilbert space isomorphism from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. On the other hand in the literature the theory of Fourier transform is extended to the space of $L^1$-Boehmians and also to the space of tempered Boehmians. In this chapter we shall introduce two types of Boehmian spaces each of which contains Banach algebra-valued square integrable functions on $\mathbb{R}$ as a dense subspace and extend the theory of Fourier transform to this set up. Finally we prove that this extended Fourier transform is a one-to-one continuous linear map of one space of Boehmians onto the other.

2.1. INTRODUCTION

The theory of Schwartz's distributions, tempered distributions and their applications are well-known in the literature. The concept of Boehmians which was motivated by Boehme's regular operators [1] was defined and systematically developed and their properties investigated.
In [32] Zemanian develops the theory of Laplace transform for a testing function space consisting of Banach space-valued functions defined on $\mathbb{R}^k$. Motivated by the above theory in this chapter we shall develop a theory of Fourier transform on a certain space of Boehmians which contains Banach Algebra-valued functions defined on $\mathbb{R}$. For simplicity we have taken a separable commutative Banach Algebra $A$ with identity $I$. (It may still be possible to take just a separable Banach space instead of a Banach Algebra. But that may lead to more complications) Let us first assume that $A$ is a Banach space. To develop our theory we need an analogue of the Plancherel theorem on the space $L^2(A)$ where $L^2(A)$ consists of $A$-valued Borel measurable functions on $\mathbb{R}$ such that $\int_{\mathbb{R}} \|f(x)\|^2 \, dx < \infty$. Since we are not aware of any such theory in the literature, we assume that $A$ is a complex Hilbert space and a separable commutative Banach Algebra with identity $I$ such that the norm induced by the inner product and the norm in the Banach Algebra are equivalent and develop a Plancherel theorem for this setup. The example $A = \mathbb{C}^n$ tells us that it is possible to assume such restrictions on $A$. On the other hand if $A$ is a Hilbert space and also a complex Algebra in which left and right multiplications are continuous we can introduce a Banach Algebra structure such that the Banach Algebra norm and the
norm induced by the inner product are equivalent. (Theorem 1.2.6.). Thus we strongly believe the existence of such spaces other than $\mathbb{C}^n$. Thus in all discussions in Section 2 and in Section 3 we can certainly take $A = \mathbb{C}^n$ for some positive integer $n$ and proceed with our theory.

We first develop a Plancherel Theorem for $L^2(A)$ and then use it to define Fourier-Plancherel transform for our space of Boehmians. Unlike the classical theories wherein the Fourier transform of elements of Boehmians space are classical distributions, we shall define the Fourier-Plancherel transform as a continuous linear map from one space of Boehmians onto another.

Since the theory of Fourier transform on Banach Algebra-valued functions has already been developed (See [27,28]) we present the required theory with minimum details in Section 2. In Section 3 and in Section 4 we introduce two different vector-valued Boehmian spaces. We shall exhibit an embedding of $L^2(A)$, the space of all $A$-valued square integrable functions on $\mathbb{R}$ in these Boehmian spaces. In Section 5 we introduce Fourier-Plancherel transform on our spaces of Boehmians and obtain its properties. Finally in Section 6 we shall make a comparative study of the theory developed here and those that are already available in the existing literature and known to us.
2.2. Fourier-Plancherel Transform On $L^2(A)$

Theorem 2.2.1. Let $1 \leq p < \infty$. If $f \in L^p(A)$ and $g \in L^q(A)$, then

$$(f*g)(x) = \int f(x-y)g(y) \ dm(y)$$

exists as a Bochner Integral.

Proof. In fact by Lemma 1.4.4. and Lemma 1.4.8. $f(x-y)g(y)$ as a function of $y$ is Bochner-measurable. Let $K = \text{supp } g$ and

$$\|g\|_0 = \sup_{x \in K} \|g(x)\|$$

then $\int f(x-y)g(y) \ dm(y) = \int f(x-y)g(y) \ dm(y)$.

Now $\int_K \|f(x-y)g(y)\| \ dm(y) \leq \|g\|_0 \int_K \|f(x-y)\| \ dm(y) \leq c \|g\|_0 < \infty$

where $c = \|f\|_1$ if $p = 1$ and $c = \|f\|_p \|g\|_q^{1/q}$ if $p > 1$ with $1/p + 1/q = 1$.

Thus by Lemma 1.4.3. $\int_K f(x-y)g(y) \ dm(y)$ exists for each $x \in \mathbb{R}$ as a Bochner Integral.

Theorem 2.2.2. Let $1 \leq p < \infty$. If $f \in L^p(A)$ and $g \in L^q(A)$, then $f*g \in L^p(A)$ and $\|f*g\|_p \leq \|f\|_p \|g\|_q$.

Proof. Let $K = \text{supp } g$ and $\|g\|_0 = \sup_{x \in K} \|g(x)\|$. Then

$$\|f*g\|_p^p = \int_{\mathbb{R}} \|f(x-y)g(y)\|^p \ dm(y)$$

$$\leq \int_{\mathbb{R}} \left( \int_{K} \|f(x-y)g(y)\| \ dm(y) \right)^p \ dm(x) \quad (1)$$

Let $\lambda = \int_{K} \|g(y)\| \ dm(y)$ and $d\mu(y) = \frac{1}{\lambda} \|g(y)\| \ dm(y)$.

Then $\mu$ is a positive Borel measure with $\int_{K} d\mu(y) = 1$ and
\[ \lambda \int_{K} \Vert f(x-y) \Vert \, d\mu(y) = \int_{K} \Vert f(x-y) \Vert \, \Vert g(y) \Vert \, dm(y) \leq \|g\|_{0} \int_{K} \|f(x-y)\| \, dm(y) < \infty. \]

Hence by Jensen's inequality (Lemma 1.4.9.), (1) becomes

\[ \|f \ast g\|_{p}^{p} \leq \lambda^{p} \int_{R} \left( \int_{K} \|f(x-y)\| \, dm(x) \right) \|g(y)\| \, dm(y) \]

\[ = \lambda^{p-1} \int_{R} \int_{K} \|f(x-y)\| \|g(y)\| \, dm(y) \, dm(x) \]

\[ = \lambda^{p-1} \int_{K} \|g(y)\| \left( \int_{R} \|f(x-y)\| \, dm(x) \right) \, dm(y) \]

\[ = \lambda^{p-1} \lambda^{p} \|f\|_{p}^{p} \]

\[ = \lambda^{p} \|f\|_{p}^{p} \]

\[ = \|f\|_{p}^{p} \cdot \|g\|_{1}^{p}. \]

Thus \( \|f \ast g\|_{p} \leq \|f\|_{p} \|g\|_{1}. \)

\[ \square \]

Remark 2.2.3. If \( f \in L^{2}(A) \), then \( f \) is locally integrable since

\[ \int_{K} \|f(x)\| \, dx \leq \left( \int_{K} \|f(x)\|^{2} \, dx \right)^{1/2} \left( m(K) \right)^{1/2} \]

for any compact subset \( K \) of \( R \) where \( m \) denotes the Lebesgue measure. Hence \( f \) can be considered as an \([A:A]-valued\) distribution, i.e. a map from \( B(A) \) to \( A \) given by \( \Lambda_{x}(\phi) = \int_{R} f(t) \phi(t) \, dt \) for all \( \phi \in B(A) \).

The right-hand side as a Bochner integral exists since \( f \phi \) is Bochner measurable. Let \( K = \text{supp } \phi \). By Schwartz's inequality
\[ \int_K \|f(t)\| \|\phi(t)\| \, dt \leq \|\phi\|_0 \int_K \|f(t)\| \, dt \leq \|\phi\|_0 \|f\|_2 \sqrt{m(K)}^{1/2} < \infty. \]

Λ_1 is clearly linear and Λ_1 is continuous as \( \|\Lambda_1(\phi)\| \leq M \|\phi\|_0 \)
where \( M = \int_K \|f(t)\| \, dt < \infty \). Thus \( \Lambda_1 \in \mathcal{B}(A) \). Let \( g \in \mathcal{B}(A) \). As in [32] we can define the convolution \( \Lambda_1 \ast g \) which is regularized by \( u \) in \( \mathcal{E}(A) \) such that

\[ u(x) = \Lambda_1(\tau_x g) = \int_R f(t)g(x-t) \, dm(t) = (f \ast g)(x). \]

Since \( f \in L^2(A) \), \( g \in \mathcal{B}(A) \) by Theorem 2.2.2., \( u = f \ast g \in L^2(A) \). In effect we have \( \Lambda_1 \ast g = \Lambda_1 \ast g \). Thus the convolution product \( f \ast g \) with \( f \in L^2(A) \), \( g \in \mathcal{B}(A) \) coincides with the convolution \( \Lambda_1 \ast g \) as defined in [32].

Lemma 2.2.4. Let \( f \in L^1(A) \), \( g \in \mathcal{E}(A) \). For compact subsets \( K_1, K_2 \) of \( R \) we get

\[ \int_{K_1} \int_{K_2} f(t)g(x-t) \, dx \, dt = \int_{K_2} \int_{K_1} f(t)g(x-t) \, dt \, dx. \]

Proof. Let \( A' \) denote the dual of \( A \). Since \( A' \) is a normed space it is sufficient to prove that for all \( \Lambda \in A' \)

\[ \Lambda \left( \int_{K_1} \int_{K_2} f(t)g(x-t) \, dx \, dt \right) = \Lambda \left( \int_{K_2} \int_{K_1} f(t)g(x-t) \, dt \, dx \right). \tag{2} \]

Now

\[ \Lambda \left( \int_{K_1} \int_{K_2} f(t)g(x-t) \, dx \, dt \right) = \int_{K_1} \Lambda \left( \int_{K_2} f(t)g(x-t) \, dx \right) \, dt \]
A quick calculation shows that Fubini's Theorem is applicable (since
\[
\int_{K_1} \int_{K_2} \| \Lambda(f(t)g(x-t)) \| dx dt \leq \| \Lambda \|_0 \int_{K_1} \int_{K_2} \| f(t) \| \| g(x-t) \| dx dt
\]
\[
= \| \Lambda \|_0 \int_{K_1} \int_{K_2} f(t) \left( \int_{K_2} \| g(x-t) \| dx \right) dt
\]
which is finite since \( f \in \mathcal{L}^1(A) \) and \( g \in \mathcal{B}(A) \). Thus we get \( (2) \) and the proof is complete. \( \square \)

**Definition 2.2.5.** For \( f \in \mathcal{L}^1(A) \) we define
\[
\hat{f}(t) = \lim_{n \to \infty} \int_{-n}^{n} f(x) e^{-itx} \, dm(x) \quad (3)
\]

**Remark 2.2.6.** First we observe that \( \int_{-n}^{n} \| f(x) \| dm(x) \leq \| f \|_1 < \infty \).

If \( s_n = \int_{-n}^{n} \| f(x) \| \, dm(x) \) for all \( n \), then \( s_n \in \mathbb{R} \) and we have
\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} \int_{-n}^{n} \| f(x) \| \, dm(x) = \| f \|_1 < \infty. \]
This means \( (s_n) \) is a convergent sequence in \( \mathbb{R} \). Now let
\[
H_n(t) = \int_{-n}^{n} f(x) e^{-itx} \, dm(x) \text{ for all } n.
\]
For each fixed \( t \), we claim that \( H_n(t) \) converges in \( A \).

Let \( a_n = \int_{-n}^{n} f(x) e^{-itx} dm(x) \) \( \forall n \).
For $n > m$ we consider

$$\|a_n - a_m\| = \left\| \int_{-n}^{n} f(x)e^{-itx} \, d\mu(x) - \int_{-m}^{m} f(x)e^{-itx} \, d\mu(x) \right\|$$

$$= \left\| \int_{-n}^{m} f(x)e^{-itx} \, d\mu(x) + \int_{m}^{n} f(x)e^{-itx} \, d\mu(x) \right\|$$

$$\leq \int_{-n}^{m} \|f(x)\| \, d\mu(x) + \int_{m}^{n} \|f(x)\| \, d\mu(x)$$

$$= \|s_n - s_m\|.$$

$(s_n)$ is Cauchy implies that $(a_n)$ is Cauchy. Since $A$ is complete we get that $(a_n)$ is convergent in $A$. This means that $\lim_{n \to \infty} \int_{-n}^{n} f(x)e^{-itx} \, d\mu(x)$ exists as an element of $A$.

Hence our claim. \hfill \Box

**Lemma 2.2.7.** If $f \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$, then $(\Lambda^* f)^\wedge = \Lambda \hat{f}$ for any arbitrary continuous linear functional $\Lambda$ on $A$.

**Proof.** Applying Lemma 1.4.2 and Dominated convergence theorem we can get

$$(\Lambda^* f)^\wedge(t) = \lim_{n \to \infty} \int_{-n}^{n} (\Lambda^* f)(x)e^{-itx} \, d\mu(x)$$

$$= \lim_{n \to \infty} \Lambda \left( \int_{-n}^{n} f(x)e^{-itx} \, d\mu(x) \right)$$

$$= \Lambda \left( \lim_{n \to \infty} \int_{-n}^{n} f(x)e^{-itx} \, d\mu(x) \right)$$

$$= \Lambda \hat{f}(t). \hfill \Box$$

**Lemma 2.2.8.** If $f \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$ and $\phi \in \mathcal{S}(A)$ then $(f \ast \phi)^\wedge = \hat{\phi}$.
Proof. Let \( K = \text{supp } \phi \)

\[
\widehat{(f*\phi)}(t) = \lim_{n \to \infty} \int_{-n}^{n} (f*\phi)(x)e^{-itx} \, dm(x)
\]

\[
= \lim_{n \to \infty} \int_{-n}^{n} \left( \int_{K} f(x-y)\Phi(y) \, dm(y) \right) e^{-itx} \, dm(x)
\]

\[
= \lim_{n \to \infty} \int_{-n}^{n} \left\{ \int_{-n}^{n} f(x-y)e^{-it(x-y)} \, dm(x) \Phi(y)e^{-ity} \right\} dm(y)
\]

by Lemma 2.2.4.

Since the integrands within parentheses are point-wise convergent to \( \hat{f}(t)\Phi(y)e^{-ity} \) as \( n \to \infty \) and bounded by \( \|f\|_1 \|\Phi(y)\|_0 \) which is in \( L^1(\mathbb{R}) \), by Dominated Convergence theorem 1.4.10. we get for all \( t \in \mathbb{R} \)

\[
(f*\phi)^{\wedge}(t) = \int_{K} \hat{f}(t)\Phi(y) e^{-ity} \, dm(y)
\]

\[
= \hat{f}(t)\hat{\Phi}(t)
\]

\[
= \hat{(f*\phi)}(t).
\]

Thus \( (f*\phi)^{\wedge} = \hat{f} \).

In the following we take \( A=\mathbb{C}^n \) for some positive integer \( n \) or any Hilbert space satisfying the conditions stated in the beginning.

We shall denote the Banach Algebra norm in \( A \) by \( \| \|_A \), the norm in \( A \) induced by the inner product by \( \| \|_H \) and the inner product of \( z \) and \( w \) by \( \langle z, w \rangle \).

For example if \( z = (z_1, z_2, \ldots, z_n) \), \( w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n \)
then $\|z\|_H = \left( \sum_{i=1}^{n} |z_i|^2 \right)^{1/2}$, $\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i$, $\|z\|_A = \max_{1 \leq i \leq n} (|z_i|)$.

If $f \in \mathcal{L}^2(A)$ we denote $\left( \int_{\mathbb{R}} \|f(x)\|_H^2 \, dm(x) \right)^{1/2}$ by $\|f\|_H$ and $\left( \int_{\mathbb{R}} \|f(x)\|_A^2 \, dm(x) \right)^{1/2}$ by $\|f\|_2$ or by $\|f\|_A$.

Definition 2.2.9. For $f, g \in \mathcal{L}^2(A)$ we define an innerproduct

$$\langle f, g \rangle = \int_{\mathbb{R}} \langle f(x), g(x) \rangle \, dm(x) \quad (4).$$

Theorem 2.2.10. $\mathcal{L}^2(A)$ is a Hilbert space under the innerproduct (4).

Proof. Let $\|a\|_H < c \|a\|_A$ for all $a \in A$. The innerproduct is well defined since

$$\int_{\mathbb{R}} |\langle f(x), g(x) \rangle| \, dm(x) \leq \int_{\mathbb{R}} \|f(x)\|_H \|g(x)\|_H \, dm(x)$$

$$\leq \left( \int_{\mathbb{R}} \|f(x)\|_H^2 \, dm(x) \right)^{1/2} \times \left( \int_{\mathbb{R}} \|g(x)\|_H^2 \, dm(x) \right)^{1/2}$$

$$\leq c^2 \left( \int_{\mathbb{R}} \|f(x)\|_A^2 \, dm(x) \right)^{1/2} \times \left( \int_{\mathbb{R}} \|g(x)\|_A^2 \, dm(x) \right)^{1/2}$$

$$\leq c^2 \|f\|_2 \|g\|_2$$

$$< \infty.$$ 

It is easy to verify that $\mathcal{L}^2(A)$ is an innerproduct space.
with respect to the inner product given by (4). Since \( L^2(A) \) with \( \| \cdot \|_A \) is complete (Theorem 1.4.6.) and since \( \| \cdot \|_A \) and \( \| \cdot \|_H \) are equivalent in A we get that \( L^2(A) \) is a Hilbert space with respect to the inner product given by (4). \( \square \)

**Lemma 2.2.11**. Let \( f \in L^1(A) \). Then

(i) \( \int_E <f(x), y> \, dx = \int_E \langle f(x) \rangle, y> \) for all \( y \in A \)

(ii) \( <y, \int_E f(x) \, dx> = \int_E <y, f(x)> \, dx \) for all \( y \in A \)

for any Borel set \( E \subseteq \mathbb{R} \) of finite measure.

**Proof.** We can easily prove (i) for any characteristic function on a finite measurable subset. If \( \chi \) denotes the \( A \)-valued characteristic function on \( F \), (i.e. taking only values 1 and 0 of \( A \) ) then

\[
\int_E \langle \chi (x), y \rangle \, dx = \int_{F \cap E} <\chi, y> \, dx + \int_{F \setminus E} <0, y> \, dx = <\chi, y> m(F \cap E)
\]

\[= \int_E \langle \chi (x), y \rangle \, dx. \] Since any simple function is a finite linear combination of characteristic functions and since inner product is linear in the first variable we get (i) for any simple Bochner integrable function. If \( f \) is any Bochner integrable function then there exists a sequence \( (f_n) \) of simple Bochner integrable functions such that

\( (f_n) \rightharpoonup f \) a.e. and \( \int_E f_n \rightharpoonup \int_E f. \)

Using the continuity of the inner product we can get (i).
Similarly (ii) can be proved.

\[\]

**Theorem 2.2.12.** If \( f \in L^1(A) \cap L^2(A) \), then \( \|f\|_H = \|f\|_H^\wedge \).

**Proof.** Let \( g(x) = \langle f, f_x \rangle \) for all \( x \in \mathbb{R} \). Then \( g: \mathbb{R} \rightarrow \mathbb{C} \) and

\[
g(x) = \int_{\mathbb{R}} \langle f(y), f(x+y) \rangle \, dm(y) \quad \forall x \in \mathbb{R}.
\]

We first note that as in the classical case \( x \rightarrow f_x \) is uniformly continuous from \( \mathbb{R} \) to \( L^2(A) \) and using the continuity of the inner product we see that \( g \) is continuous on \( \mathbb{R} \). Now

\[
|g(x)| \leq \int_{\mathbb{R}} |\langle f(y), f(x+y) \rangle| \, dm(y)
\]

\[
\leq \int_{\mathbb{R}} \|f(y)\|_H \|f(x+y)\|_H \, dm(y)
\]

\[
\leq c^2 \int_{\mathbb{R}} \|f(y)\|_A \|f(x+y)\|_A \, dm(y)
\]

\[
\leq c^2 \|f\|_2^2
\]

using Holder's inequality and the fact that \( \|f\|_2 = \|f_x\|_2 \).

Thus \( g \) is bounded. As \( g \) is continuous it is Borel measurable.

Moreover,

\[
\int_{\mathbb{R}} |g(x)| \, dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\langle f(y), f(x+y) \rangle| \, dm(y) \right) \, dm(x)
\]

\[
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle f(y), f(x+y) \rangle| \, dm(y) \, dm(x)
\]

\[
\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \|f(y)\|_H \|f(x+y)\|_H \, dm(y) \, dm(x)
\]

\[
\leq c^2 \|f\|_1^2
\]

by Fubini's theorem.
So we get that $g \in L^1(\mathbb{R})$. Using the classical techniques as in [30] we can get

$$\lim_{\lambda \to 0} (g * h_{\lambda})(0) = g(0)$$

(5)

and

$$\lim_{\lambda \to 0} (g * h_{\lambda})(0) = \int_{\mathbb{R}} \hat{g}(t) \, dm(t)$$

(6)

where $h_{\lambda}(x) = \int e^{-\lambda |t|} e^{itx} \, dm(t)$, $\lambda > 0$.

Using the definition of $g$ in (5) we get

$$\lim_{\lambda \to 0} (g * h_{\lambda})(0) = \langle f,f \rangle = \|f\|_H^2$$

(7)

Now using Fubini's theorem, Lemma 2.2.11. and the Dominated Convergence Theorem wherever necessary we get

$$\hat{g}(t) = \int_{\mathbb{R}} g(x) e^{-itx} \, dm(x)$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle f(y), f(x+y) \rangle \, dm(y) \right) e^{-itx} \, dm(x)$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle f(y) e^{ity}, f(x+y) e^{it(x+y)} \rangle \, dm(x) \right) \, dm(y)$$

$$= \lim_{n \to \infty} \lim_{k \to \infty} \left( \int_{-n}^{n} \langle f(y) e^{ity}, f(x+y) e^{it(x+y)} \rangle \, dm(x) \right) \, dm(y)$$

$$= \lim_{n \to \infty} \lim_{k \to \infty} \left[ \langle f(y) e^{ity}, \int_{-k}^{k} f(x+y) e^{it(x+y)} \, dm(x) \rangle \right] \, dm(y)$$

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\[
\begin{aligned}
&= \lim_{n \to \infty} \int_{-n}^{n} \left( \left\langle f(y)e^{iy}, \lim_{k \to \infty} \int_{-k}^{k} f(x+y)e^{it(x+y)} \, dm(x) \right\rangle \right) \, dm(y) \\
&= \lim_{n \to \infty} \int_{-n}^{n} \left\langle f(y)e^{iy}, f(-t) \right\rangle \, dm(y) \\
&= \left\| \hat{f}(-t) \right\|_H^2.
\end{aligned}
\]

Now (6) gives
\[
\lim_{\lambda \to 0} \left( g * h_{\lambda} \right)(0) = \left\| \hat{f} \right\|_H^2.
\]

(7) and (8) together imply \( \left\| f \right\|_H^2 = \left\| \hat{f} \right\|_H^2 \).

Lemma 2.2.13. If \( f \in L^1(A) \cap L^2(A) \), then \( \hat{\tilde{f}} = \hat{f} \) where \( \tilde{f}(x) = f(-x) \) for all \( x \in \mathbb{R} \).

Proof. If \( f \in L^1(A) \cap L^2(A) \), then for any continuous linear functional \( \Lambda \) on \( A \) we have \( \Lambda f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and by classical Plancherel theorem on \( L^2(\mathbb{R}) \) we get \( \left( \Lambda f \right)^\sim = \left( \Lambda f \right)^\hat{} \). By repeated application of Lemma 2.2.7. we get \( \Lambda \tilde{f} = \Lambda \hat{f} \). Since \( \Lambda \) is an arbitrary continuous linear functional on \( A \) we get that \( \tilde{f} = f \).

We now develop (in the following) a Plancherel Theorem on \( L^2(A) \) as an analogue of Theorem 1.2.4.
Theorem 2.2.14. To each \( f \in L^2(A) \) we assign \( \hat{f} \in L^2(A) \) such that

(i) If \( f \in L^1(A) \cap L^2(A) \) then \( \|f\|_H = \|\hat{f}\|_H \)

(ii) \( f \rightarrow \hat{f} \) is a Hilbert space isomorphism of \( L^2(A) \) onto \( L^2(A) \).

Proof. (i) follows from Theorem 2.2.12.

(ii) Let \( f \in L^2(A) \) and \( f_n = \chi_{[-n,n]} f \) for all \( n = 1, 2, \ldots \) where \( \chi_{[-n,n]} \) denotes the characteristic function on \([−n,n]\). It is clear that \( (f_n) \in L^1(A) \cap L^2(A) \) and \( \|f_n - f\|_A \rightarrow 0 \) as \( n \rightarrow \infty \). Since the norms are equivalent we get \( \|f_n - f\|_H \rightarrow 0 \) as \( n \rightarrow \infty \). By (i) \( \|\hat{f}_n\|_H = \|f_n\|_H \). Since \( (f_n) \) is Cauchy with respect to \( \|\cdot\|_H \) we get that \( (\hat{f}_n) \) is Cauchy with respect to \( \|\cdot\|_H \) and therefore with respect to \( \|\cdot\|_A \). By the completeness of \( L^2(A) \) we get that \( (\hat{f}_n) \) is convergent in \( L^2(A) \) (say) to \( \hat{f} \) with respect to \( \|\cdot\|_A \) and therefore with respect to \( \|\cdot\|_H \).

Moreover

\[
\|\hat{f}\|_H = \lim_{n \rightarrow \infty} \|\hat{f}_n\|_H = \lim_{n \rightarrow \infty} \|f_n\|_H = \|f\|_H. \tag{9}
\]

Now using Lemma 2.2.13 and the continuity of Fourier transform we can obtain \( \hat{f} = f \) for any \( f \in L^2(A) \) and this implies that the mapping \( f \rightarrow \hat{f} \) from \( L^2(A) \) to \( L^2(A) \) is onto. Hence the theorem.

\[2.3.\] Bohemian space \( B(L^2(A), \Delta) \)

We take \( A \) as before. Let \( G = L^2(A) \) and \( S = \mathcal{A}(A) \). For \( f \in G \),
we define \( f \ast g \) as in Theorem 2.2.1. and \( f \ast g \in G \) by Theorem 2.2.2. We now obtain a number of preliminary results for the construction of our Boehmian spaces.

Lemma 2.3.1. (i) If \( g_1, g_2 \in S \), then \( g_1 \ast g_2 \in S \) and \( g_1 \ast g_2 = g_2 \ast g_1 \).

(ii) If \( f, g \in S \) and \( h \in S \), then \( (f \ast g) \ast h = f \ast h + g \ast h \).

(iii) If \( f \in G \) and \( g, h \in S \), then \( (f \ast g) \ast h = f \ast (g \ast h) \).

Proof. Proofs of (i) to (iii) are simple analogues of the classical case and so we prefer to omit them.

Definition 2.3.2. A sequence of \( A \)-valued functions \((\delta_n) \in S \) is said to be in \( \Delta \) if

(i) \( \int_R \delta_n(x) \, dm(x) = I \) where \( I \) denotes the identity in \( A \).

(ii) \( \int_R \| \delta_n(x) \| \, dm(x) \leq M \), for all \( n \) for some \( M > 0 \) and

(iii) \( \text{supp} \delta_n \to 0 \) as \( n \to \infty \).

Theorem 2.3.3. Let \( f, g \in G \) and \((\delta_i) \in \Delta \) be such that \( f \ast \delta_i = g \ast \delta_i \) for all \( i = 1, 2, \ldots \). Then \( f = g \) in \( G \).

Proof. We first claim that \( f \ast \delta_i \to f \) as \( i \to \infty \) in \( L^2(A) \).

Let \( \text{supp} \delta_i \subseteq K \) for all \( i \). Consider

\[
\| f \ast \delta_i - f \|_2^2 = \int_R \left( \int_K (f(x-y) - f(x)) \delta_i(y) \, dm(y) \right)^2 \, dm(x)
\]

\[
\leq \int_R \left( \int_K \| f(x-y) - f(x) \| \delta_i(y) \, dm(y) \right)^2 \, dm(x). \quad (10)
\]
Let \( \lambda = \int_K \|\delta_1(y)\| dm(y) \leq M \) and \( d\mu(y) = \frac{1}{\lambda} \|\delta_1(y)\| dm(y) \). By Jensen's inequality (10) becomes

\[
\|f \ast \delta_1 - f\|^2 \leq \lambda \left( \int_R \left( \int_K \|f(x-y)-f(x)\|^2 \|\delta_1(y)\| dm(y) \right) dm(x) \right) 
\]

\[
\leq M \int_K \|\delta_1(y)\| \left( \int_R \|f(x-y)-f(x)\|^2 dm(x) \right) dm(y) \tag{11}
\]

If \( f \in L^2(A) \) the mapping \( y \mapsto f_y \) where \( f_y(x) = f(x-y) \) is uniformly continuous from \( R \to L^2(A) \). Therefore for given \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that \( |y| < \eta \Rightarrow \|f_y - f\| < \varepsilon / M \). Since \( \text{supp} \delta_1 \to 0 \) as \( i \to \infty \), we can have \( \text{supp} \delta_1 \subseteq [-\eta, \eta] \) for sufficiently large \( i \). Hence for sufficiently large \( i \) (11) becomes

\[
\|f \ast \delta_1 - f\|^2 \leq M \int_{|y| < \eta} \|\delta_1(y)\| \|f_y - f\|^2 dm(y)
\]

\[
< \left( \frac{\varepsilon^2}{M} \right) \int_{|y| < \eta} \|\delta_1(y)\| dm(y)
\]

\[
< \varepsilon^2.
\]

In a similar manner \( g \ast \delta_1 \to g \) in \( L^2(A) \) as \( i \to \infty \). The proof of the theorem now follows by taking \( L^2 \)-limits in the equalities \( f \ast \delta_i = g \ast \delta_i \) for all \( i \). \( \square \)

**Theorem 2.3.4.** Let \( \delta = (\delta_1, \delta_2, \delta_3, \ldots, \) \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \) be in \( \Delta \). Then \( \delta \ast \varepsilon = (\delta_1 \ast \varepsilon_1, \delta_2 \ast \varepsilon_2, \delta_3 \ast \varepsilon_3, \ldots, \) \( \varepsilon \Delta \).
Proof. We have (i) \( \int_{\mathbb{R}} \delta_i(x) dm(x) = \int_{\mathbb{R}} \varepsilon_i(x) dm(x) = 1 \) for all \( i \).

(ii) \( \int_{\mathbb{R}} \|\delta_i(x)\| dm(x) \leq M_1, \int_{\mathbb{R}} \|\varepsilon_i(x)\| dm(x) \leq M_2 \) for all \( i \)
for some \( M_1, M_2 > 0 \).

(iii) \( \text{supp} \, \delta_i \rightarrow 0, \text{supp} \, \varepsilon_i \rightarrow 0 \) as \( i \rightarrow \infty \).

We first prove that \( \int_{\mathbb{R}} (\delta_i * \varepsilon_i)(x) \ dm(x) = 1 \) for all \( i \).

Let \( \text{supp} \, \delta_i \subseteq K_1 \) for all \( i \), \( \text{supp} \, \varepsilon_i \subseteq K_2 \) for all \( i \),
\( G_i = \text{supp} \, \delta_i * \varepsilon_i \) so that \( G_i \subseteq K_1 + K_2 \). Now
\[
\int_{\mathbb{R}} (\delta_i * \varepsilon_i)(x) \ dm(x) = \int_{G_i} (\delta_i * \varepsilon_i)(x) \ dm(x) \\
= \int_{G_i} \left( \int_{K_1} \delta_i(t) \varepsilon_i(x-t) \ dm(t) \right) \ dm(x) \\
= \int_{K_1} \left( \int_{G_i} \delta_i(t) \varepsilon_i(x-t) \ dm(x) \right) \ dm(t) \\
= 1
\]
by Fubini's theorem.

Further,
\[
\int_{\mathbb{R}} \|\delta_i * \varepsilon_i(x)\| \ dm(x) = \int_{G_i} \|\delta_i * \varepsilon_i(x)\| \ dm(x) \\
\leq \int_{G_i} \left( \int_{K_1} \|\delta_i(t)\| \|\varepsilon_i(x-t)\| \ dm(t) \right) \ dm(x) \\
\leq \int_{K_1} \|\delta_i(t)\| \left( \int_{G_i} \|\varepsilon_i(x-t)\| \ dm(x) \right) \ dm(t)
\]
\[ \leq \int_{K_1} \| \delta_i(t) \| \left( \int_{K_2} \| \epsilon_1(s) \| \, dm(s) \right) \, dm(t) \]

\[ \leq M_1 M_2 \quad \text{for all } i. \]

Since \( \text{supp} (\delta_i * \epsilon_i) \subseteq \text{supp} \delta_i + \text{supp} \epsilon_i \), we get that \( \text{supp} (\delta_i * \epsilon_i) \to 0 \) as \( i \to \infty \). Hence the theorem. \( \Box \)

In view of Theorem 2.3.3 and Theorem 2.3.4, the family \( \Delta \) can be called as delta sequences in the sense of [11].

We now verify that the convergence in \( L^2(A) \) satisfies the following conditions.

Theorem 2.3.5. (i) If \( \lim_{n \to \infty} f_n = f \) in \( L^2(A) \) and \( \delta \in S \), then
\[ \lim_{n \to \infty} f_n * \delta = f * \delta. \]

(ii) If \( \lim_{n \to \infty} f_n = f \) in \( L^2(A) \) and \( (\delta_n) \in \Delta \), then \( \lim_{n \to \infty} f_n * \delta = f. \)

Proof. (i) By Theorem 2.2.2, we get that
\[ \| f_n * \delta - f * \delta \|_2^2 = \| (f_n - f) * \delta \|_2^2 \leq \| f_n - f \|_2 \| \delta \|_1. \]

By hypothesis this tends to zero as \( n \to \infty \).

(ii) Consider \( \| f_n * \delta_n - f \|_2 \leq \| f_n - f \|_2 \| \delta_n \|_1 + \| f * \delta_n - f \|_2. \) The first term of the right-hand side tends to zero as \( n \to \infty \) by hypothesis and by the property (ii) of delta sequences. The second term of the above inequality also tends to zero as observed in the proof of Theorem 2.3.3. \( \Box \)

In view of the above three theorems we can construct
the Boehmian space in the canonical way using \( L^2(A) \) and \( \Delta \). (See [11]). This space we denote by \( \mathcal{B}(L^2(A), \Delta) \). Convergence in this space is taken as \( \delta \)-convergence.

In the following theorem we have shown that \( L^2(A) \) can be imbedded in the Boehmian space \( \mathcal{B}(L^2(A), \Delta) \).

**Theorem 2.3.6.** The mapping \( f \rightarrow \left[ \frac{f \ast \delta_i}{\delta_i} \right] \) where \((\delta_i) \in \mathcal{D}(A)\) is a continuous imbedding of \( L^2(A) \) into \( \mathcal{B}(L^2(A), \Delta) \).

**Proof.** The mapping is one-to-one since \( \left[ \frac{f \ast \delta_i}{\delta_i} \right] = \left[ \frac{g \ast \delta_i}{\delta_i} \right] \) implies \((f \ast \delta_i) \ast \delta_j = (g \ast \delta_i) \ast \delta_j \) for all \( i, j \). In particular since \( \delta_i \ast \delta_i = \delta_i^2 \) we have \( f \ast \delta_i = g \ast \delta_i \). Applying Lemma 2.3.1., Theorem 2.3.4. and Theorem 2.3.3. successively we get \( f = g \) in \( L^2(A) \). We now show that this map is continuous.

Let \( (f_n) \rightarrow 0 \) in \( L^2(A) \). We claim that \( x_n = \left[ \frac{f_n \ast \delta_i}{\delta_i} \right] \rightarrow 0 \) in \( \mathcal{B}(L^2(A), \Delta) \). We need only to observe that for each \( i \) \( x_n \ast \delta_i = f_n \ast \delta_i \) which tends to 0 in \( L^2(A) \) as \( n \rightarrow \infty \) by Theorem 2.3.5. and thus the proof is complete. \( \Box \)

Now we extend Lemma 2.2.8. to elements of \( L^2(A) \).

**Lemma 2.3.7.** If \( f \in L^2(A) \) and \( g \in \mathcal{D}(A) \), then \((f \ast g)^\wedge = f \ast g\).
Proof. Let \((f_n) \to f\) in \(L^2(A)\) where \(f_n \in L^1(A) \cap L^2(A)\). Such a sequence exists as we saw in the proof of Theorem 2.2.14. By Theorem 2.3.5. \(f_n * g \to f * g\) in \(L^2(A)\). So by Plancherel theorem (Theorem 2.2.14.)

\[
(f_n * g) \to (f * g) \quad \text{in} \quad L^2(A).
\]

Thus by Lemma 2.2.8.

\[
\hat{f_n} \hat{g} \to (f * g)^\wedge \quad \text{in} \quad L^2(A).
\] (12)

On the other hand \(f_n \to f\) implies \(\hat{f_n} \to \hat{f}\) in \(L^2(A)\). Now as in the classical case \(g\) as a function of \(t\) is bounded. Hence

\[
\hat{f_n} \hat{g} \to \hat{f} \hat{g} \quad \text{in} \quad L^2(A).
\] (13)

(12) and (13) imply \((f * g)^\wedge = \hat{f} \hat{g}\).

\[\square\]

### 2.4. Boehmian Space \(B(L^2(A), \hat{\Delta})\)

We shall now describe yet another Boehmian space which contains \(L^2(A)\). Let \(G = L^2(A)\) and \(S_1 = \hat{S} = \{\hat{\delta}/\delta \in S\}\) where \(S = \mathcal{B}(A)\).

For \(f \in G\), \(\hat{\delta} \in S_1\) we define \((\hat{f} \hat{\delta})(x) = f(x) \hat{\delta}(x)\) for all \(x \in \mathbb{R}\).

**Lemma 2.4.1.** If \(f \in G\) and \(\hat{\delta} \in S_1\), then \(\hat{f} \hat{\delta} \in G\).

**Proof.** We observe that \(\hat{f} \hat{\delta}\) is Borel-measurable and for all \(t\),

\[
\|\hat{\delta}(t)\|_A \leq \|\delta\|_1.
\]

So we get that

\[
\int_{\mathbb{R}} \|\hat{f} \hat{\delta}\|_A^2 \, dm(t) = \int_{\mathbb{R}} \|f(t)\|_A^2 \|\hat{\delta}(t)\|_A^2 \, dm(t) \leq \int_{\mathbb{R}} \|f(t)\|_A^2 \|\delta\|_1^2 \, dm(t),
\]

\[
= \|f\|_2^2 \|\delta\|_1^2,
\]

41
which is finite. Hence \( f \hat{\in} G \). The lemma follows.

Lemma 2.4.2. The mapping \( (f, \hat{\delta}) \rightarrow f \hat{\delta} \) from \( G \times S_1 \rightarrow G \) satisfies the following properties.

(i) If \( \hat{\delta}_1, \hat{\delta}_2 \in S_1 \), then \( \hat{\delta}_1 \hat{\delta}_2 \in S_1 \) and \( \hat{\delta}_1 \hat{\delta}_2 = \hat{\delta}_2 \hat{\delta}_1 \).

(ii) If \( f, g \in G \) and \( \hat{\delta} \in S_1 \), then \( (f+g) \hat{\delta} = f \hat{\delta} + g \hat{\delta} \).

(iii) If \( f \in G \) and \( \hat{\delta}, \hat{\varepsilon} \in S_1 \), then \( (f \hat{\delta}) \hat{\varepsilon} = f(\hat{\delta} \hat{\varepsilon}) \).

Proof. Since \( A \) is a commutative Banach Algebra the required results are immediate.

Definition 2.4.3. The set of all sequences \( (\hat{\delta}_i) \) such that \( (\hat{\delta}_i) \in \hat{\Delta} \) is denoted by \( \hat{\Delta} \).

Lemma 2.4.4. Let \( f, g \in G \) and \( (\hat{\delta}_i) \in \hat{\Delta} \) such that \( f \hat{\delta}_i = g \hat{\delta}_i \) for all \( i \). Then \( f = g \) in \( G \).

Proof. Since \( f \in L^2(A) \), \( f \) also belongs to \( L^2(A) \). As in the proof of Theorem 2.3.3. for any \( (\hat{\delta}_i) \in \hat{\Delta} \) we have \( \hat{f} * \hat{\delta}_i \rightarrow \hat{f} \) in \( L^2(A) \) as \( i \rightarrow \infty \). Since the Plancherel transform is continuous on \( L^2(A) \), we get \( \hat{f} \hat{\delta}_i \rightarrow f \). Equivalently we get \( \hat{f} \hat{\delta}_i \rightarrow f \). In a similar way we get \( \hat{g} \hat{\delta}_i \rightarrow g \).

We note that if \( (\hat{\delta}_i), (\hat{\varepsilon}_i) \) are two delta sequences in \( \hat{\Delta} \), then by definition \( (\delta_i), (\varepsilon_i) \) are delta sequences in \( \Delta \). So by Theorem 2.3.4. \( (\delta_i \ast \varepsilon_i) \in \Delta \). Thus by Lemma 2.3.7. \( (\delta_i \varepsilon_i) \in \Delta \).
In view of the above lemmas the elements of $\hat{\Delta}$ can be
called as delta sequences (in the sense of [11]).

Now we shall verify that the convergence in $G$ satisfies
the following conditions.

Lemma 2.4.5. (i) If $\lim_{n} f_n = f$ in $L^2(A)$ and $\delta \in S_1$, then

$$\lim_{n} f_n \delta = f\delta \text{ in } L^2(A).$$

(ii) If $\lim_{n} f_n = f$ in $L^2(A)$ and $(\delta_n) \in \Delta$, then $\lim_{n} f_n \delta_n = f$
in $L^2(A)$.

Proof. (i) Since $\delta(t)$ as a function of $t$ is bounded we get the result (i).

(ii) follows by Theorem 2.2.14. and Theorem 2.3.5(ii).

The following theorem proves that $L^2(A)$ can also be imbedded in our second Boehmian space $\mathfrak{B}(L^2(A),\hat{\Delta})$.

Theorem 2.4.6. The mapping

$$i: f \mapsto \left[ \begin{array}{c} f\delta_i \\ \delta_i \end{array} \right], \quad (\delta_i) \in \hat{\Delta}$$

is a continuous embedding of $L^2(A)$ into $\mathfrak{B}(L^2(A),\hat{\Delta})$.

Proof. If $f \in L^2(A)$ we note that $f\delta_n\delta_n$ is a quotient...
in the sense of 1.5. as \( f_{n m} \delta_n = f_{m n} \delta_m \). So \( \left[ \frac{f_{n m} \delta_n}{\delta_n} \right] \) belongs to \( \mathfrak{B}(L^2(A), \Delta) \). The map \( i \) is one-to-one since

\[
\left[ \frac{f_{n m} \delta_n}{\delta_n} \right] = \left[ \frac{g_{n m} \delta_n}{\delta_n} \right]
\]

implies \( f_{n m} \delta_n = g_{n m} \delta_n \) for all \( n, m \). In particular \( f_{n n} \delta_n = g_{n n} \delta_n \) for all \( n \). As usual letting \( n \to \infty \) we get \( f = g \) in \( L^2(A) \). We now claim that the map \( i \) is continuous. Let \( (f_n) \to 0 \) in \( L^2(A) \) as \( n \to \infty \). We claim that \( x_n = \left[ \frac{f_{n i} \delta_i}{\delta_i} \right] \to 0 \) in \( \mathfrak{B}(L^2(A), \Delta) \). For each \( i \) we have \( x_{n i} \delta_i = f_{n i} \delta_i \). By Lemma 2.4.5.

for each \( i \) we get \( f_{n i} \delta_i \to 0 \) in \( L^2(A) \) as \( n \to \infty \). This implies that \( x_n \delta_i \to 0 \).

In view of the above lemmas we see that \( \mathfrak{B}(L^2(A), \Delta) \) is a Boehmian space containing \( L^2(A) \). We shall equip this with its usual \( \delta \)-convergence.
2.5. Fourier-Plancherel Transform for Boehmians

Definition 2.5.1. Let \( x = \left[ \frac{f_n}{\phi_n} \right] \in \mathcal{B}(L^2(A), \Delta) \). We define the Fourier transform of \( x \) as \( \left[ \frac{\hat{f}_n}{\hat{\phi}_n} \right] \) and we denote it by \( \hat{x} \).

Remark 2.5.2. The Fourier transform is well defined. For if \( x \) has two representations (say) \( x = \left[ \frac{f_n}{\phi_n} \right] = \left[ \frac{g_n}{\xi_n} \right] \), where \( f_n, g_n \in L^2(A) \) and \( (\phi_n), (\xi_n) \in \Delta \), then \( f_n \ast \xi_m = g_m \ast \phi_n \). Taking Fourier-Plancherel transform on both sides and using Lemma 2.3.7, we get \( \frac{\hat{f}_n}{\hat{\phi}_n} = \frac{\hat{g}_n}{\hat{\xi}_n} \). This gives that \( \left[ \frac{\hat{f}_n}{\hat{\phi}_n} \right] = \left[ \frac{\hat{g}_n}{\hat{\xi}_n} \right] \).

Theorem 2.5.3. Let \( \mathcal{B}(L^2(A), \Delta) \rightarrow \mathcal{B}(L^2(A), \Delta) \) be defined by \( F(x) = \hat{x} \). Then \( F \) is a continuous one-to-one map from \( \mathcal{B}(L^2(A), \Delta) \) onto \( \mathcal{B}(L^2(A), \Delta) \).

Proof. Let \( (x_n) \rightarrow 0 \) in \( \mathcal{B}(L^2(A), \Delta) \). (say) \( x_n = \left[ \frac{f_n}{\phi_n} \right] \).

(By remark 1.5.5. we can take a common \( \delta \)-sequence for the denominators of all \( x_n \)'s.)
We claim that \( x_n = \left[ \frac{\hat{f}_{n,i}}{\phi_i} \right] \stackrel{\delta}{\longrightarrow} 0 \) in \( \mathcal{B}(L^2(A), \Delta) \). By hypothesis for each fixed \( i \) as \( n \rightarrow \infty \), \( (f_{n,i}) \rightarrow 0 \) in \( L^2(A) \) with respect to \( \| \cdot \|_A \). So for each fixed \( i \) as \( n \rightarrow \infty \), 
\( (f_{n,i}) \rightarrow 0 \) in \( L^2(A) \) with respect to \( \| \cdot \|_H \). By Theorem 2.2.14. for each fixed \( i \) as \( n \rightarrow \infty \), 
\( \left( \hat{f}_{n,i} \right) \rightarrow 0 \) in \( L^2(A) \) with respect to \( \| \cdot \|_A \) and since \( \| \cdot \|_A \) and \( \| \cdot \|_H \) are equivalent we get for each fixed \( i \) as \( n \rightarrow \infty \), 
\( \left( \hat{f}_{n,i} \right) \rightarrow 0 \) in \( L^2(A) \) with respect to \( \| \cdot \|_A \). This implies that \( x_n \stackrel{\delta}{\longrightarrow} 0 \) in \( \mathcal{B}(L^2(A), \Delta) \).

We now prove that the map \( F \) is one-to-one. Let \( x_1 = x_2 \). Then 
\[
\begin{bmatrix}
\hat{f}_n \\
\hat{g}_n
\end{bmatrix}
= 
\begin{bmatrix}
\hat{g}_n \\
\hat{\xi}_n
\end{bmatrix},
\]
so \( \hat{f}_n \hat{\xi}_m = \hat{g}_n \hat{\phi}_n \). By Lemma 2.3.7, we get \( (f_n \ast \xi_m) \hat{=} (g_m \ast \phi_n) \). Since Fourier-Plancherel transform is one to one it follows that \( f_n \ast \xi_m = g_m \ast \phi_n \). So we get \( x_1 = x_2 \). We now claim that the map \( F \) is onto. Since Fourier-Plancherel transform is onto by Theorem 2.2.14 given \( y = \begin{bmatrix} g_n \\ \xi_n \end{bmatrix} \) in
\[ \mathbb{B}(L^2(A), \Delta) \) we take \( x = \left[ \frac{f_n}{\phi_n} \right] \) where \( \hat{f}_n = g_n \) and \( \hat{\phi}_n = \xi_n \). It can be easily verified that \( x \in \mathbb{B}(L^2(A), \Delta) \) and \( \hat{x} = y \). \( \square \)

Lemma 2.5.4. If \( x_1, x_2 \in \mathbb{B}(L^2(A), \Delta) \), then (i) \( \hat{x_1 + x_2} = \hat{x_1} + \hat{x_2} \).
(ii) \( (\lambda x)^{\wedge} = \lambda \hat{x} \), \( \lambda \in \mathbb{C} \) where addition and multiplication are defined as usual for Boehmians.

Proof. Results are immediate from the definitions.

From the above theorems we see that the Fourier-Plancherel transform is a one-to-one continuous linear mapping from \( \mathbb{B}(L^2(A), \Delta) \) onto \( \mathbb{B}(L^2(A), \hat{\Delta}) \).

Remark 2.5.5. \( f \in L^2(A) \) can be identified with the element \( x = \left[ \frac{f * \delta_1}{\delta_1} \right] \in \mathbb{B}(L^2(A), \Delta) \) where \( (\delta_1) \) is any delta sequence in \( \Delta \).

Its Plancherel transform as a Boehmian is \( \left[ \frac{(f * \delta_1)^{\wedge}}{\delta_1} \right] = \left[ \frac{\hat{f} * \delta_1}{\delta_1} \right] \).

This later Boehmian is nothing but the identification of \( \hat{f} \) in \( \mathbb{B}(L^2(A), \hat{\Delta}) \). So the Plancherel transform on \( \mathbb{B}(L^2(A), \Delta) \) is an extension of the Plancherel transform on \( L^2(A) \).

(ii) For \( x = \left[ \frac{f_n}{\phi_n} \right] \in \mathbb{B}(L^2(A), \Delta) \) and \( y = \left[ \frac{g_n}{\xi_n} \right] \), \( g_n \in \mathbb{B}(A), (\xi_n) \in \Delta \)

we define \( x * y = \left[ \frac{f_n * g_n}{\phi_n * \xi_n} \right] \). By Lemma 2.3.7, we get \( (x * y)^{\wedge} = \hat{x} \hat{y} \).
2.6. COMPARATIVE STUDY

As already observed Plancherel transform theory on $L^2(A)$ developed here is an extension of the classical Plancherel theorem. In the literature there are three types of Boehmian spaces on which the theory of Fourier transform is developed viz. $L^1$-Boehmians [14], tempered Boehmians [13] and more general tempered Boehmians[16]. In all these cases the Fourier transform was defined as a classical distribution. Since in an arbitrary separable Banach Algebra division does not make sense we have chosen the natural approach to define our Plancherel transform. However, the space $L^2(A)$, the space of compactly supported $[A:A]$ valued distributions belonging to $[B(A):A]$ are all subspaces of $B(L^2(A),\Delta)$. Thus the space $B(L^2(A),\Delta)$ is larger than $L^2(A)$. Moreover, in the classical case where $A$ is replaced by $C$, one can identify each element of $B(L^2(A),\Delta)$ as an element of $B$ defined in [13] and also verify that the original definition of $\hat{x}$ for $x \in B$ coincides with our definition.

However, when $A \neq C$ these two definitions are qualitatively different (since there is no natural way in which $B$ can be identified as a subspace of $B(L^2(A),\Delta)$). Further in the case of vector-valued functions the classical technique in which the Fourier transform can be identified as a distribution, no longer works.